Journal of Applied Nonlinear Dynamics 9(3) (2020) 391-400

Journal of Applied Nonlinear Dynamics

https://lhscientificpublishing.com/Journals/JAND-Default.aspx

Lower Bound of Blow up Time for Three Species Cooperating Model

V. Bhuvaneswari[1†], K. Balachandran[2]

[1] Department of Mathematics with Computer Applications, PSG College of Arts and Science, Coimbatore 641 014, India

[2] Department of Mathematics, Bharathiar University, Coimbatore 641 046, India

Submission Info

Communicated by A.C.J. Luo
Received 17 February 2019
Accepted 31 May 2019
Available online 1 October 2020

Keywords

Lower bound
Blow up time
Sobolev type inequalities

Abstract

In this paper, we consider the initial boundary value problem for the three species cooperating model under various boundary conditions in which the solution may blow up in finite time. Explicit lower bound for blow up time is being obtained by using techniques based on Sobolev type and first order differential inequalities.

1 Introduction

The role of population ecology in mathematics is not to predict the change of specific populations, but to provide the general insight into how biological processes, often driven by individual behavior, affect population change over time. First of all, the ecology is reduced to differential equation model which represents the essential phenomenon and the dynamic rules of mathematics and biology will be studied.

In this paper, let us consider the following three species cooperating model under different boundary conditions in $\mathbb{R}^3$:

$$\left.\begin{array}{ll}
u_{1t} = d_1\Delta u_1 + u_1(a_1 - c_1u_1 + e_1u_2), & x \in \Omega, t > 0,\\
u_{2t} = d_2\Delta u_2 + u_2(a_2 + b_2u_1 - c_2u_2 + e_2u_3), & x \in \Omega, t > 0,\\
u_{3t} = d_3\Delta u_3 + u_3(a_3 + b_3u_2 - c_3u_3), & x \in \Omega, t > 0,\\
u_1(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), u_3(x,0) = u_{30}(x), & x \in \Omega,\\
u_1(x,t) = u_2(x,t) = u_3(x,t) = 0, & x \in \partial\Omega, t > 0,\\
\dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} = \dfrac{\partial u_3}{\partial n} = 0, & x \in \partial\Omega, t > 0,\\
\dfrac{\partial u_1}{\partial n} = \alpha_1 u_1, \dfrac{\partial u_2}{\partial n} = \alpha_2 u_2, \dfrac{\partial u_3}{\partial n} = \alpha_3 u_3, & x \in \partial\Omega, t > 0,
\end{array}\right\} \tag{1}$$

†Corresponding author.
Email address: 88bhuvana@gmail.com

ISSN 2164-6457, eISSN 2164-6473/$-see front materials
DOI:10.5890/JAND.2020.09.004

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, α_i, $i = 1,2,3$ are positive, $T > 0$ is the maximal existence time. The population densities of three cooperating species are respectively represented by $u_i = u_i(x,t)$, $i = 1,2,3$. The real constants a_i, c_i, $i = 1,2,3$, represents the intrinsic growth rates and intra-specific competition respectively. The positive constants e_1, e_2, b_2, b_3 are coefficients for inter-specific cooperation. The problem (1) is a simple food chain model which describes the three interacting species in which the presence of one species encourages the growth of preceding one and vice versa.

The nonlinear parabolic partial differential equations are being attracted by many researchers due to its applications in the broad fields such as engineering, biology, physics and so on. During the past decades, existence and blow up phenomena of solutions to various classes of nonlinear mathematical biology problems have been studied by several authors (refer [1–4]). The books by Straughan [5], Quittner and Souplet [6] as well as the survey paper of Bandle and Brunner [7] and references therein provide us a deeper knowledge on blow up. Kim and Lin [8] obtained the blow up estimates for the above food chain model under Dirichlet boundary conditions. They found the upper bound estimates for any n and lower bound to blow up rate for $n = 1$. They also investigated the three species food chain model in [9] and showed that the global existence of solutions exists if the intra-specific competitions are strong, whereas blowing up of solutions exists under certain conditions if the intra-specific competitions are weak. One can follow the articles [10–12] for recent developments in three species models.

Moreover, when $u_3 = 0$, the system (1) gets reduced to the following cooperating two species Lotka Volterra model

$$\left.\begin{aligned} u_{1t} &= d_1\Delta u_1 + u_1(a_1 - b_1u_1 + c_1u_2), \;\; x \in \Omega, t > 0, \\ u_{2t} &= d_2\Delta u_2 + u_2(a_2 + b_2u_1 - c_2u_2), \;\; x \in \Omega, t > 0, \end{aligned}\right\} \tag{2}$$

Lou et al. [13] gave the sufficient condition for the solution to blow up in finite time for the problem (2) under homogeneous Neumann boundary conditions. Lin [14] derived the upper and lower bounds of blowup rate for the two species Lotka-Volterra cooperating model (2) under homogeneous Dirichlet boundary conditions. Lin et al. [15] considered the above two species model and they obtained blow up properties of the system (2). They also proved that the periodic solutions exist if the intra-specific conditions are strong, whereas blow up of solutions exists under certain conditions if the intra-specific conditions are weak. Pao [16] showed that the blow up of solutions for the system (2) is possible provided the two species are strongly mutualistic ($c_1b_2 > c_2b_1$) which means that the geometric mean of the interaction coefficients exceeds that of population regulation coefficients. For recent developments, one can refer [17–20].

All the above mentioned works are related in finding the existence and blow up of solutions for various models. But our main interest is to evaluate t^*, the blow up time. Since t^* is not easily computable, we need to derive its upper and lower bounds. Many methods which are used in proving blow up of solutions provide an upper bound for blow up time. For application purposes, due to the explosive nature of solutions, it is important to determine the lower bound for blow up time. Over the last few years, beginning with the work of Payne and Schaefer [21, 22], lower bound to blow up time are obtained for various types of problems and also under different boundary conditions by using first order differential inequality technique. In [23], Bhuvaneswari et al. obtained the explicit lower bound for blow up time for two species chemotaxis model in $\mathbb{R}^3$ along with Dirichlet, Neumann and Robin boundary conditions. Marras in [24], considered blow-up solutions to parabolic systems, coupled through their nonlinearities under various boundary conditions with nonlinearities depending on the gradient solution and obtained lower bound for blow up time. One can refer [25–29] for recent articles on lower bound.

Therefore, in line with these motivations, in this work the lower bound for blow up time for the solution of the system (1) under Dirichlet boundary condition is derived in section 2. Since the Sobolev

Talenti inequality does not hold true for the case of Neumann boundary condition, we derive lower bound for the same by using Sobolev type inequality which provides the restriction for convex domain $\Omega \in \mathbb{R}^3$ in section 3. In section 4, we obtain a blow up estimate from below for Robin type boundary condition.

2 Dirichlet boundary condition

In this section, we obtain the lower bound for blow time in $\mathbb{R}^3$ for the system (1) under Dirichlet boundary condition $u_i(x,t) = 0,\ i = 1,2,3,$ on the boundary. And also let us assume that the initial values $u_{i0}(x), i = 1,2,3$ satisfy the compatibility condition on the boundary $u_{i0}(x) = 0$ on $\partial\Omega$.

Let us assume that the solutions for the system (1) blow up in finite time t^*. For $n \geq 1$, define the auxiliary function

$$\begin{aligned} \varphi(t) &= \int_\Omega u_1^{2n} dx + \int_\Omega u_2^{2n} dx + \int_\Omega u_3^{2n} dx \\ &= \varphi_1(t) + \varphi_2(t) + \varphi_3(t), \end{aligned} \tag{3}$$

where $\varphi_0(x) = \int_\Omega u_{10}^{2n} dx + \int_\Omega u_{20}^{2n} dx + \int_\Omega u_{30}^{2n} dx$ satisfies the first order differential inequality of the form $\varphi'(t) \leq \psi(\varphi)$ for some compatible function $\psi(\varphi)$. It then follows that t^* is bounded below by $t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{\psi(\eta)}$. The following Lemma is useful for us in proving our main results.

Lemma 1. *[24] Let w be a real valued function defined in $\mathbb{R}^3$. Then the following inequality holds true for all $x \in \Omega$:*

$$\int_\Omega w^3 dx \leq \frac{C_T}{4\tau^3} (\int_\Omega w^2 dx)^3 + \frac{3\tau C_T}{4} \int_\Omega |\nabla w|^2 dx, \tag{4}$$

where $C_T = 2\pi^{-1} 3^{-\frac{3}{4}}$ is the Sobolev constant (see [30], $q = 6, p = 2, m = 3$).

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, star-shaped and convex domain in two orthogonal directions with smooth boundary $\partial\Omega$. Assume that (u_1, u_2, u_3) are nonnegative classical solutions of the system (1) in Ω under Dirichlet boundary condition with compatible initial data. Moreover, let the solution (u_1, u_2, u_3) blows up in φ measure (3) at time t^*. Then t^* satisfies*

$$t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{A\eta + B\eta^3 + C}, \tag{5}$$

for some positive constants A, B and C.

Proof. Differentiating $\varphi_1(t)$, upon substituting (1) and simplifying, we get

$$\begin{aligned} \varphi_1'(t) &= 2nd_1 \int_\Omega u_1^{2n-1} \Delta u_1 dx + 2na_1 \int_\Omega u_1^{2n} dx - 2nc_1 \int_\Omega u_1^{2n+1} dx + 2ne_1 \int_\Omega u_1^{2n} u_2 dx \\ &= 2nd_1 \int_{\partial\Omega} u_1^{2n-1} \frac{\partial u_1}{\partial n} ds - 2n(2n-1) d_1 \int_\Omega u_1^{2n-2} |\nabla u_1|^2 dx + 2na_1 \int_\Omega u_1^{2n} dx \\ &\quad -2nc_1 \int_\Omega u_1^{2n+1} dx + 2ne_1 \int_\Omega u_1^{2n} u_2 dx. \end{aligned} \tag{6}$$

Using Young's and Hölder's inequalities to the last term, one can get

$$\int_\Omega u_1^{2n} u_2 dx \ \leq \ (\int_\Omega u_1^{2n+1} dx)^{\frac{2n}{2n+1}} (\int_\Omega u_2^{2n+1} dx)^{\frac{1}{2n+1}}$$

$$\leq \frac{2n}{2n+1}\int_\Omega u_1^{2n+1}dx + \frac{1}{2n+1}\int_\Omega u_2^{2n+1}dx. \tag{7}$$

Inserting (7) in (6), one gets

$$\begin{aligned}\varphi_1'(t) \leq & -\frac{2(2n-1)d_1}{n}\int_\Omega |\nabla u_1^n|^2 dx + 2na_1\int_\Omega u_1^{2n}dx + [\frac{2n}{2n+1}\big(2n(e_1-c_1)-c_1\big)]\\ & \times \int_\Omega u_1^{2n+1}dx + \frac{2ne_1}{2n+1}\int_\Omega u_2^{2n+1}dx.\end{aligned} \tag{8}$$

Analogously proceeding for $\varphi_2(t), \varphi_3(t)$, one can obtain the following:

$$\begin{aligned}\varphi_2'(t) \leq & -\frac{2(2n-1)d_2}{n}\int_\Omega |\nabla u_2^n|^2 dx + 2na_2\int_\Omega u_2^{2n}dx + [\frac{2n}{2n+1}\big(2n(b_2+e_2-c_2)-c_2\big)]\\ & \times \int_\Omega u_2^{2n+1}dx + \frac{2nb_2}{2n+1}\int_\Omega u_1^{2n+1}dx + \frac{2ne_2}{2n+1}\int_\Omega u_3^{2n+1}dx,\end{aligned} \tag{9}$$

$$\begin{aligned}\varphi_3'(t) \leq & -\frac{2(2n-1)d_3}{n}\int_\Omega |\nabla u_3^n|^2 dx + 2na_3\int_\Omega u_3^{2n}dx + [\frac{2n}{2n+1}\big(2n(b_3-c_3)-c_3\big)]\\ & \times \int_\Omega u_3^{2n+1}dx + \frac{2nb_3}{2n+1}\int_\Omega u_2^{2n+1}dx.\end{aligned} \tag{10}$$

Thus adding (8), (9) and (10), we get

$$\begin{aligned}\varphi'(t) \leq & -\frac{2(2n-1)d_1}{n}\int_\Omega |\nabla u_1^n|^2 dx - \frac{2(2n-1)d_2}{n}\int_\Omega |\nabla u_2^n|^2 dx - \frac{2(2n-1)d_3}{n}\int_\Omega |\nabla u_3^n|^2 dx\\ & +2na_1\int_\Omega u_1^{2n}dx + 2na_2\int_\Omega u_2^{2n}dx + 2na_3\int_\Omega u_3^{2n}dx\\ & +[\frac{2n}{2n+1}\big(2n(b_2+e_1-c_1)-c_1\big)]\int_\Omega u_1^{2n+1}dx\\ & +[\frac{2n}{2n+1}\big(2n(b_2+b_3+e_1+e_2-c_2)-c_2\big)]\int_\Omega u_2^{2n+1}dx\\ & +[\frac{2n}{2n+1}\big(2n(b_3+e_2-c_3)-c_3\big)]\int_\Omega u_3^{2n+1}dx.\end{aligned} \tag{11}$$

Making use of Young's and Hölder's inequalities, we obtain

$$\begin{aligned}\int_\Omega u_i^{2n+1}dx \leq & \,(\int_\Omega u_i^{3n}dx)^{\frac{2n+1}{3n}}|\Omega|^{\frac{n-1}{3n}}\\ \leq & \,(\frac{2n+1}{3n})\int_\Omega u_i^{3n}dx + (\frac{n-1}{3n})|\Omega|,\end{aligned} \tag{12}$$

for $i = 1,2,3$. Inserting (12) in (11) and simplifying, we get

$$\begin{aligned}\varphi'(t) \leq & -\frac{2(2n-1)d_1}{n}\int_\Omega |\nabla u_1^n|^2 dx - \frac{2(2n-1)d_2}{n}\int_\Omega |\nabla u_2^n|^2 dx - \frac{2(2n-1)d_3}{n}\int_\Omega |\nabla u_3^n|^2 dx\\ & +2na_1\int_\Omega u_1^{2n}dx + 2na_2\int_\Omega u_2^{2n}dx + 2na_3\int_\Omega u_3^{2n}dx\\ & +C_1\int_\Omega u_1^{3n}dx + C_2\int_\Omega u_2^{3n}dx + C_3\int_\Omega u_3^{3n}dx + C,\end{aligned} \tag{13}$$

where, $C_i, i = 1,2,3$, and C are constants that depend on b_i, e_i, c_i, n and $|\Omega|$. From Lemma (1) taking $w = u^n$, one can yield

$$\int_\Omega u_i^{3n}dx \leq \frac{C_T}{4\varepsilon_i^3}(\int_\Omega u_i^{2n}dx)^3 + \frac{3\varepsilon_i C_T}{4}\int_\Omega |\nabla u_i^n|^2 dx,\ i = 1,2,3, \tag{14}$$

where $\varepsilon_i, i = 1,2,3$, are positive constants whose values need to be determined later. Inserting (14) in (13), one gets

$$\begin{aligned}\varphi'(t) \ \le\ & -[\frac{2(2n-1)d_1}{n} - \frac{3\varepsilon_1 C_T C_1}{4}]\int_\Omega |\nabla u_1^n|^2 dx - [\frac{2(2n-1)d_2}{n} - \frac{3\varepsilon_2 C_T C_2}{4}]\int_\Omega |\nabla u_2^n|^2 dx \\ & -[\frac{2(2n-1)d_3}{n} - \frac{3\varepsilon_3 C_T C_3}{4}]\int_\Omega |\nabla u_3^n|^2 dx + 2na_1 \int_\Omega u_1^{2n} dx + 2na_2 \int_\Omega u_2^{2n} dx \\ & + 2na_3 \int_\Omega u_3^{2n} dx + \frac{C_T C_1}{4\varepsilon_1^3}(\int_\Omega u_1^{2n} dx)^3 + \frac{C_T C_2}{4\varepsilon_2^3}(\int_\Omega u_2^{2n} dx)^3 \\ & + \frac{C_T C_3}{4\varepsilon_3^3}(\int_\Omega u_3^{2n} dx)^3 + C. \end{aligned} \tag{15}$$

Choose the values of ε_i, $i = 1,2,3$, such that

$$\frac{2(2n-1)d_i}{n} - \frac{3\varepsilon_i C_T C_i}{4} = 0, \quad i = 1,2,3.$$

Solving the above, we get

$$\varepsilon_i = \frac{8(2n-1)d_i}{3nC_T C_i}, \quad i = 1,2,3.$$

Thus (15) now gets reduced to the differential inequality

$$\varphi'(t) \ \le\ A\varphi + B\varphi^3 + C, \tag{16}$$

where $A = 2n(a_1 + a_2 + a_3)$ and $B = \frac{C_T}{4}[\frac{C_1}{\varepsilon_1^3} + \frac{C_2}{\varepsilon_2^3} + \frac{C_3}{\varepsilon_3^3}]$ and we have made use of the fact that, for any $\gamma > 1$ and non negative a and b,

$$a^\gamma + b^\gamma \le (a+b)^\gamma. \tag{17}$$

Integrating the inequality (16) from 0 to t, we get

$$t \ge \int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{A\eta + B\eta^3 + C}.$$

From our assumption that the solution blows up in time t^* in some finite measure $\varphi(t)$, we obtain

$$t^* \ge \int_{\varphi(0)}^{\infty} \frac{d\eta}{A\eta + B\eta^3 + C}.$$

Thus we get the desired result.

3 Neumann boundary condition

In this section, let us obtain the lower bound for blow up time for the system (1) under Neumann condition $\frac{\partial u_i}{\partial n} = 0, i = 1,2,3$ on $\partial\Omega$ that satisfies the compatibility condition $\frac{\partial u_{i0}}{\partial n} = 0$ on the boundary.

Theorem 3. *Let* (u_1, u_2, u_3) *be the solution of* (1) *under Neumann condition in a bounded, convex and star-shaped domain* $\Omega \subset \mathbb{R}^3$*. If the solution becomes unbounded in the finite measure* φ *at some finite time* t^**, then* t^* *is bounded below by*

$$t^* \ \ge\ \int_{\varphi(0)}^{\infty} \frac{d\eta}{C + D\eta + E\eta^{3/2} + F\eta^3}, \tag{18}$$

where C, D, E, F *are positive constants.*

Proof. The proof is same as that of Theorem (2) till (13). Since the Sobolev Talenti inequality is valid only for the Dirichlet case, that cannot be applied here. So in order to bound the 7th to 9th terms we make use of the following Sobolev type inequality which was derived by Payne and Schaefer in [21, (2.8)-(2.16)] in a restricted convex domain Ω in $\mathbb{R}^3$.

$$\int_\Omega u_i^{3n}dx \leq \frac{1}{3^{3/4}}\{\frac{3}{2\rho_0}\int_\Omega u_i^{2n}dx+(\frac{d}{\rho_0}+1)(\int_\Omega u_i^{2n}dx)^{\frac{1}{2}}(\int_\Omega |\nabla u_i^n|^2dx)^{\frac{1}{2}}\}^{\frac{3}{2}}. \tag{19}$$

Making use of the following inequalities,

$$\begin{aligned}(a+b)^{\frac{j+1}{j}} &\leq 2^{\frac{1}{j}}(a^{\frac{j+1}{j}}+b^{\frac{j+1}{j}}),\\ a^p+b^q &\leq pa+bq, \quad p+q=1,\end{aligned}$$

where a and b are some positive constants, (19) now becomes, for $i=1,2,3$,

$$\begin{aligned}\int_\Omega u_i^{3n}dx &\leq \frac{2^{1/2}}{3^{3/4}}\{m_1^{3/2}(\int_\Omega u_i^{2n}dx)^{3/2}+m_2^{3/2}(\int_\Omega u_i^{2n}dx)^{3/4}(\int_\Omega |\nabla u_i^n|^2dx)^{3/4}\}\\ &\leq \frac{2^{1/2}m_1^{3/2}}{3^{3/4}}(\int_\Omega u_i^{2n}dx)^{3/2}+\frac{2^{1/2}m_2^{3/2}}{3^{3/4}4\varepsilon_i^3}(\int_\Omega u_i^{2n}dx)^3\\ &\quad+\frac{2^{1/2}m_2^{3/2}\varepsilon_i 3^{1/4}}{4}(\int_\Omega |\nabla u_i^n|^2dx),\end{aligned} \tag{20}$$

where we have made use of the arithmetic geometric inequality for positive weight function ε_i, $i=1,2,3$, whose values need to be determined later on, $m_1=\frac{3}{2\rho_0}$ and $m_2=(\frac{d}{\rho_0}+1)$. Thus inserting (20) in (13), we get

$$\begin{aligned}\varphi'(t) &\leq -[\frac{2(2n-1)d_1}{n}-\frac{2^{1/2}m_2^{3/2}\varepsilon_1 C_1 3^{1/4}}{4}]\int_\Omega |\nabla u_1^n|^2dx\\ &\quad-[\frac{2(2n-1)d_2}{n}-\frac{2^{1/2}m_2^{3/2}\varepsilon_2 C_2 3^{1/4}}{4}]\int_\Omega |\nabla u_2^n|^2dx\\ &\quad-[\frac{2(2n-1)d_3}{n}-\frac{2^{1/2}m_2^{3/2}\varepsilon_3 C_3 3^{1/4}}{4}]\int_\Omega |\nabla u_3^n|^2dx\\ &\quad+2na_1\int_\Omega u_1^{2n}dx+2na_2\int_\Omega u_2^{2n}dx+2na_3\int_\Omega u_3^{2n}dx\\ &\quad+\frac{2^{1/2}m_1^{3/2}C_1}{3^{3/4}}(\int_\Omega u_1^{2n}dx)^{3/2}+\frac{2^{1/2}m_1^{3/2}C_2}{3^{3/4}}(\int_\Omega u_2^{2n}dx)^{3/2}\\ &\quad+\frac{2^{1/2}m_1^{3/2}C_3}{3^{3/4}}(\int_\Omega u_3^{2n}dx)^{3/2}+\frac{2^{1/2}m_2^{3/2}C_1}{3^{3/4}\cdot 4\varepsilon_1^3}(\int_\Omega u_1^{2n}dx)^3\\ &\quad+\frac{2^{1/2}m_2^{3/2}C_2}{3^{3/4}\cdot 4\varepsilon_2^3}(\int_\Omega u_2^{2n}dx)^3+\frac{2^{1/2}m_2^{3/2}C_3}{3^{3/4}\cdot 4\varepsilon_3^3}(\int_\Omega u_3^{2n}dx)^3+C.\end{aligned} \tag{21}$$

Let us choose the values of $\varepsilon_i, i=1,2,3,$ in such a way that the first three terms of the above inequality vanish.

$$\varepsilon_i=\frac{2^{5/2}(2n-1)d_i}{m_2^{3/2}nC_i 3^{1/4}}, \quad i=1,2,3.$$

Thus (21) now gets reduced to the differential inequality

$$\varphi'(t)\leq C+D\varphi+E\varphi^{3/2}+F\varphi^3, \tag{22}$$

where we have made use of the inequality (17) and

$$\begin{aligned} D &= 2n(a_1+a_2+a_3), \\ E &= \frac{2^{1/2}m_1^{3/2}}{3^{3/4}}(C_1+C_2+C_3), \\ F &= \frac{2^{1/2}m_2^{3/2}}{3^{3/4}\cdot 4}(\frac{C_1}{\varepsilon_1^3}+\frac{C_2}{\varepsilon_2^3}+\frac{C_3}{\varepsilon_3^3}). \end{aligned}$$

Solving (22), one gets

$$t \geq \int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{C+D\eta+E\eta^{3/2}+F\eta^3}.$$

Since the solution blows up in finite time t^* in finite measure φ,

$$t^* \geq \int_{\varphi(0)}^{\infty} \frac{d\eta}{C+D\eta+E\eta^{3/2}+F\eta^3}.$$

Thus we get the desired result.

4 Robin boundary condition

We consider in this section the system (1) along with the Robin boundary condition $\frac{\partial u_i}{\partial n} = \alpha_i u_i$, $i = 1,2,3$, where α_i are positive constants. And also we assume that the initial values satisfy the compatibility condition $\frac{\partial u_{i0}}{\partial n} = 0$ on the boundary $\partial\Omega$. Moreover the theorem in the above section cannot be applied here since we need to calculate the values of the integral over the boundary. We need the help of the following lemma to prove our main result.

Lemma 4. *For any $\mathscr{C}^1$ function $w(x) > 0$ in a bounded star shaped convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we have the following inequality*

$$\int_{\partial\Omega} w^n d\sigma \leq \frac{N}{\rho_0}\int_\Omega w^n dx + \frac{nd}{\rho_0}\int_\Omega w^{n-1}|\nabla w|dx, \tag{23}$$

where $\rho_0 = \min_{\partial\Omega}(x\cdot n)$ *and* $d = \max_{\Omega}|x|$.

Proof. Since Ω is a bounded star-shaped domain, we have $\rho_0 > 0$. Consider the identity

$$\nabla(w^n x) = Nw^n + nw^{n-1}(x\cdot\nabla w).$$

On integrating the identity over Ω and applying Gauss divergence theorem, we obtain

$$\begin{aligned} \int_{\partial\Omega} w^n (x\cdot n) d\sigma &= N\int_\Omega w^n dx + n\int_\Omega w^{n-1}|x\cdot\nabla w|dx, \\ \int_{\partial\Omega} w^n d\sigma &\leq \frac{N}{\rho_0}\int_\Omega w^n dx + \frac{nd}{\rho_0}\int_\Omega w^{n-1}|\nabla w|dx, \end{aligned}$$

which follow from the definition of ρ_0 and d.

Theorem 5. *Let $u_i(x,t)$ be the nonnegative classical solution of the system (1) with Robin boundary condition in a bounded star-shaped domain $\Omega \subset \mathbb{R}^3$ which is assumed to be convex in two orthogonal directions. Then assuming that the solution blows up in finite time t^* in finite measure $\varphi(t)$, we need to show that t^* is bounded from below by (18).*

Proof. Differentiating the finite measure $\varphi(t)$, we get

$$\begin{aligned}\varphi'(t) &= 2nd_1\alpha_1\int_{\partial\Omega}u_1^{2n}ds+2nd_2\alpha_2\int_{\partial\Omega}u_2^{2n}ds+2nd_3\alpha_3\int_{\partial\Omega}u_3^{2n}ds\\ &-\frac{2(2n-1)d_1}{n}\int_\Omega|\nabla u_1^n|^2dx-\frac{2(2n-1)d_2}{n}\int_\Omega|\nabla u_2^n|^2dx-\frac{2(2n-1)d_3}{n}\int_\Omega|\nabla u_3^n|^2dx\\ &+2na_1\int_\Omega u_1^{2n}dx+2na_2\int_\Omega u_2^{2n}dx+2na_3\int_\Omega u_3^{2n}dx\\ &+C_1\int_\Omega u_1^{3n}dx+C_2\int_\Omega u_2^{3n}dx+C_3\int_\Omega u_3^{3n}dx+C. \end{aligned}\tag{24}$$

From (23), the first three terms now become, for $N=3$,

$$\begin{aligned}\int_{\partial\Omega}u_i^{2n}ds &\le \frac{3}{\rho_0}\int_\Omega u_i^{2n}dx+\frac{2nd}{\rho_0}\int_\Omega u_i^{2n-1}|\nabla u_i|dx\\ &\le \frac{3}{\rho_0}\int_\Omega u_i^{2n}dx+\frac{2nd}{\rho_0}(\int_\Omega u_i^{2n}dx\int_\Omega u_i^{2n-2}|\nabla u_i|^2dx)^{1/2}\\ &\le \frac{3}{\rho_0}\int_\Omega u_i^{2n}dx+\frac{d}{\rho_0}\int_\Omega u_i^{2n}dx+\frac{d}{\rho_0}\int_\Omega|\nabla u_i^n|^2dx\\ &\le m_3\int_\Omega u_i^{2n}dx+m_4\int_\Omega|\nabla u_i^n|^2dx, \end{aligned}\tag{25}$$

where we have used Cauchy Schwarz inequality, $m_3=\frac{3+d}{\rho_0}$ and $m_4=\frac{d}{\rho_0}$. Inserting (25) in (24) and simplifying one gets

$$\begin{aligned}\varphi'(t) &\le -(2d_1[\frac{2n-1}{n}-n\alpha_1m_4])\int_\Omega|\nabla u_1^n|^2dx-(2d_2[\frac{2n-1}{n}-n\alpha_2m_4])\int_\Omega|\nabla u_2^n|^2dx\\ &-(2d_3[\frac{2n-1}{n}-n\alpha_3m_4])\int_\Omega|\nabla u_3^n|^2dx+2n(a_1+d_1\alpha_1m_3)\int_\Omega u_1^{2n}dx\\ &+2n(a_2+d_2\alpha_2m_3)\int_\Omega u_2^{2n}dx+2n(a_3+d_3\alpha_3m_3)\int_\Omega u_3^{2n}dx\\ &+C_1\int_\Omega u_1^{3n}dx+C_2\int_\Omega u_2^{3n}dx+C_3\int_\Omega u_3^{3n}dx+C. \end{aligned}\tag{26}$$

Inserting (20) in the above inequality, we obtain

$$\begin{aligned}\varphi'(t) &\le -(2d_1[\frac{2n-1}{n}-n\alpha_1m_4]-\frac{2^{1/2}m_2^{3/2}\varepsilon_1C_13^{1/4}}{4})\int_\Omega|\nabla u_1^n|^2dx\\ &-(2d_2[\frac{2n-1}{n}-n\alpha_2m_4]-\frac{2^{1/2}m_2^{3/2}\varepsilon_2C_23^{1/4}}{4})\int_\Omega|\nabla u_2^n|^2dx\\ &-(2d_3[\frac{2n-1}{n}-n\alpha_3m_4]-\frac{2^{1/2}m_2^{3/2}\varepsilon_3C_33^{1/4}}{4})\int_\Omega|\nabla u_3^n|^2dx\\ &+2n(a_1+d_1\alpha_1m_3)\int_\Omega u_1^{2n}dx+2n(a_2+d_2\alpha_2m_3)\int_\Omega u_2^{2n}dx\\ &+2n(a_3+d_3\alpha_3m_3)\int_\Omega u_3^{2n}dx+\frac{2^{1/2}m_1^{3/2}C_1}{3^{3/4}}(\int_\Omega u_1^{2n}dx)^{3/2}\\ &+\frac{2^{1/2}m_1^{3/2}C_2}{3^{3/4}}(\int_\Omega u_2^{2n}dx)^{3/2}+\frac{2^{1/2}m_1^{3/2}C_3}{3^{3/4}}(\int_\Omega u_3^{2n}dx)^{3/2} \end{aligned}$$

$$+\frac{2^{1/2}m_2^{3/2}C_1}{3^{3/4}\cdot 4\varepsilon_1^3}(\int_\Omega u_1^{2n}dx)^3+\frac{2^{1/2}m_2^{3/2}C_2}{3^{3/4}\cdot 4\varepsilon_2^3}(\int_\Omega u_2^{2n}dx)^3$$
$$+\frac{2^{1/2}m_2^{3/2}C_3}{3^{3/4}\cdot 4\varepsilon_3^3}(\int_\Omega u_3^{2n}dx)^3+C. \tag{27}$$

Choose the values of ε_i, $i=1,2,3$, in the following way such that the first three terms of the above inequality vanish

$$\varepsilon_i = 2d_i[\frac{2n-1}{n}-n\alpha_i m_4]\frac{4}{2^{1/2}m_2^{3/2}C_i 3^{1/4}}.$$

Thus the above inequality (27) is now reduced to

$$\varphi'(t)\le C+D\varphi+E\varphi^{3/2}+F\varphi^3, \tag{28}$$

where

$$D = 2n(a_1+a_2+a_3)+2nm_3(d_1\alpha_1+d_2\alpha_2+d_3\alpha_3),$$
$$E = \frac{2^{1/2}m_1^{3/2}}{3^{3/4}}(C_1+C_2+C_3),$$
$$F = \frac{2^{1/2}m_2^{3/2}}{3^{3/4}\cdot 4}(\frac{C_1}{\varepsilon_1^3}+\frac{C_2}{\varepsilon_2^3}+\frac{C_3}{\varepsilon_3^3}).$$

Solving (28), we get the resultant lower bound.

5 Conclusion

In this article, explicit estimates from below for blow up time were obtained for different types of boundary conditions for an ecological system namely, the three species cooperating model in $\mathbb{R}^3$ by choosing suitable type of auxiliary function and compatible initial data using first order differential inequality technique.

Acknowledgement

The first author would like to thank PSG College of Arts and Science, Coimbatore for constant support towards research work and second author is supported by UGC BSR. Further the authors thank the referees for useful comments and valuable suggestions which led to improvement in the quality of this article.

References

[1] Bhuvaneswari, V., Shangerganesh, L., and Balachandran, K. (2015), Global existence and blow up of solutions of quasilinear chemotaxis system, *Mathematical Methods in the Applied Sciences*, **38**, 3738-3746.

[2] Bhuvaneswari, V. and Balachandran, K. (2016), Global existence and blow up of solutions of two species chemotaxis model, *Journal of Applied Nonlinear Dynamics*, **5**, 459-471.

[3] KwangIk, K., ZhiGui, L., and Zhi, L. (2010), Global existence and blow up of solutions to a free boundary problem for mutualistic model, *Science China, Mathematics*, **53**, 2085-2095.

[4] Zhu, L., Ling, Z., and Lin, Z. (2014), Global existence and blow up analysis to a cooperating model with self diffusion, *Advances in Difference Equations*, **2014**, 166.

[5] Straughan, B. (1998), *Explosive Instabilities in Mechanics*, Springer, Berlin.
[6] Quittner, P. and Souplet, P. (2007), *Superlinear parabolic problems. Blow up, global existence and steady states*, Birkhäuser, Basel.
[7] Bandle, C. and Brunner, H. (1998), Blow up in diffusion equations: a survey, *Journal of Computational and Applied Mathematics*, **97**, 3-22.
[8] Kim, K.I. and Lin, Z. (2004), Blow up estimates for a parabolic system in a three species cooperating model, *Journal of Mathematical Analysis and Applications*, **293**, 663-676.
[9] Kim, K.I. and Lin, Z. (2004), Blow up in a three species cooperating model, *Applied Mathematics Letters*, **17**, 89-94.
[10] Xie, Z. (2012), Cross diffusion induced Turing instability for a three species food chain model, *Journal of Mathematical Analysis and Applications*, **388**, 539-547.
[11] Mimura, M. and Tohma, M. (2015), Dynamic coexistence in a three species competition diffusion system, *Ecological Complexity*, **21**, 215-232.
[12] Wang, C.Y., Wang, S., and Yan, X.P. (2009), Global asymptotic stability of three species mutualism models with diffusion and delay effects, *Discrete Dynamics in Nature and Society*, **2009**, Article ID: 317298, 20 pages.
[13] Lou, Y., Nagylaki, T., and Ni, W.M. (2001) On diffusion induced blowups in a mutualistic model, *Nonlinear Analysis*, **45**, 329-342.
[14] Lin, Z. (2002), Blow up estimates for a mutualistic model in ecology, *Electronic Journal of Qualitative Theory of Differential Equations*, **8**, 1-14.
[15] Lin, Z., Liu, J., and Pedersen, M. (2011), Periodicity and blow up in a two species cooperating model, *Nonlinear Analysis: Real World Applications*, **12**, 479-486.
[16] Pao, C.V. (1992), *Nonlinear Parabolic and Elliptic Equations*, Plenum, New York.
[17] Pao, C.V. (2014), A Lotka Volterra cooperating reaction diffusion system with degenerate density dependent diffusion, *Nonlinear Analysis*, **95**, 460-467.
[18] Pao, C.V. (2014), Dynamics of Lotka Volterra competition reaction diffusion systems with degenerate diffusion, *Journal of Mathematical Analysis and Applications*, **421**, 1721-1742.
[19] Feng, P. (2008), Global and blow up solutions for a mutualistic model, *Nonlinear Analysis*, **68**, 1898-1908.
[20] Pao, C.V. (2015), Dynamics of Lotka Volterra cooperation systems governed by degenerate quasilinear reaction diffusion equations, *Nonlinear Analysis, Real World Applications*, **23**, 47-60.
[21] Payne, L.E. and Schaefer, P.W. (2006), Lower bounds for blow up time in parabolic problems under Neumann conditions, *Applicable Analysis*, **85**, 1301-1311.
[22] Payne, L.E. and Schaefer, P.W. (2007), Lower bounds for blow up time in parabolic problems under Dirichlet conditions, *Journal of Mathematical Analysis and Applications*, **328**, 1196-1205.
[23] Bhuvaneswari, V. and Balachandran, K. (2017), Estimates from below for blow up time for two species chemotaxis system, *Discontinuity, Nonlinearity and Complexity*, **6**, 305-315.
[24] Marras, M. (2011), Bounds for blow up time in nonlinear parabolic systems under various boundary conditions, *Numerical Functional Analysis and Optimization*, **32**, 453-468.
[25] Marras, M., Piro, S.V., and Viglialoro, G. (2014), Estimates from below of blow up time in a parabolic system with gradient term, *International Journal of Pure and Applied Mathematics*, **93**, 297-306.
[26] Farina, M.A., Marras, M., and Viglialoro, G. (2015), On explicit lower bounds and blow up times in a model of chemotaxis, *Dynamical Systems, Differential Equations and Applications*, 409-417, doi:10.3934/proc.2015.0409.
[27] Tao, Y. and Piro, S.V. (2016), Explicit lower bound of blow up time in a fully parabolic chemotaxis system with nonlinear cross diffusion, *Journal of Mathematical Analysis and Applications*, **436**, 16-28.
[28] Ding, J. and Shen, X. (2018), Blow up analysis in quasilinear reaction diffusion problems with weighted nonlocal source, *Computers and Mathematics with Applications*, **75**, 1288-1301.
[29] Tao, X. and Fang, Z.B. (2017), Blow up phenomena for a nonlinear reaction diffusion system with time dependent coefficients, *Computers and Mathematics with Applications*, **74**, 2520-2528.
[30] Talenti, G. (1976), Best constant in Sobolev inequality, *Annali di Matematica Pura ed Applicata*, **110**, 353-372.