Neurocomputing 483 (2022) 322–332

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Event-triggered impulsive control design for synchronization of inertial neural networks with time delays

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ARTICLE INFO

Article history:
Received 6 September 2021
Revised 8 December 2021
Accepted 3 February 2022
Available online 9 February 2022
Communicated by Zidong Wang

Keywords:
Inertial neural networks
Time delay
Event-triggered impulsive control
Synchronization

ABSTRACT

This paper investigates the synchronization problem of inertial neural networks (INNs) with time delays by virtue of event-triggered (E-T) impulsive control, in which a Lyapunov function based E-T mechanism is used to determine impulsive instants. The synchronization analysis of INNs through E-T impulsive control technique unlike time-triggered impulsive control, which would be activated when certain well-designed conditions exist, E-T impulsive control is only allowed when certain well-defined events occur. Besides that, control input is only required at triggered instants and also no control input is required for two triggered instants in a sequence. Considering the E-T mechanism, the synchronization analysis of the INNs is discussed by reduced and non-reduced order approaches and constructing the suitable Lyapunov functionals. Finally, two simulation results are provided to illustrate the efficacy of the theoretical results.

1. Introduction

The dynamical behaviours of neural networks (NNs) with inertia, i.e. inertial neural networks (INNs), have been extensively studied so far and many attractive results on the asymptotic and exponential stability of equilibrium for delayed INNs have been obtained. The synchronization problem of inertial delayed neural networks has become a prominent topic in recent years. Many potential applications for synchronization exist, including secure communications [1,2], information science, and image encryption [3,4]. With increased interest in INNs, many researchers are investigating the synchronization problem of delayed INNs. Zhang and Cao [5] investigated the finite-time synchronization of delayed INNs by using integral inequality method. In [6], global exponential stabilization and lag-synchronization of the delayed INNs are considered. By utilizing matrix measure and Halanay inequalities, some sufficient conditions for the stability and synchronization results are derived for a class of delayed memristive INNs [7], inertial bidirectional associative memory NNs [8]. In [2], a class of INNs with multi-proportional delays was considered. First, the original INNs with multi-proportional delays can be expressed as a first-order differential equation by designing a suitable variable substitution. Second, certain new and effective criteria for achieving finite-time and fixed-time synchronization of the INNs are established by using Lyapunov functionals and analytical techniques, in connection with unique control algorithms. Tang and Jian [9] investigated the exponential synchronization of INNs with discrete and finite distributed time-varying delays through periodically intermittent control. Although the switching speed of the power amplifiers is finite, time delays in neural systems are common when NNs are used in embedded systems, which affects the dynamical behaviors of NNs [1,2,4,9,10]. Therefore, it is required to investigate the synchronization of time-delayed INNs. Meanwhile, several control systems, including sliding mode control [11], adaptive control [12,13], feedback control [14], sampled-data control [15,16], pinning control [17,18], periodically intermittent control [9], impulsive control [19,20], event-triggered control (E-TC) [21,22], have been developed to investigate synchronization and stability problems.

Impulsive control has received a lot of interest because of its many applications in real-world networks. The convergence rate of the systems may become quicker or slower, or even non-convergent, if the state of nodes is subject to instantaneous change at certain impulsive instants [23]. Therefore, it is necessary to investigate the role of impulsive control in the synchronization and stability of the dynamical systems. Many studies are currently being researched on the impact of time-triggered (T-T) impulsive control on neural networks. But the working period of impulsive controller cannot be changed until controller is formatted. That

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https://doi.org/10.1016/j.neucom.2022.02.023
0925-2312/

is, the impulsive instants are pre-designed, making impulsive instants stationary [24–26]. In addition, the memristor based delayed NNs was explored by impulsive controller in [24], where the impulsive instants cannot be changed until the controller is formatted. In current technological advancements, the selection of a controllers such as sampled data control [27,28], H_∞ control [29], optimum control [30], sliding mode control [31], impulsive control [32], has a significant impact on the output of the system. There are two broad groups of state feedback control techniques for synchronization that have been studied thus far. One is continuous or piecewise continuous control, for example, sampled data control [27], optimum control [30], or sliding mode control [31], requires the controller to react as the state error changes. Moreover, continuous controller upgrades, would require a large amount of power. It's possible that the other, which is discontinuous, can aid with issues like impulse control. Furthermore, the impulsive control technique has sparked the interest of many scholars, because of its applications in finance, biological models, and medicine. It is a discontinuous control approach with a range of benefits over continuous control (like state-feedback control) systems, such as high reliability, efficiency, easy installation, and minimal maintenance costs. In general, the sample period of a time-triggered control (T-TC) cannot be set too high in order to ensure that delayed NNs are synchronized. However, if delayed NNs are in perfect working order to be dependable, a shorter sampling interval may result in resource waste. Furthermore, on a network channel with limited bandwidth, the T-TC technique dramatically increases the likelihood of packet waiting and collision. Thus, researchers have devised a control that is triggered by events in order to overcome the limitations imposed by existing T-TC techniques. This control is called E-TC. Unlike T-TC, the E-T mechanism has proven significant benefits in reducing controller update times in order to achieve the desired results [33–39]. For instance, in [36], the synchronization issue of delayed switched NNs with communication delays has been researched by E-TC. It significantly reduces the number of control updates required for coupled switched NNs synchronization tasks involving embedded microprocessors with limited on-board resources. To reduce the control updates a distributed E-TC technique under the periodic sampling has been introduced in [40] for leader less synchronization of delayed coupled NNs. In [41], secure communication based quantized synchronization results has been given for master–slave NNs under the E-TC. An E-TC strategy is a type of control application approach that is inserted between both the sampler and the controller by an E-T strategy that has been predesigned. When using E-TC, sampling is only accomplished when the state-dependent error reaches a tolerable level of severity. When compared to a T-TC procedure, the E-TC method has a lower likelihood of redundant information transfer. E-TC laws govern whether sampling should be performed rapidly or slowly, and whether sampling information should be conveyed for control update purposes. The approach was then applied to networks, and event-triggered impulsive control (E-TIC) approaches were presented as a result of the application. Whenever the E-TIC strategy is violated, the combination of delayed impulsive control and the E-T mechanism produces E-T instants, and delayed impulsive control is only applied at these E-T instants. As a result, in order to increase the performance of systems under E-TC, it is necessary to improve the approach used to determine whether an event has occurred. From the perspective of application, the E-TIC technique is appropriate, and the control approach will effectively increase the utilization of resources in systems with restricted bandwidth. Following up on the aforementioned investigations, the authors discuss INNs with E-TIC, and thoroughly investigate the synchronization criteria of this network. As far as the authors are aware, few literature studies have devoted to the synchronization problem for delayed INNs with both order reduced and non reduced method through the E-TIC scheme, which is another motivation of our research. Motivated by the preceding discussions, our paper's main conclusions are summarized below:

1. Up until now, no related synchronization results have been reported for delayed INNs by using reduced and non-reduced order method. Therefore, to shorten such a gap, we will present several new results guaranteeing synchronization of delayed INNs in this brief. To solve this problem an E-TIC approach is proposed. The NNs in this paper are with inertial term, which extend the earlier publications [34–40] without inertial term.
2. Several researchers in [1,5,6,9] apply the reduced-order method and in [42,43] apply the non reduced-order method to study the dynamical behavior of delayed INNs, as we known, the INNs system with reduced-order method may rigor in real world applications. Different from those works, in this brief learn about the two methods namely reduced order and non- reduced order (Direct method) methods are clearly described, as well as how their shortcomings are exposed and how non reduced order method is optimal to meet them.
3. Besides, by designing a E-TIC scheme, the leader–follower synchronization of the addressed delayed INNs is studied and some effective conditions are given. It is important to note that the dynamic behaviors of a class of INNs are analyzed in this paper by employing some new Lyapunov functions without using the variable transformation, this differs totally from the classical reduced order method.

Notations: Let $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and real n-dimensional space equipped with the Euclidean norm $||\cdot||$. $\mathbb{R}^{n\times m}$ the set of all real $n\times m$ matrices. For any matrix $M\in\mathbb{R}^{n\times n}, M^T$ represents the transpose of a square matrix M and the symbol $\star$ denotes the transpose of a block matrices and $diag\{\dots\}$ express a diagonal matrix. 0 and I are respectively denotes the zero and identity matrix with appropriate dimension. $\Gamma=\{1,2,\dots,N\}$.

2. Preliminaries and model description

Graph Theory: Let $\mathbb{G}=(\mathbb{V},\mathbb{E},\mathbb{A})$ be an N-order digraph, where the edge set is $\mathbb{E}$, and the node set is $\mathbb{V}$. The ordered pair of node (i,j) represents the directed edge $\mathscr{E}_{ij}\in\mathbb{E}$ in digraph $\mathbb{G}$, where node j is a node i's neighbor. Denote the set of neighbors of node i as $\Xi_i=\{j\in\mathbb{V}|\mathscr{E}_{ij}\in\mathbb{E}\}$. The definitions of matrices $\mathbb{A},\mathbb{B},\mathbb{D},\mathbb{L}$ and $\mathbb{H}$ can be found in [44] and thus are omitted here.

Consider the INNs consisting of N identical nodes with the dynamics of the i^{th} node is described by the following second-order differential equation:

$$\frac{d^2x_i(t)}{dt^2}=-A\frac{dx_i(t)}{dt}-Bx_i(t)+Cf(x_i(t))+Df(x_i(t-\varphi))+I(t)+v_i(t),\quad i\in\Gamma. \tag{1}$$

where $d^2x_i(t)/dt^2$ is called as the inertial term of the INNs (1); $x_i(t)=(x_{i1}(t),\dots,x_{in}(t))^T\in\mathbb{R}^n$ is the vector state of the i^{th} node at time t; $A=diag\{a_1,\dots,a_n\}, B=diag\{b_1,\dots,b_n\}$ are constant matrices with $a_p,b_p>0$; $C=(c_{pq})_{n\times n}$ and $D=(d_{pq})_{n\times n}$, $p,q=1,2,\dots,n$ denote the connection weight matrix and time delayed connection weight matrix respectively; The nonlinear function $f(x_i(t))=(f(x_{i1}(t)),\dots,f(x_{in}(t)))^T$ and $f(x_i(t-\varphi))=(f(x_{i1}(t-\varphi)),\dots,f(x_{in}(t-\varphi)))^T$ are the activations function for the INNs (1); φ is the constant time delay; $I(t)=(I_1(t),\dots,I_n(t))^T$ is the input vector; $v_i(t)$ represents the control input to realize leader-following synchronization to be defined later.

The initial conditions of the INNs(1) is represented as follows:

$$x_i(r) = \upsilon_i(r), \quad \frac{dx_i(r)}{ds} = \varrho_i(r), \quad -\varphi \leqslant r \leqslant 0,$$

where $\upsilon_i(r), \varrho_i(r) \in \mathscr{PC}^1([-\varphi,0],\mathbb{R}^n), \mathscr{PC}^1([-\tau,0],\mathbb{R}^n)$is the set of functions which are continuously differentiable defined on $[-\varphi,0]$.

To continue, consider the following assumptions, definitions, and lemmas.

Assumption 2.1. The activation functions $f_p(\cdot):\mathbb{R}\to\mathbb{R}$ in (1) satisfy Lipschitz condition, that is there exists a constants $F_p > 0$ for all $p = 1,\ldots,n, u, v \in \mathbb{R}$ such that

$$| f_p(u) - f_p(v) | \leqslant F_p \mid u - v \mid .$$

Definition 2.2. [45] Suppose that $\mathscr{V}:\mathbb{R}^n \to \mathbb{R}^+$ is locally Lipschitz function, the upper right-hand Dini derivative of $\mathscr{V}$ along a nonlinear impulsive delay system is represented by

$$D^+\mathscr{V}[g] = \lim_{r\to 0^+}\sup \frac{1}{h}[\mathscr{V}(\theta + rg) - \mathscr{V}(\theta)].$$

Lemma 2.3. [45] For the impulsive system with time delay $\dot{\theta}(t) = f(\theta_t), t \geqslant t_0, t \neq t_n, \theta(t) = g_n(\theta(t^-)), t = t_n, n \in \mathbb{Z}_+, \theta - t_0 = \phi$ with E-T mechanism $t_n = \inf\{t > t_{n-1} : \Omega(t,\theta(t)) \geqslant 0\}$the Lipschitz function $\mathscr{V}:\mathbb{R}^n \to \mathbb{R}_+$ and there exist constants $c > 0, a_n > 0, d_n > 0, \eta > 0, \gamma > 0$, with $a_n \in (0,\eta], n \in \mathbb{Z}_+$, (a) $\mathscr{V}(g_n(\theta)) \leqslant e^{-d_n}\mathscr{V}(\theta), \forall \theta \in \mathbb{R}^n, \forall n \in \mathbb{Z}_+$; (b) $D^+\mathscr{V}[f] \leqslant c\mathscr{V}(\theta(t))$; wherever $e^{\gamma s - \eta}\mathscr{V}(\theta(t+s)) \leqslant \mathscr{V}(\theta(t)), \forall s \in [-\varphi,0], t \neq t_n$,where $\theta(t)$ represents the solution of the impulsive model; (c) Sequence $\{a_n\}$ in E-T mechanism satisfies $a_n + \sum_{k=1}^{l}(a_{n-k} - d_{n-k}) \leqslant \eta, \forall \quad l \in \mathbb{S}_n, \forall n \geqslant 2$, where $\mathbb{S}_n = 1,2,\ldots,n-2,n-1$. Then the delayed impulsive control model does not reveal Zeno behavior (ZB) under E-T mechanism. Moreover, the triggered impulsive sequence $\{t_n\}$ satisfies

$$t_n - t_{n-1} \geqslant \frac{a_n}{\gamma + c}, \quad \forall n \in \mathbb{Z}_+.$$

Lemma 2.4. Assume that $\mathscr{V}(\theta(t))$ is a positive definite and radially unbounded function under the conditions of Lemma 2.3. Then, for any constant $\gamma > 0$ and triggering parameters $a_n \in \mathbb{R}_+$ such that $\sum_{n=1}^{m} a_n \to +\infty$ as $m \to +\infty$ the system (1) is globally exponentially stable under E-T mechanism:

$$t_n = \inf\{t > t_{n-1} : \Omega(t,\theta(t)) \geqslant 0\} \tag{2}$$

with $\Omega(t,\theta(t)) = \mathscr{V}(\theta(t)) - \max\{e^{\gamma t_{n-1}}\mathscr{V}(\theta(t_{n-1})), e^{\gamma t_0}\mathscr{V}_0\}e^{a_n - \gamma t}$, where $\mathscr{V}_0 = \sup\{\mathscr{V}(\theta(t_0+s)), s \in [-\tau,0]\}, \mathscr{V}(\theta(t))$ and $\mathscr{V}(\theta(t_{n-1}))$ denote the Lyapunov functions.

In this paper, we will make leader-following synchronization of delayed INNs that synchronize by the suitable designed controller. Therefore, take INNs (1) as the followers and consider the leader of the delayed INNs (1) is represented as follows:

$$\frac{d^2s(t)}{dt^2} = -A\frac{ds(t)}{dt} - Bs(t) + Cf(s(t)) + Df(s(t-\varphi)) + I(t). \tag{3}$$

in which $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^T \in \mathbb{R}^n$, denotes the vector state of the leader node and the remaining parameters and functions are the same as those in (1). The initial values of the INNs (3) is represented as follows:

$$s(r) = \delta(r), \ \frac{ds(r)}{ds} = \theta(r), \ -\varphi \leqslant r \leqslant 0,$$

where $\delta(r), \theta(r) \in \mathscr{PC}^1([-\varphi,0],\mathbb{R}^n)$. Now letting the suitable variable transformation $y_i(t) = (dx_i(t)/dt) + x_i(t)$.for $i \in \Gamma$ the INNs (1) can be rewritten into the following first order equations:

$$\begin{cases} \frac{dx_i(t)}{dt} = & -x_i(t) + y_i(t) + u_i(t), \\ \frac{dy_i(t)}{dt} = & \alpha x_i(t) - \beta y_i(t) + Cf(x_i(t)) + Df(x_i(t-\varphi)) + I(t) + \upsilon_i(t), i \in \Gamma, \end{cases} \tag{4}$$

where $\alpha = A - B - I, \beta = A - I, \Gamma = \{1,\ldots,N\}$ and letting $z(t) = (ds(t)/dt) + s(t)$ then the INNs (3) can be written as

$$\begin{cases} \frac{ds(t)}{dt} = & -s(t) + z(t), \\ \frac{dz(t)}{dt} = & \alpha s(t) - \beta z(t) + Cf(s(t)) + Df(s(t-\varphi)) + I(t). \end{cases} \tag{5}$$

Define the synchronization error $e_{1i}(t) = x_i(t) - s(t), e_{2i}(t) = y_i(t) - z(t)$for $i \in \Gamma$. Therefore, the error dynamics of the i^{th} node can be described as:

$$\begin{cases} \frac{de_{1i}(t)}{dt} = & -e_{1i}(t) + e_{2i}(t) + u_i(t), \\ \frac{de_{2i}(t)}{dt} = & \alpha e_{1i}(t) - \beta e_{2i}(t) + Cf(e_{1i}(t)) + Df(e_{1i}(t-\varphi)) + \upsilon_i(t), \end{cases} \tag{6}$$

where $f(e_{1i}(t)) = f(x_i(t)) - f(s(t)), f(e_{2i}(t)) = f(y_i(t)) - f(z(t))$.

It is obvious that the synchronization problem of leader (3) and follower (1) is equivalent to the synchronization between (5) and (4). To realize the leader following synchronization between (1) and (3), we design the following E-T impulsive control:

$$\begin{cases} u_i(t) = \sum_{n\in\mathbb{Z}_+} \overline{K}_i\overline{E}_i(t)\delta(t-t_n), \\ \upsilon_i(t) = \sum_{n\in\mathbb{Z}_+} \widetilde{K}_i\widetilde{E}_i(t)\delta(t-t_n), \end{cases} \tag{7}$$

where, $\overline{E}_i(t) = \sum_{j=1,j\neq i}^{N}\hat{a}_{ij}(x_j(t) - x_i(t)) + \hat{b}_i(s(t) - x_i(t))$, $\widetilde{E}_i(t) = \sum_{j=1,j\neq i}^{N}\hat{a}_{ij}(y_j(t) - y_i(t)) + \hat{b}_i(z(t) - y_i(t))$, $\overline{K}_i > 0$ and $\widetilde{K}_i > 0$ denotes the control gain of the i^{th} follower node dynamics to be designed, $\hat{a}_{ij}$ are the elements of weighted adjacency matrix $\mathbb{A}$ of the directed graph $\mathbb{G}, \hat{b}_i$ are the elements of leader adjacency matrix $\mathbb{B}$ of the directed graph $\mathbb{G}$, the Dirac delta function is $\delta(.)$ and the E-T impulsive sequence is denoted by $\{t_n, n \in \mathbb{Z}^+\}$. Then, the equivalent vector form of the system (6) can be written s

$$\begin{cases} \frac{de_1(t)}{dt} = & -(I_N \otimes I_n)e_1(t) + (I_N \otimes I_n)e_2(t), \\ \frac{de_2(t)}{dt} = & (I_N \otimes \alpha)e_1(t) - (I_N \otimes \beta)e_2(t) \\ & +(I_N \otimes C)f(e_1(t)) + (I_N \otimes D)f(e_1(t-\varphi)), t \neq t_n, \\ \Delta e_1(t) = & -(\overline{K}\mathbb{H} \otimes I_n)e_1(t^-), \\ \Delta e_2(t) = -\left(\widetilde{K}\mathbb{H} \otimes I_n\right)e_2(t^-), t = t_n, n \in \mathbb{Z}_+ & . \end{cases} \tag{8}$$

where $e_1(t) = (e_{11}(t), e_{12}(t), \ldots, e_{1N}(t))^T, e_2(t) = (e_{21}(t), e_{22}(t), \ldots, e_{2N}(t))^T$, $\Delta e_h(t) = e_h(t) - e_h(t^-), h = 1,2, \overline{K} = diag\{\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_N\}$and $\widetilde{K} = diag\left\{\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_N\right\}$ to be designed.

Remark 2.5. The structure of an impulse control scheme is simple, and it just needs discrete control to produce the required result, recently E-TIC has gotten a lot of attention as a mixture of impulsive control and E-TC. By incorporating E-T mechanism into impulse control, E-T impulse governs the system states by instantly modifying the states at specific times, which are estimated by an well-designed E-T mechanism.

3. Main results

In this section, we are going to study the leader-following synchronization of reduced and non-reduced order INNs by using E-TIC. The synchronization criteria is given by the following two theorems.

Theorem 3.1. Suppose that there exists constant matrices $\alpha > 0, \beta > 0$, and constants $\gamma > 0, c, a, \vartheta, n \times n$ positive symmetric matrices $P_1, P_2, n \times n$ diagonal matrices Q_1, Q_2 and positive diagonal matrices $\overline{K}, \widetilde{K}$ such that $FQ_1F < \beta P_1, FQ_2F < \beta P_2$ and

$$\wedge = \begin{bmatrix} I_N \otimes (2P_1 - P_2\beta - \beta^T P_2 - l_1 P_2) & P_2 C & P_2 D & P_1\alpha^T + P_2\alpha \\ \star & -Q_1 & 0 & 0 \\ \star & \star & -Q_2 & 0 \\ \star & \star & \star & \Theta \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} -e^{-a} I_N & I_N - \mathbb{H}^T \overline{K} \\ \star & -I_N \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} -e^{-a} I_N & I_N - \mathbb{H}^T \widetilde{K} \\ \star & -I_N \end{bmatrix} < 0. \tag{11}$$

where $\Theta = I_N \otimes (-2P_1 + FQ_1F + \vartheta e^{\gamma\varphi + \eta} P_2 - l_1 P_1), F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\}$. Then the leader-following synchronization problem of INN (1) and 3 is solved under the control gains $\overline{K}, \widetilde{K}$ and E-T mechanism $t_n = \inf\{t > t_{n-1} : \Omega(t, e(t)) \geqslant 0\}$ with $\Omega(t, e(t)) = e_1^T(t)(I_N \otimes P_1)e_1(t) + e_2^T(t)(I_N \otimes P_2)e_2(t) - \max\{e^{\gamma t_{n-1}} e_1^T(t_{n-1})(I_N \otimes P_1) e_1(t_{n-1}), e^{\gamma t_0} V_0\} e^{a - \gamma t}$, where $V_0 = \sup\{e_1^T(t_0 + s)(I_N \otimes P_1)e_1(t_0 + s) + e_2^T(t_0 + s)(I_N \otimes P_2)e_2(t_0 + s), s \in [\varphi, 0]\}$.

Proof: Let us consider the following Lyapunov function candidate

$$\mathscr{V}(t) = \mathscr{V}_1(t) + \mathscr{V}_2(t), \tag{12}$$

where $\mathscr{V}_1(t) = e_1^T(t)(I_N \otimes P_1)e_1(t), \mathscr{V}_2(t) = e_2^T(t)(I_N \otimes P_2)e_2(t)$.

Find the derivative of $\mathscr{V}(t)$ along the solutions of (8)

$$\begin{aligned} D^+\mathscr{V}_1(t) \leqslant\ & 2e_1^T(t)(I_N \otimes P_1)\dot{e_1}(t), \\ =\ & 2e_1^T(t)(I_N \otimes P_1)[-(I_N \otimes I_n)e_1(t) + (I_N \otimes I_n)e_2(t)], \\ =\ & e_1^T(t)[I_N \otimes 2P_1]e_1(t) + e_1^T(t) \\ & [(I_N \otimes 2P_1]e_2(t).D^+\mathscr{V}_2(t) \\ & \quad \leqslant 2e_2^T(t)(I_N \otimes P_2)\dot{e_2}(t), = e_2^T(t)(I_N \otimes 2P_2)[(I_N \otimes \alpha)e_1(t) \\ & -(I_N \otimes \beta)e_2(t) + (I_N \otimes C)f(e_1(t)) + (I_N \otimes D)f(e_1(t - \varphi(t)))], \end{aligned}$$

$$\begin{aligned} =\ & e_2^T(t)\left[I_N \otimes \left(P_2\alpha + \alpha^T P_2\right)\right]e_1(t) \\ & -e_2^T(t)\left[I_N \otimes \left(P_2\beta + \beta^T P_2\right)\right]e_2(t) \\ & +e_1^T(t)[I_N \otimes (FQ_1F)]e_1(t) + e_2^T(t)\left[I_N \otimes \left(P_2 D Q_2^{-1} D^T P_2\right)\right]e_2(t) \\ & e_1^T(t)[I_N \otimes (2P_1 + FQ_1F)]e_1(t) \\ & +e_1^T(t - \varphi)[I_N \otimes FQ_2F]e_1(t - \varphi) \\ & +e_1^T(t)[I_N \otimes 2P_1]e_2(t) + e_2^T(t)\left[I_N \otimes \left(P_2\alpha + \alpha^T P_2\right)\right]e_1(t). \end{aligned}$$

When $e_1^T(t - \varphi)[I_N \otimes P_1]e_1(t - \varphi) \leqslant e^{\gamma\varphi + \eta} e_1^T(t)(I_N \otimes P_1)e_1(t)$, where $\eta \geqslant a_n + \sum_{k=1}^{l}(a_{n-k} - d_{n-k})$ and $FQ_2F < \vartheta P_2, FQ_1F < \vartheta P_1$, by (9) it is not difficult to compute that

$$\begin{aligned} D^+\mathscr{V}(t) \leqslant\ & e_1^T(t)[I_N \otimes (2P_1 + FQ_1F + \vartheta e^{\gamma\varphi + \eta} P_2)]e_1(t) \\ & +e_2^T(t)\left[I_N \otimes \left(-P_2\beta - \beta^T P_2 + P_2 C Q_1^{-1} C^T P_2 + P_2 D Q_2^{-1} D^T P_2\right)\right]e_2(t) \\ & +e_1^T(t)[I_N \otimes 2P_1]e_2(t) + e_2^T(t)\left[I_N \otimes \left(P_2\alpha + \alpha^T P_2\right)\right]e_1(t), \end{aligned}$$

$$\begin{aligned} =\ & e_1^T(t)[I_N \otimes (2P_1 + FQ_1F + \vartheta e^{\gamma\eta + \eta} P_2 - l_1 P_1)]e_1(t) + e_1^T(t)[I_N \otimes l_1 P_1]e_1(t) \\ & +e_2^T(t)\left[I_N \otimes \left(-P_2\beta - \beta^T P_2 + P_2 C Q_1^{-1} C^T P_2 + P_2 D Q_2^{-1} D^T P_2 - l_1 P_2\right)\right]e_2(t) \\ & +e_2^T(t)[I_N \otimes l_1 P_2]e_2(t) + e_1^T(t)[I_N \otimes 2P_1]e_2(t) + e_2^T(t)\left[I_N \otimes \left(P_2\alpha + \alpha^T P_2\right)\right]e_1(t), \\ =\ & \xi^T(t) \wedge \xi(t) + l_1\left[e_1^T(t)(I_N \otimes P_1)e_1(t) + e_2^T(t)(I_N \otimes P_2)e_2(t)\right] \\ \leqslant\ & l_1\mathscr{V}(t), t \in [t_{n-1}, t_n), n \in \mathbb{Z}^+. \end{aligned}$$

Consequently from Eqs. (11) and (10), we have

$$-e^{-a} I_N + \left(I_N - \overline{K}\mathbb{H}\right)^T \left(I_N - \overline{K}\mathbb{H}\right) \leqslant 0,$$

$$-e^{-a} I_N + \left(I_N - \widetilde{K}\mathbb{H}\right)^T \left(I_N - \widetilde{K}\mathbb{H}\right) \leqslant 0.$$

which implies that

$$\begin{aligned} \mathscr{V}(t_n) =\ & e_1^T(t_n)(I_N \otimes P_1)e_1(t_n) + e_2^T(t)(I_N \otimes P_2)e_2(t_n), \\ \leqslant\ & e_1^T(t_n^-)\left[(I_N - \overline{K}\mathbb{H}) \otimes I_n\right]^T (I_N \otimes P_1)\left[(I_N - \overline{K}\mathbb{H}) \otimes I_n\right]e_1(t_n^-) \\ & +e_2^T(t_n^-)\left[\left(I_N - \widetilde{K}\mathbb{H}\right) \otimes I_n\right]^T (I_N \otimes P_2)\left[\left(I_N - \widetilde{K}\mathbb{H}\right) \otimes I_n\right]e_2(t_n^-), \\ =\ & e_1^T(t_n^-)\left[\left((I_N - \overline{K}\mathbb{H})^T (I_N - \overline{K}\mathbb{H})\right) \otimes P_1\right]e_1(t_n^-) \\ & +e_2^T(t_n^-)\left[\left(\left(I_N - \widetilde{K}\mathbb{H}\right)^T \left(I_N - \widetilde{K}\mathbb{H}\right)\right) \otimes P_2\right]e_2(t_n^-), \\ \leqslant\ & e^{-a}\mathscr{V}(t_n^-). \end{aligned}$$

Then it is easy to see that all the required conditions in Lemma (2.3)-(2.4) are satisfied, then the synchronization problem of INNs (1) and (3) is solved. The proof is now completed.

Remark 3.2. The first-order differential Eqs. (4) and (5) are produced via the variable substitutions $y_i(t) = (dx_i(t)/dt) + x_i(t)$ and $z(t) = (ds(t)/dt) + s(t)$ have double the dimension of the given second-order INNs (1) and (3) this enormously increase the difficulties of analytical calculations and the complexity of the obtained result. Moreover, designing a second-order neural network using control inputs other than a reduced order system may be more significant and helpful. Therefore, to avoid to overcome these problems rising from the reduced order variable substitutions, a new method is directly investigating the synchronization problems of the INNs. Next we are going to investigate the leader-following synchronization of non-reduced order INNs by using E-TIC.

To synchronize the INNs (1) with the leader node (3) the E-TIC input is designed as follows:

$$u_i(t) = \sum_{n \in \mathbb{Z}_+} \tilde{\tilde{k}}_i g_i(t)\delta(t - t_n), \tag{13}$$

where $g_i(t) = \sum_{j=1, j \neq i}^{N} \hat{a}_{ij}(x_j(t) - x_i(t)) + \hat{b}_i(s(t) - x_i(t)), \tilde{\tilde{k}}_i$ denotes the control gain of the i^{th} followers node which is to be design, $\hat{a}_{ij}$ are the elements of weighted adjacency matrix $\mathbb{A}$ of the directed graph $\mathbb{G}, \hat{b}_i$ are the elements of leader adjacency matrix $\mathbb{B}$ of the directed graph $\mathbb{G}, \delta(\cdot)$ denotes Dirac delta function and event-based impulse sequence is $\{t_n, n \in \mathbb{Z}_+\}$. The synchronization error between INNs (1) and leader node (3) is described as the following second order differential equation:

$$\frac{d^2 e_i(t)}{dt^2} = -A\frac{de_i(t)}{dt} - Be_i(t) + Cf(e_i(t)) + Df(e_i(t - \varphi)) + u_i(t). \tag{14}$$

Thus, under the E-TIC input (13), the compact form of the error (16) can be written as

$$\begin{aligned}\ddot{e}(t) &= -(I_N\otimes A)\dot{e}(t) - (I_N\otimes B)e(t) + (I_N\otimes C)f(e(t))\\ &\quad + (I_N\otimes D)f(e(t-\varphi)),\\ \Delta e(t) &= -\left(\tilde{\tilde{K}}\mathbb{H}\otimes I_n\right)e(t^-), t=t_n, n\in\mathbb{Z}_+.\end{aligned} \tag{15}$$

where $\Delta e(t) = e(t) - e(t^-), \tilde{\tilde{K}} = diag\left\{\tilde{\tilde{k}}_1,\ldots,\tilde{\tilde{k}}_N\right\}$ to be designed.

In order to investigate our synchronization results for non-reduced order INNs, the following notations are given.

$$\begin{aligned}\mathfrak{P} &= \alpha\gamma - A\alpha^2 + \tfrac{1}{2}\alpha^2|C| + \tfrac{1}{2}\alpha^2|D|,\\ \mathfrak{Q} &= B\alpha\gamma + \tfrac{1}{2}\alpha^2(|C|F + |D|F) + (|C|F + |D|F)|\alpha\gamma|,\\ \mathfrak{R} &= \omega + \gamma^2 - B\alpha^2 - A\alpha\gamma,\end{aligned}$$

where, α,γ,ω are some non-zero constants.

Assumption 3.3. There exist some non-zero constants α,γ,ω such that the conditions $(\mathfrak{Q} - \mathfrak{R}/4\mathfrak{P} - \bar{l}_1\omega/2) < 0, (1/4\mathfrak{P} - \bar{l}_1/2) < 0, \mathfrak{R} - \gamma < 0, 2\mathfrak{P} - \alpha < 0$ are satisfied.

Theorem 3.4. *Suppose that the Assumptions (2.1) and (3.3) hold. Then there exists non-zero constants $\alpha,\gamma,\omega,\bar{l}_1$ and $N\times N$ positive diagonal matrices $\tilde{\tilde{K}}$such that $\left[\left(I_N - \tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]\left[\left(I_N - \tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]/2 < 0, \left[\left(I_N - \tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]\left[\left(I_N - \tilde{\tilde{K}}\mathbb{H}\right)\otimes\gamma I_n\right] < 0$ and*

$$\begin{bmatrix} -e^{-a}I_N & I_N - \mathbb{H}^T\tilde{\tilde{K}}\\ * & -I_N\end{bmatrix} < 0. \tag{16}$$

Therefore, the synchronization problem of INNs (1) and (3) is obtained under the gain matrix $\tilde{\tilde{K}} = \text{diag}\left\{\tilde{\tilde{k}}_1,\ldots,\tilde{\tilde{k}}_N\right\}$.

Proof. Construct the following Lyapunov function candidate

$$\mathscr{V}(t) = \frac{1}{2}\sum_{i=1}^{N}\omega e_i^2(t) + \frac{1}{2}\sum_{i=1}^{N}(\alpha\dot{e}_i(t) + \gamma e_i(t))^2.$$

Calculating the derivatives of $\mathscr{V}(t)$ along the solutions of (14)

$$D^+\mathscr{V}(t) = \sum_{i=1}^{N}\omega e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}(\alpha\dot{e}_i(t) + \gamma e_i(t))(\alpha\ddot{e}_i(t) + \gamma\dot{e}_i(t)),$$

$$\begin{aligned}&= \sum_{i=1}^{N}\omega e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}\left[\alpha^2\dot{e}_i(t)\ddot{e}_i(t) + \alpha\gamma\dot{e}_i^2(t) + \alpha\gamma e_i(t)\ddot{e}_i(t) + \gamma^2 e_i(t)\dot{e}_i(t)\right],\\ &= \sum_{i=1}^{N}\omega e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}\gamma^2 e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}(\alpha^2\dot{e}_i(t) + \alpha\gamma e_i(t))\\ &\quad\left[-A\dot{e}_i(t) - Be_i(t) + Cf(e_i(t)) + Df(e_i(t-\varphi))\right] + \sum_{i=1}^{N}\alpha\gamma\dot{e}_i^2(t),\\ &= \sum_{i=1}^{N}\omega e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}\gamma^2 e_i(t)\dot{e}_i(t)\\ &\quad + \sum_{i=1}^{N}\{-A\alpha^2\dot{e}_i^2(t) - B\alpha^2 e_i(t)\dot{e}_i(t) + C\alpha^2\dot{e}_i(t)f(e_i(t)) + D\alpha^2\dot{e}_i(t)f(e_i(t-\varphi))\\ &\quad -A\alpha\gamma e_i(t)\dot{e}(t) - B\alpha\gamma e_i^2(t) + C\alpha\gamma e_i(t)f(e_i(t)) + D\alpha\gamma e_i(t)f(e_i(t-\varphi))\}\\ &\quad \sum_{i=1}^{N}\alpha\gamma\dot{e}_i^2(t), \leqslant \sum_{i=1}^{N}(\alpha\gamma - A\alpha^2)\dot{e}_i^2(t) - \sum_{i=1}^{N}(B\alpha\gamma)e_i^2(t)\\ &\quad + \sum_{i=1}^{N}(\omega + \gamma^2 - B\alpha^2 - A\alpha\gamma)e_i(t)\dot{e}_i(t) + \sum_{i=1}^{N}(\alpha^2|\dot{e}_i(t)| + |\alpha\gamma||e_i(t)|)|C||f(e_i(t))|\\ &\quad + \sum_{i=1}^{N}(\alpha^2|\dot{e}_i(t)| + |\alpha\gamma||e_i(t)|)|D||f(e_i(t-\varphi))|.\end{aligned} \tag{17}$$

By Assumption (2.1) and $uv \leqslant \frac{1}{2}(u^2 + v^2)$, we have

$$\begin{aligned}\sum_{i=1}^{N}(\alpha^2|\dot{e}_i(t)| + |\alpha\gamma||e_i(t)|)|C||f(e_i(t))| \leqslant\; & \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|C|F(\dot{e}_i^2(t) + e_i^2(t))\\ & + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||C|F(e_i^2(t) + e_i^2(t)),\\ =\; & \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|C|F\dot{e}_i^2(t)\\ & + \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|C|Fe_i^2(t)\\ & + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||C|Fe_i^2(t)\\ & + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||C|Fe_i^2(t),\\ =\; & \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|C|F\dot{e}_i^2(t)\\ & + \tfrac{1}{2}\sum_{i=1}^{N}(\alpha^2|C|F + |\alpha\gamma||C|F\\ & + |\alpha\gamma||C|F)e_i^2(t).\end{aligned} \tag{18}$$

$$\begin{aligned}&\sum_{i=1}^{N}(\alpha^2|\dot{e}_i(t)| + |\alpha\gamma||e_i(t)|)|D||f(e_i(t-\varphi))|\\ &\leqslant \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|D|F(\dot{e}_i^2(t) + e_i^2(t-\varphi)) + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||D|F(e_i^2(t) + e_i^2(t-\varphi)),\\ &= \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|D|F\dot{e}_i^2(t) + \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|D|Fe_i^2(t-\varphi)\\ &+ \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||D|Fe_i^2(t) + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||D|Fe_i^2(t-\varphi),\\ &= \tfrac{1}{2}\sum_{i=1}^{N}\alpha^2|D|F\dot{e}_i^2(t) + \tfrac{1}{2}\sum_{i=1}^{N}|\alpha\gamma||D|Fe_i^2(t)\\ &+ \tfrac{1}{2}\sum_{i=1}^{N}(\alpha^2|D|F + |\alpha\gamma||D|F)e_i^2(t-\varphi).\end{aligned} \tag{19}$$

From (17)-(19), we get

$$\begin{aligned}D^+\mathscr{V}(t) \leqslant\; & \sum_{i=1}^{N}\left(\alpha\gamma - A\alpha^2 + \frac{1}{2}\alpha^2|C| + \frac{1}{2}\alpha^2|D|\right)\dot{e}_i^2(t)\\ & - \sum_{i=1}^{N}\left(B\alpha\gamma + \frac{1}{2}|\alpha\gamma||C|F + \frac{1}{2}|\alpha\gamma||D|F + \frac{1}{2}\alpha^2|C|F\right.\\ & \left. + \tfrac{1}{2}\alpha^2|D|F + \tfrac{1}{2}|\alpha\gamma||C|F + \tfrac{1}{2}|\alpha\gamma||D|F\right)e_i^2(t)\\ & + \sum_{i=1}^{N}(\omega + \gamma^2 - B\alpha^2 - A\alpha\gamma)e_i(t)\dot{e}_i(t),\\ =\; & \sum_{i=1}^{N}(\alpha\gamma - A\alpha^2 + \tfrac{1}{2}\alpha^2|C| + \tfrac{1}{2}\alpha^2|D|)\dot{e}_i^2(t)\\ & - \sum_{i=1}^{N}(B\alpha\gamma + \tfrac{1}{2}\alpha^2(|C|F + |D|F) + (|C|F + |D|F)|\alpha\gamma|)e_i^2(t)\\ & + \sum_{i=1}^{N}(\omega + \gamma^2 - B\alpha^2 - A\alpha\gamma)e_i(t)\dot{e}_i(t),\\ =\; & \sum_{i=1}^{N}\{\mathfrak{P}\dot{e}_i^2(t) + \mathfrak{Q}e_i^2(t) + \mathfrak{R}e_i(t)\dot{e}_i(t)\},\\ \leqslant\; & \bar{l}_1\mathscr{V}(t).\end{aligned}$$

As a result from (16), we have

$$-e^{a}I_N+\left(I_N\otimes\tilde{\tilde{K}}\mathbb{H}\right)^T\left(I_N\otimes\tilde{\tilde{K}}\mathbb{H}\right)\leqslant 0 \tag{20}$$

when $t=t_n$,

$$\begin{aligned}
\mathscr{V}(t_n) = {} & \tfrac{1}{2}(I_N\otimes\omega I_n)e^2(t_n)+\tfrac{1}{2}((I_N\otimes\alpha I_n)\dot{e}_i(t_n)+(I_N\otimes\gamma I_n)e(t_n))^2,\\
\leqslant {} & \tfrac{1}{2}(I_N\otimes\omega I_n)\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]e^2(t_n^-)\\
& +\tfrac{1}{2}\left[(I_N\otimes\gamma I_n)\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]e^2(t_n^-)\right]^2,\\
= {} & \tfrac{1}{2}\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right](I_N\otimes\omega I_n)e^2(t_n^-)\\
& +\tfrac{1}{2}\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[(I_N\otimes\gamma I_n)^2e^2(t_n^-)\right],\\
= {} & \tfrac{1}{2}\left[\left(I_N-\tilde{\tilde{K}}H\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\\
& \left[(I_N\otimes\omega I_n)e^2(t_n^-)+(I_N\otimes\gamma I_n)^2e^2(t_n^-)\right],\\
= {} & \tfrac{1}{2}\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\\
& \left[(I_N\otimes\omega I_n)e^2(t_n^-)+\left[(I_N\otimes\gamma I_n)^2e^2(t_n^-)+(I_N\otimes\alpha I_n)^2\dot{e}^2(t_n^-)\right.\right.\\
& +2(I_N\otimes\alpha I_n)\dot{e}(t_n^-)(I_N\otimes\gamma I_n)e(t_n^-)]\\
& -(I_N\otimes\alpha I_n)^2\dot{e}^2(t_n^-)-2(I_N\otimes\alpha I_n)\dot{e}(t_n^-)(I_N\otimes\gamma I_n)e(t_n^-)],\\
= {} & \left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\\
& \left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes I_n\right]\mathscr{V}(t_n^-)\\
& -\tfrac{1}{2}\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]\dot{e}^2(t_n^-)\\
& -\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes\alpha I_n\right]\left[\left(I_N-\tilde{\tilde{K}}\mathbb{H}\right)\otimes\gamma I_n\right]\dot{e}(t_n^-)e(t_n)\\
\leqslant {} & e^{-a}\mathscr{V}(t_n^-).
\end{aligned}$$

Then it can be conclude that all the required conditions in Lemma (2.3)-(2.4) are satisfied then the synchronization problem of INNs (1) and (3) is obtained and the proof is completed. □

Remark 3.5. The novelty and technical difficulty of this paper can be illustrated as follows:

1. It is proposed to use an E-TIC design to accomplish synchronization in a network of INNs that has a delay in time. We derive the primary results of this paper in terms of lower-dimensional linear matrix inequalities, which are used to accomplish leader-following synchronization of INNs with time delays, based on particular assumptions about node dynamics and graph theory ideas. One of the most significant contributions of this paper is the investigation of leader-following synchronization of INNs with E-TIC, which differs from the majority of previously published works in the field. The findings of this paper contribute to the improvement of previously published results in the field of leader-following synchronization of delayed INNs with other controllers.
2. In the E-TC scheme, sampling occurs only when the state-dependent error reaches a tolerable level. When compared to the T-TC process, the E-TC method has a lower likelihood of redundant information transfer. The E-TC law specifies whether sampling shall be done rapidly or slowly, and whether sampling data must be transmitted for control updates. After then, the approach was applied to networks, and E-TC approaches were proposed.
3. When delayed impulsive control is used in connection with the E-T mechanism, E-T instants are produced when the E-TIC strategy is violated, and delayed impulsive control is only performed at the E-T instants in this design. As a result, in order to increase the performance of systems operating under E-TC conditions, the strategy must be modified in accordance with the E-T condition. From the perspective of application, the E-TIC technique is appropriate, and the control approach will effectively increase the utilization of resources in bandwidth-constrained environments.

4. Numerical simulations

In this section, we present two simulation results and their figures to illustrate the efficacy of the analytical solutions of this paper.

Example 4.1. In this example we consider the following isolated delayed INNs with two neurons:

$$\frac{d^2s(t)}{dt^2}=-A\frac{ds(t)}{dt}-Bs(t)+Cf(s(t))+Df(s(t-\varphi(t)))+I(t). \tag{21}$$

and the corresponding followers system is represented by the following second order equations with six nodes and each node has two neurons:

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2}&=-A\frac{dx_i(t)}{dt}-Bx_i(t)+Cf(x_i(t))+Df(x_i(t-\varphi(t)))\\&\quad+I(t)+v_i(t),\quad i=1,\ldots,4,\end{aligned} \tag{22}$$

with the controllers to be designed as (4) and (5), where $x_i(t)=(x_{i1}(t),x_{i2}(t))^T, i=1,\ldots,6.$

The coefficient matrices are given as

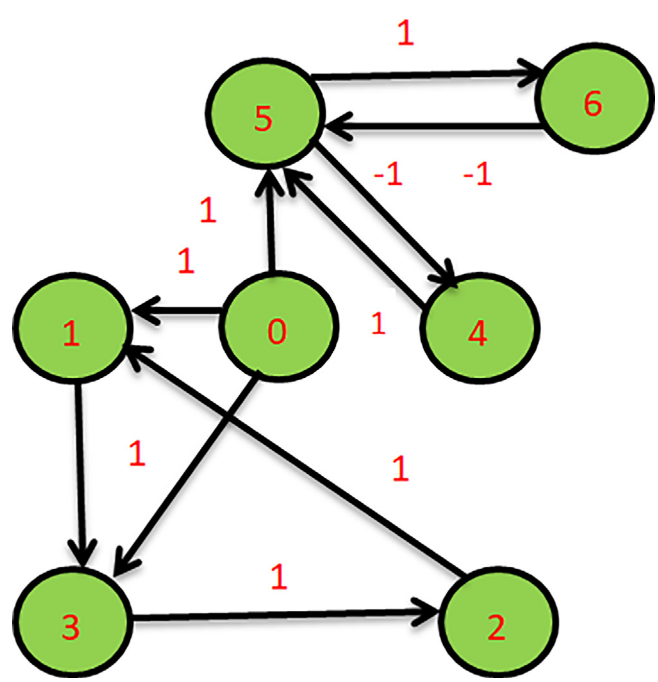

Fig. 1. Network communication topology of Examples 4.1 and 4.2.

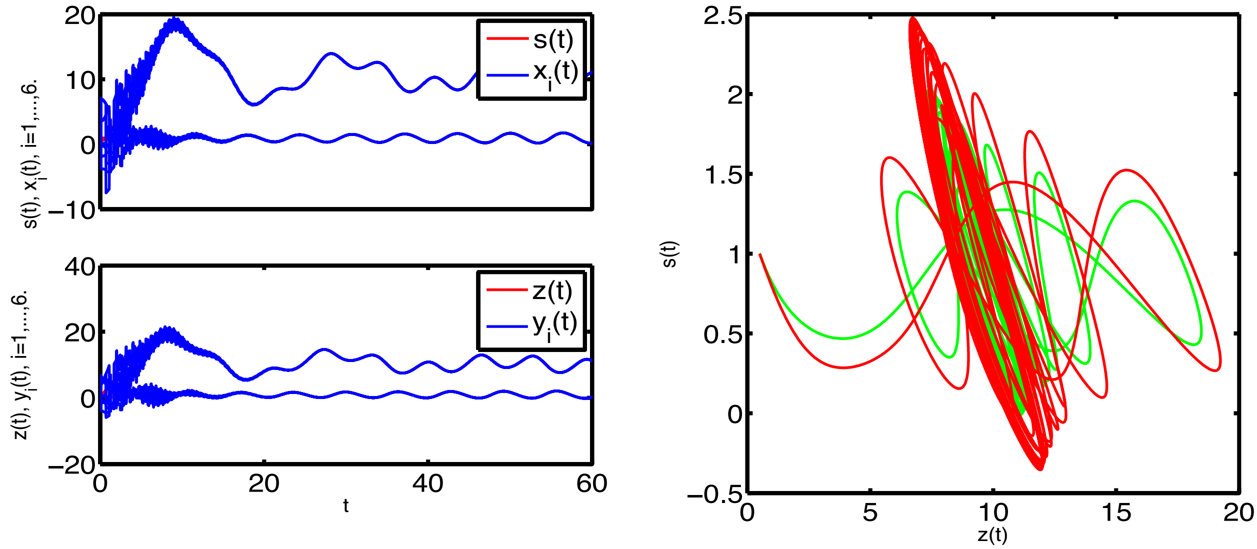

Fig. 2. State trajectories of leader and follower nodes and phase portrait graph of leader in Example 4.1.

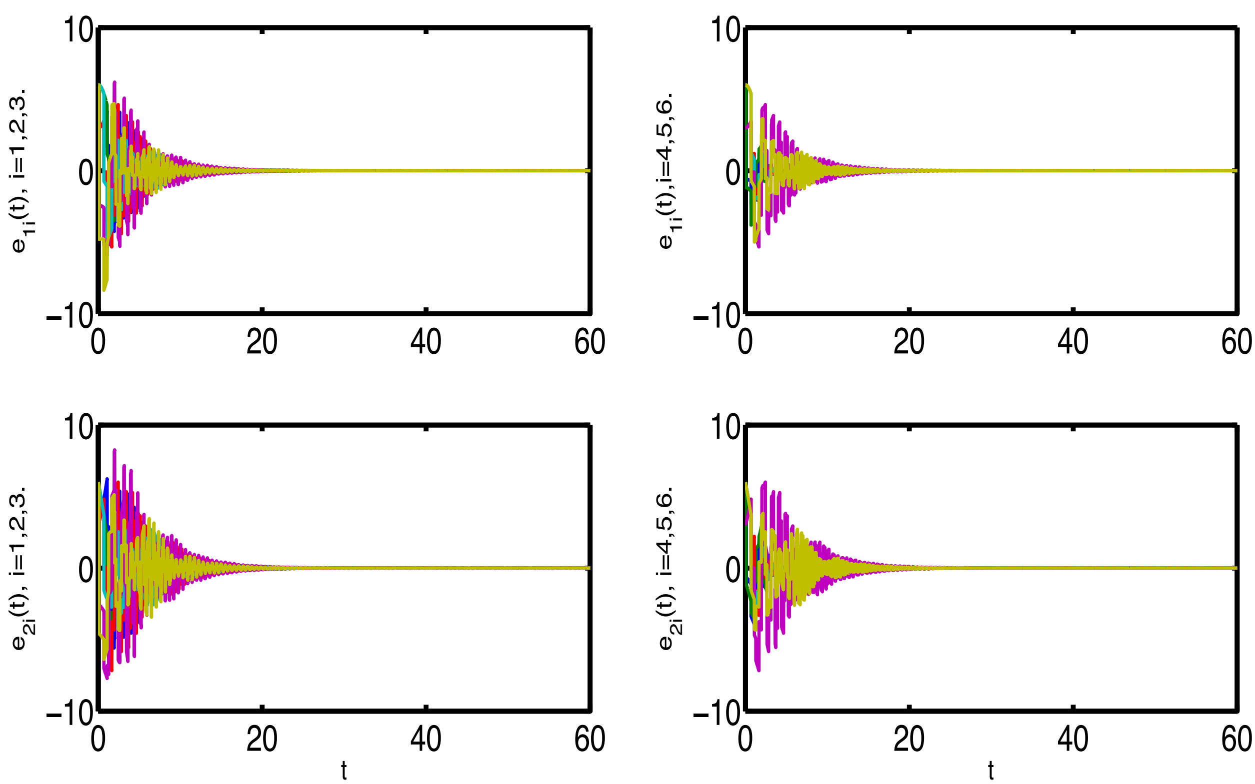

Fig. 3. State trajectories of leader and following synchronization error between (21) and (22) in Example 4.1.

$A = [0.1000.5], \quad B = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad C = \begin{bmatrix} -0.75 & 2.3 \\ 1.8 & -0.25 \end{bmatrix},$
$D = \begin{bmatrix} -0.68 & 1.5 \\ -0.4 & -1.2 \end{bmatrix}.$

Also, $f(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T, f(x_i(t-\varphi(t))) = (\tanh(x_{i1}(t-\varphi(t))), \tanh(x_{i2}(t-\varphi(t))))^T$, are the activations without and with time delays that satisfy the Assumption 2.1 with Lipschitz constant $F_i = 0.5$. The time varying delay $\varphi(t) = 3.3t$. The network communication topology of the INNs (21) and (22) is given in Figure. 1 and the Laplacian and leader adjacency matrix of the corresponding network topology are obtained as:

$$\mathscr{L} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \mathscr{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us take the initial values $s(t_0) = (0.5, 1)^T, x_i(t_0) = i \times (3, 6)^T, i = 1, \ldots, 6$. Now we have to design the E-T mechanism to realize the leader–follower synchronization between (21) and (22). Choose the constant $\gamma = 0.08, a = 0.2, \beta = 8$ and $c = 8$. Then we are infer the LMI conditions in Theorem 3.1 has solution that are feasible and the impulsive gain matrix is $\overline{K} = \widetilde{K} = 0.53I_6$ and

$$P_1 = \begin{pmatrix} 6.7 & -0.02 \\ -0.02 & 6.3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.84 & 0.04 \\ 0.04 & 0.822 \end{pmatrix}.$$

Under the E-T mechanism, the leader-following synchronization of systems (21) and (22) can be achieved. Fig. 2 shown that the states of the one leader and six followers. Under the designed ET impulsive control strategy, each followers node follows the leader, as can be shown in Fig. 2. Under the same conditions the synchronization error between (21) and (22) can converges to zero, which is shown in Fig. 3. The triggered instants on the interval [0, 10] is shown in Fig. 4.

Example 4.2. Consider the INNs (21) and (22) with the following coefficient matrices:

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.65 & 2.3 & 0.1 \\ 1.8 & -0.25 & 0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} -0.68 & 1.5 & -0.4 \\ -0.4 & -1.2 & 0 \\ 0 & 0.2 & 0.2 \end{bmatrix}.$$

Also, $f(x_i(t)) = 0.5(|x_i(t) + 1| - |x_i(t) - 1|), f(x_i(t - \varphi(t))) = 0.5(|x_i(t - \varphi(t)) + 1| - |x_i(t - \varphi(t)) - 1|)$, are the monotonically increasing increasing activation functions for without and with time delays, which are monotone increasing and globally Lipschitz continuous. The time varying delay $\varphi(t) = 3.3t$. The network communication topology of the INNs (21) and (22) is given in Figure. 1 and the Laplacian and leader adjacency matrix of the corresponding network topology are obtained as in Example 1.

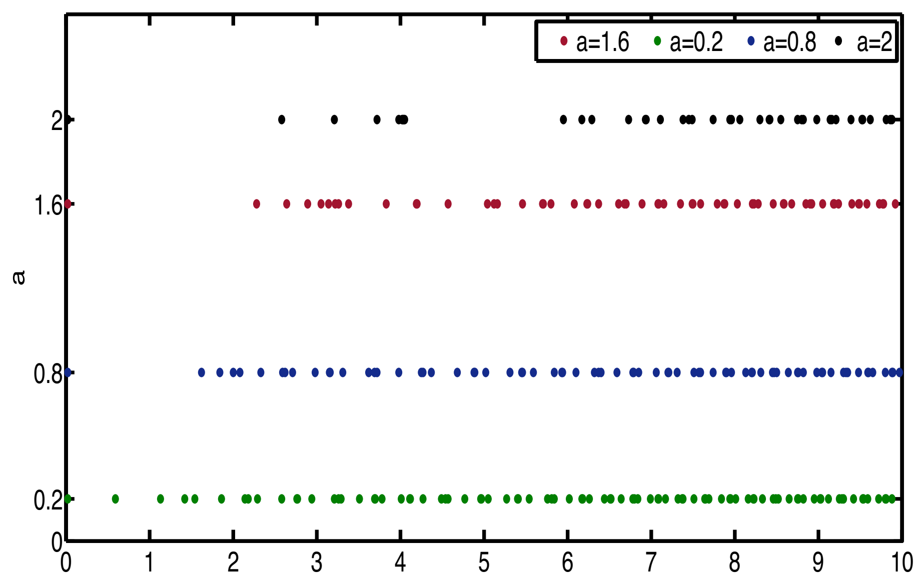

Fig. 4. Triggered instance of the INN (22) with $a = 0.2, 0.8, 1.6, 2$ in Example 4.1.

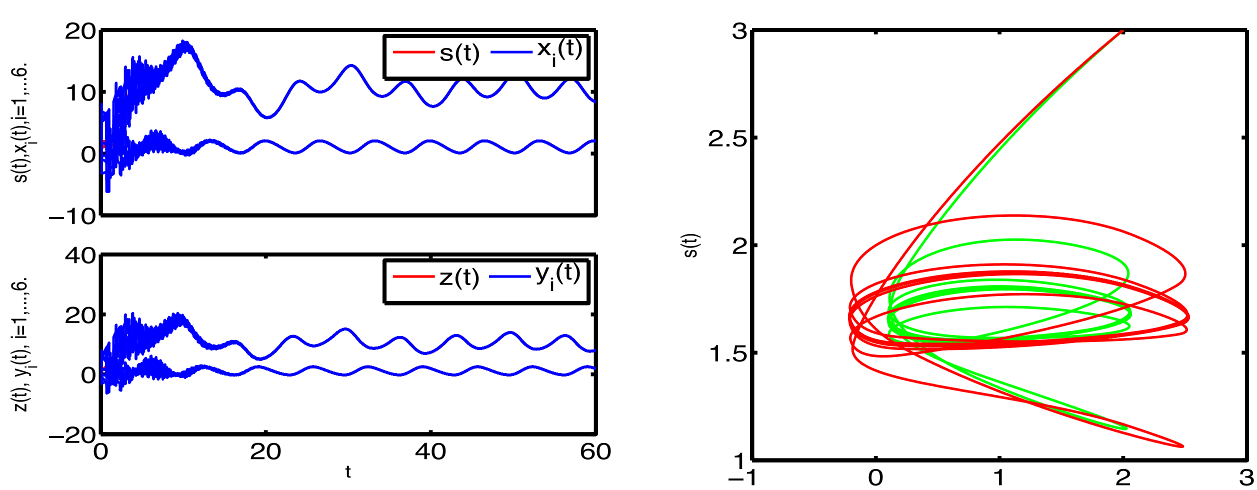

Fig. 5. State trajectories of leader and follower nodes and phase portrait graph of leader in Example 4.2.

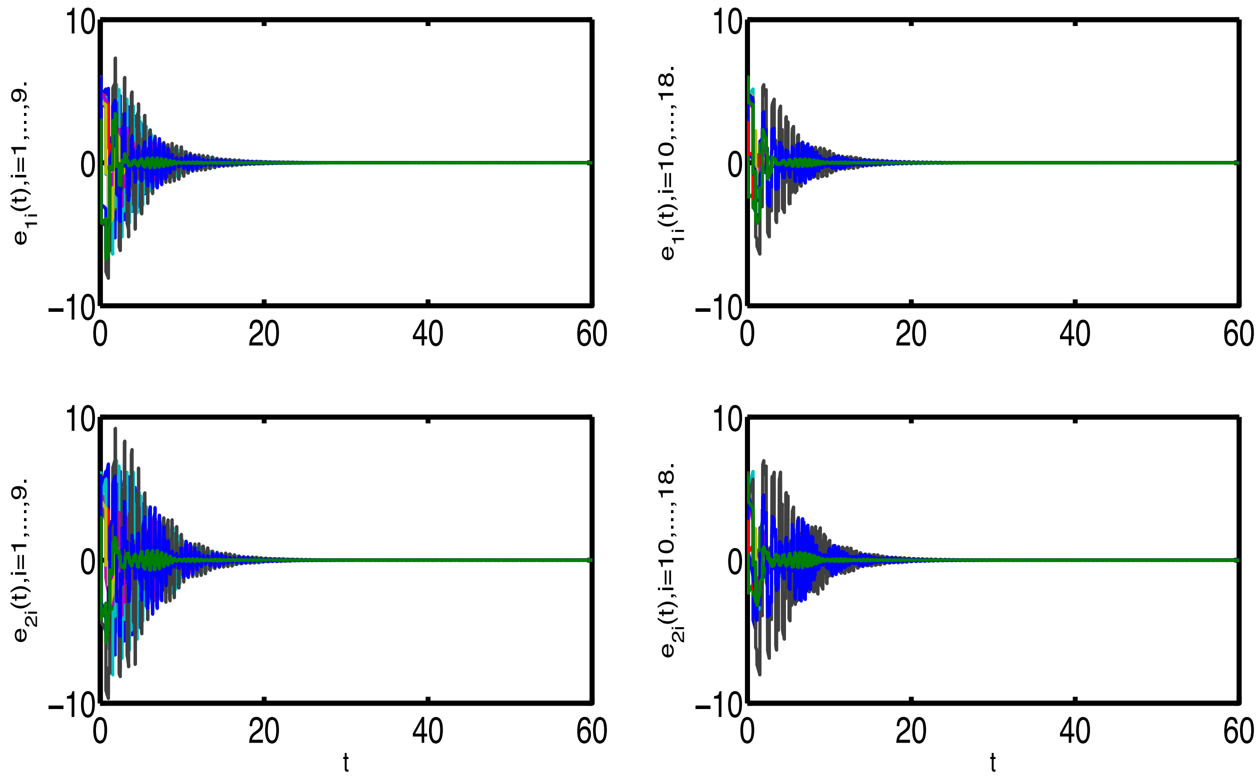

Fig. 6. State trajectories of leader and following synchronization error between (21) and (22) in Example 4.2.

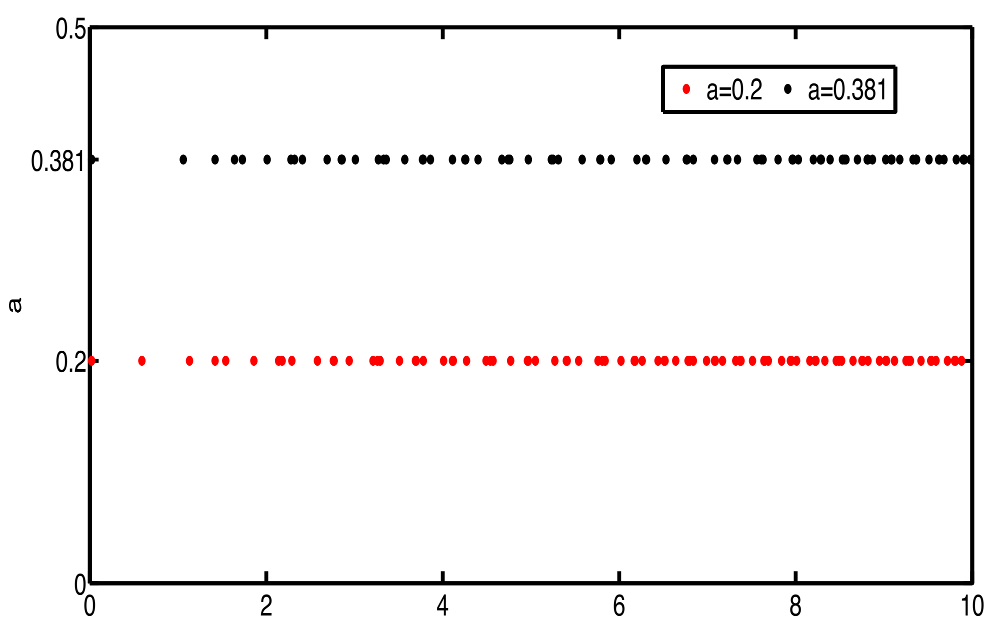

Fig. 7. Triggered instance of the INN (22) in Examples 4.1 and 4.2.

Let us take the initial values $s(t_0) = (1,2,3)^T, x_i(t_0) = i \times (3,6,3)^T, i = 1,\dots,6$. Now we have to design the E-T mechanism to realize the leader–follower synchronization between (21) and (22). Choose the constant $c = 8.2, \gamma = 0.072, a = 0.381$ and$\beta = 8.55$. Then we are infer the LMI conditions in Theorem 3.1 has solution that are feasible and the impulsive gain matrix is $\overline{K} = \widetilde{K} = 0.48I_6$ and

$$P_1 = \begin{pmatrix} 6 & -0.02 & 0.4 \\ -0.02 & 6.1 & 0.3 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}, P_2 = \begin{pmatrix} 0.74 & 0.04 & 0.5 \\ 0.04 & 0.822 & 0.3 \\ 0.04 & 0.7 & 0.2 \end{pmatrix}.$$

Under the E-T mechanism, the leader-following synchronization of systems (21) and (22) can be achieved. Fig. 5 shows the states of the one leader and six followers. Under the designed ET impulsive control technique, each followers node follows the leader, as can be shown in Fig. 5. Under the same conditions the synchronization error between (21) and (22) can converge to zero, which is shown in Fig. 6. The triggered instants of Examples 4.1 and 4.2 on the interval [0, 10] is shown in Fig. 7.

5. Conclusion

In this paper, the synchronization analysis of INNs with constant time delays has been studied. Firstly, the INNs are taken with reduced order using some variable transformations and by using E-T mechanism with impulsive control mechanism the synchronization conditions are obtained for the addressed networks. Secondly, by constructing suitable Lyapunov functional candidate for non reduced order INNs to obtain some new leader-following synchronization criteria with the help of E-T mechanism. Finally, two numerical results are offered to highlight the usefulness of theoretical conclusions. Future work will focus on solving leader-following fixed-time synchronization problem of discontinuous coupled inertial neural network with E-T mechanism and indefinite functionals.

CRediT authorship contribution statement

S. Shanmugasundaram: Conceptualization, Writing – original draft. **K. Udhayakumar:** Conceptualization. **R. Rakkiyappan:** Conceptualization, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

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