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NEUTRAL IMPULSIVE STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH FINITE DELAY AND POISSON JUMPS

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Abstract. In this paper, we present the existence, uniqueness and asymptotic behaviour of the mild solution for neutral impulsive stochastic differential equations driven by Poisson jumps, and fractional Brownian motion with the Hurst index $H > \frac{1}{2}$. The results are obtained by using Banach fixed point principle in a Hillbert space.

1. Introduction

Impulsive stochastic differential equations are effectively used to describe the real life phenomena in the fields of ecology, chemical technology, electrical engineering, etc. So many researchers showed interest in investigating neutral stochastic differential equations. (Refer [10], [12], [4],[11])

Fractional Brownian motions are widely used in modelling many complex phenomena in applications when the systems are subject to rough external forcing. An fbm is differs from the standard Brownian motion, semi-martingales and others classically used in the theory of stochastic processes. It is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $H \in (0, 1)$. It is a self similar process with stationary increments and has a long-memory when $H > \frac{1}{2}$.

Initially, Ferrante and Rovira established the existence and uniqueness of solutions to delayed SDEs with fbm for $H > \frac{1}{2}$ and constant delay by using the skorohod integral based on the malliavin calculus [5]. Existence and continuability of solutions for differential equations with delays and state-dependent impulses is established by Xinzhi Liu and George Ballinger [16]. Many researchers studied equations driven by fractional Brownian motion ([1], [6], [8], [13]). Stochastic differential equations with Poisson jumps have been considered by many authors ([4], [3], [7], [9]). Caraballo.et.al have studied the existence, uniqueness and exponential asymptotic behaviour of mild solutions by using wiener integral [2]. Nguyen Tien

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Dung studied existence, uniqueness and exponential stability of neutral stochastic differential equations using Banach-fixed point theory[14].

By the motivation of the above works, we establish the existence, uniqueness and asymptotic behaviour of mild solution to neutral impulsive stochastic differential equations with finite delay and Poisson jumps of the following form driven by fractional Brownian motion in a Hilbert space

$$\begin{cases} d[x(t)+g(t,x_t)] = [Ax(t)+f(t,x_t)]dt + \sigma(t)dW^H(t) + \\ \int_Z h(t,x_t,y)\tilde{N}(dt,dy), \quad t \geq 0, t \neq t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = I_k(x(t_k^-)), \quad k \in \mathbb{N} \\ x(t) = \phi(t), \quad t \in (-\tau, 0] \quad (0 < \tau \leq \infty) \end{cases} \tag{1}$$

where $Z \in \mathscr{L}_2^0(U-\{0\})$, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$ in a Hilbert space X with norm $\|.\|$, W^H is a fractional Brownian motion with $H > \frac{1}{2}$ on a real and separable Hilbert space Y, $\mathbb{N}$ denotes the set of positive integers, the impulsive moments satisfy $0 < t_1 < t_2 < \cdots, \lim_{k\to\infty} t_k = \infty$, and $f, g : [0,\infty)X \to X,\ \sigma : [0,\infty) \to \mathscr{L}_2^0(Y,X),\ h : [0,\infty)X \times U \to X,\ I_k : X \to X$ are defined later, the initial data $\phi \in C((-\tau,0],X)$ the space of all continuous functions from $(-\tau, 0]$ to X and has finite second moments. The space $\mathscr{L}_2^0(Y,X)$ will be defined later. We have used Banach fixed point theorem and semigroup theory as a major tool.

This paper is constructed as follows. In section 2 we present some basic notations, definition and preliminary facts. In section 3 we mentioned hypotheses to establish the main result. In section 4 we studied the existence, uniqueness and asymptotic behaviour of mild solution.

2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $T > 0$ be an arbitrary fixed horizon. An one-dimensional fractional Brownian motion(fbm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\beta^H = \{\beta^H(t), 0 \leq t \leq T\}$ with the covariance function $R(t,s) = E[\beta^H(t)\beta^H(s)]$

$$R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is known that $\beta^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$\beta^H(t) = \int_0^t K(t,s)d\beta(s), \tag{2}$$

Where β is a standard Brownian motion and the Volterra kernal $K(t,s)$ is given by

$$K(t,s) = c_H \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t \geq s.$$

for the deterministic function $\varphi \in L^2([0,T])$, the fractional Wiener integral of φ with respect to β^H is defined by

$$\int_0^T \varphi(s)d\beta^H(s) = \int_0^T K_H^*\varphi(s)d\beta(s),$$

where $K_H^*\varphi(s) = \int_s^T \varphi(r)\frac{\partial K}{\partial r}(r,s)dr$.
Let X and Y be two real, separable Hilbert spaces and let $\mathscr{L}(Y,X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathscr{L}(Y,X)$. Let $\{e_n, n = 1, 2, \ldots\}$ be a complete orthonormal basis in Y and $Q \in \mathscr{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n, n = 1, 2, \ldots$ are non-negative real numbers. We define the infinite dimensional fbm on Y with covariance Q as

$$W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H(t)$ are real, independent fbm's. This process is a $Y-$valued Gaussian, it starts from 0, has zero mean and covariance:

$$E\left\langle W^H(t), x\right\rangle \left\langle W^H(s), y\right\rangle = R(t,s)\left\langle Q(x), y\right\rangle \quad \text{for all} \quad x, y \in Y \quad \text{and} \quad t, s \in [0,T].$$

In order to define Wiener integrals with respect to the $Q-$fbm $W^H(t)$, we introduce the space $\mathscr{L}_2^0 := \mathscr{L}_2^0(Y,X)$ of all $Q-$Hilbert-Schmidt operators $\psi : Y \to X$. and $\psi \in \mathscr{L}(Y,X)$ is called a $Q-$Hilbert-Schmidt operator if

$$\|\psi\|_{\mathscr{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty$$

and that the space $\mathscr{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathscr{L}_2^0} := \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space. The fractional Wiener integral of the function $\psi : [0,T] \to \mathscr{L}_2^0(Y,X)$ with respect to $Q-$fbm is defined by

$$\int_0^t \psi(s)dW^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\psi(s)e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(s) d\beta_n(s), \tag{3}$$

where β_n is the standard Brownian motion used to present β_n^H as in equation 2.

The counting measure of stationary Poisson process $p(t)_{t>0}$ is denoted by $N(t, du)$ and $\widehat{N}(t, A) = \mathbb{E}(N(t, A)) = t\nu A$ for $A \in \mathcal{E}$, where ν is the characteristic measure. The Poisson martingale measure is defined as $\tilde{N}(t, du) = N(t, du) - t\nu(du)$, generated by p_t. If $\psi : [0,T] \to \mathscr{L}_2^0(Y,X)$ satisfies $\int_0^t \|\psi(s)\|^2_{\mathscr{L}_2^0} ds < \infty$ then the above sum in equation 3 is well defined as an $X-$valued random variable and we have

$$E\left\| \int_0^t \psi(s)dW^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathscr{L}_2^0} ds. \tag{4}$$

Lemma 1 Suppose that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, and the semigroup $S(t)$ is uniformly bounded, $\|S(t)\| \leq M$ for some constant $M \geq 1$ and every $t \geq 0$. Then, for $0 < \alpha \leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$ as a closed linear operator on its domain $\mathscr{D}(-A)^\alpha$. Furthermore, the subspace $\mathscr{D}(-A)^\alpha$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in \mathscr{D}(-A)^\alpha$$

defines a norm on $X_\alpha := \mathscr{D}(-A)^\alpha$.

Lemma 2 Under the above conditions the following properties hold.
(i) X_α is a Banach space for $0 < \alpha \leq 1$.
(ii) If the resolvent operator of A is compact, then the embedding $X_\beta \subset X_\alpha$ is continuous and compact for $0 < \alpha \leq \beta$.
(iii) For every $0 < \alpha \leq 1$, there exists M_α such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad \lambda > 0, \quad t \geq 0. \tag{5}$$

Definition 1 An $X-$valued stochastic process $\{x(t), t \in (-\tau, \infty)\}$ is called a mild solution of equation 1 if $x(t) = \phi(t)$ on $(-\tau, 0]$, and the following conditions hold:
(i) $x(.)$ is continuous on $(0, t_1]$ and each interval $(t_k, t_{k+1}], k \in \mathbb{N}$,
(ii) for each t_k, $x(t_k^+) = \lim\limits_{t \to t_k^+} x(t)$ exists,
(iii) for each $t \geq 0$, we have a.s.

$$\begin{aligned} x(t) &= S(t)(\phi(0) + g(0, \phi)) - g(t, x_t) - \int_0^t AS(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)f(s, x_s)ds + \int_0^t S(t-s)\sigma(s)dW^H(s) \\ &+ \int_0^t S(t-s)\int_Z h(s, x_s, y)\tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)). \end{aligned} \tag{6}$$

3. Hypotheses

In order to prove the required results, we assume the following conditions:
(H_1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t\geq 0}$, of bounded linear operators on X. Moreover, $S(t)$ satisfies the condition that there exists positive constants M, λ such that

$$\|S(t)\| \leq Me^{-\lambda t}, \quad t \geq 0.$$

(H_2) There exists $L_1 > 0$ such that, for all $t \geq 0, x, y \in X$.

$$\|f(t, x) - f(t, y)\|^2 \leq L_1\|x - y\|^2.$$

(H_3) There exist constants $0 < \beta < 1, L_2 > 0$ such that the function g is $X_\beta-$ valued and satisfies for all $t \geq 0, \quad x, y \in X$

$$||(-A)^\beta g(t, x) - (-A)^\beta g(t, y)||^2 \leq L_2||x - y||^2.$$

(H_4) The function $(-A)^\beta g$ is continuous in the quadratic mean square:
For all functions x,

$$\lim_{t\to s} E\|(-A)^\beta g(t, x(t)) - (-A)^\beta g(s, x(s))\|^2 = 0.$$

(H_5) There exists some positive numbers $q_k, k \in N$ such that

$$\|I_k(x) - I_k(y)\| \leq q_k\|x - y\|$$

for all $x, y \in X$ and $\displaystyle\sum_{k=1}^{\infty} q_k < \infty$.

(H_6) The function $\sigma : [0, \infty) \to \mathscr{L}_2^0(Y, X)$ satisfies $\displaystyle\int_0^\infty e^{2\gamma s}\|\sigma(s)\|^2_{\mathscr{L}_2^0} ds < \infty$ for some $\gamma > 0$.

(H_7)The measurable mappings $f(.)$,$\sigma(.)$ and $h(.)$ satisfy the following conditions:
$(7a)$ for all $t \in (-\tau, 0], \phi_1, \phi_2 \in C((-\tau, 0], X)$,

$$\begin{aligned}&|f(t,\phi_1) - f(t,\phi_2)|^2 \vee |\sigma(t,\phi_1) - \sigma(t,\phi_2)|^2_{\mathcal{L}_2^0}\\ &\leq \mathcal{K}(\|\phi_1 - \phi_2\|^2_{\mathcal{L}_2^0}).\end{aligned}$$

$(7b)$ for any $H-$ valued processes $x(t), y(t), t \in (-\tau, 0]$,

$$\int_0^t \int_Z |h(s,x_s,z) - h(s,y_s,z)|^2 v(dz)ds \vee \left(\int_0^t \int_Z |h(s,x_s,z) - h(s,y_s,z)|^4 v(dz)ds\right)^{\frac{1}{2}}$$
$$\leq \int_0^t \mathcal{K}(|x(s) - y(s)|^2)ds,$$
$$\left(\int_0^t \int_Z |h(s,x_s,z)|^4 v(dz)ds\right)^{\frac{1}{2}} \leq \int_0^t \mathcal{K}(|x(s)|^2)ds,$$

where $\mathcal{K}(.)$ is a concave nondecreasing function from $\mathcal{R}_+$ to $\mathcal{R}_+$ such that $\mathcal{K}(0) = 0, \mathcal{K}(u) > 0$ for $u > 0$ and $\displaystyle\int_{0+} \frac{du}{\mathcal{K}(u)} = +\infty$.

(H_8) For all $t \in (-\tau, 0]$, there exists a constant $L_3 > 0$ such that

$$|f(t,0)|^2 \vee |\sigma(t,0)|^2 \vee \int_Z |h(t,0,z)|^2 v(dz) \leq L_3$$

4. Existence and Uniqueness Results

Theorem 1 Assume that $f(t,0) = g(t,0) = I_k(0) = 0, \qquad \forall t \geq 0, \;\; k \in \mathbb{N}$. The assumptions $(H_1) - (H_8)$ hold and that

$$\begin{aligned}&4\Bigg(L_2\|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_2 \Gamma^2(\beta)\lambda^{-2\beta} + M^2 L_1 \lambda^{-2} + M^2\lambda^{-2}\\ &+ M^2\left(\sum_{k=1}^{\infty} q_k\right)^2\Bigg) < 1, \end{aligned} \tag{7}$$

where $\Gamma(.)$ is the Gamma function, $M_{1-\beta}$ is the corresponding constant in Lemma 2. Then the mild solution to equation 1 exists uniquely and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants $a > 0$ and $M^* = M^*(\phi, a) > 0$ such that

$$E\|x(t)\|^2 \leq M^* e^{-at}, \;\; t \geq 0. \tag{8}$$

Proof: Denote by $\mathbb{S}$ the space of all stochastic processes $x(t,\omega) : (-\tau, \infty) \times \Omega \to X$ satisfying $x(t) = \phi(t), t \in (-\tau, 0]$ and the conditions (i), (ii) in Definition 1 and there exist some constants $a > 0$ and $M^* = M^*(\phi, a) > 0$ such that

$$E\|x(t)\|^2 \leq M^* e^{-at}, \;\; t \geq 0. \tag{9}$$

Now we check that $\mathbb{S}$ is a banach space endowed with a norm $|x|^2_{\mathbb{S}} = \sup_{t\geq 0} E|x(t)|^2$. Without loss of generality, we may assume that $a < \lambda$. We define the operator Φ on $\mathbb{S}$ by

$(\Phi x)(t) = \phi(t), t \in (-\tau, 0]$ and

$$\begin{aligned}(\Phi x)(t) &= S(t)(\phi(0) + g(0,\phi)) - g(t, x_t)\\ &\quad - \int_0^t AS(t-s)g(s,x_t)ds + \int_0^t S(t-s)f(s,x_s)ds\\ &\quad + \int_0^t S(t-s)\sigma(s)dW^H(s) + \int_0^t S(t-s)\int_Z h(s,x_s,y)\tilde{N}(ds,dy)\\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)) := \sum_{i=1}^{6} P_i(t), \quad t \geq 0.\end{aligned}$$

It is enough to show that the operator Φ has a unique fixed point in $\mathbb{S}$. To prove this we use the contraction mapping principle.

Step 1: Lets check that $\Phi(\mathbb{S}) \subset \mathbb{S}$. We denote by $M_i^*, i = 1, 2, \dots$ the finite positive constants depending on ϕ, a. By the assumption (H_1) we have

$$E\|P_1(t)\|^2 \leq M^2 E\|\phi(0) + g(0,\phi)\|^2 e^{-\lambda t} \leq M_1^* e^{-\lambda t} \tag{10}$$

To analyze $P_i(t), i = 2, \dots, 6$, we found that for $x \in \mathbb{S}$ the following evaluation holds

$$\begin{aligned}E\|x_t\|^2 &\leq (M^* e^{-at} + E\|\phi_t\|^2)\\ &\leq (M^* e^{-at} + E\|\phi\|_C^2 e^{-at})\\ &\leq (M^* + E\|\phi\|_C^2)e^{-at},\end{aligned}$$

where $\|\phi\|_C = \sup_{-\tau<s\leq 0} \|\phi(s)\| < \infty$. Then by assumption (H_3) we have

$$\begin{aligned}E\|P_2(t)\|^2 &\leq \|(-A)^{-\beta}\|^2 E\|(-A)^\beta g(t,x_t) - (-A)^\beta g(t,0)\|^2\\ &\leq L_2\|(-A)^{-\beta}\|^2 E\|x_t\|^2\\ &\leq L_2\|(-A)^{-\beta}\|^2 (M^* + E\|\phi\|_C^2)e^{-at}\\ &\leq M_2^* e^{-at}.\end{aligned} \tag{11}$$

Using Lemma 2, Holder's inequality and assumption (H_3) we get that

$$\begin{aligned}E\|P_3(t)\|^2 &= E\left\|\int_0^t AS(t-s)g(s,x_s))ds\right\|^2\\ &\leq \int_0^t \|(-A)^{1-\beta}S(t-s)\|ds \int_0^t \|(-A)^{1-\beta}S(t-s)\|E\|(-A)^\beta g(s,x_s)\|^2 ds\\ &\leq M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}ds \int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}E\|x_s\|^2 ds\\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta}\int_0^t (t-s)^{\beta-1}e^{-\lambda(t-s)}(M^* + E\|\phi\|_C^2)e^{-as}ds\\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta}(M^* + E\|\phi\|_C^2)e^{-at}\int_0^t (t-s)^{\beta-1}e^{(a-\lambda)(t-s)}ds\\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma^2(\beta)}{\lambda^\beta(\lambda-a)^\beta}(M^* + E\|\phi\|_C^2)e^{-at}.\end{aligned}$$

Hence we retrieve that

$$E\|P_3(t)\|^2 \leq M_3^* e^{-at}. \tag{12}$$

we acquire by assumption (H_2) that

$$\begin{aligned}
E\|P_4(t)\|^2 &= E\left\| \int_0^t S(t-s)f(s,x_s)ds \right\|^2 \\
&\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)} E\|x_s\|^2 ds \\
&\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|_C^2)e^{-as}ds \\
&\leq M^2 L_1 \lambda^{-1}(\lambda-a)^{-1}(M^* + E\|\phi\|_C^2)e^{-at} \\
&\leq M_4^* e^{-at}. \qquad (13)
\end{aligned}$$

By using Lemma 1 we get that

$$E\|p_5(t)\|^2 \leq 2M^2 H t^{2H-1} \int_0^t e^{-2\lambda(t-s)} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds \qquad (14)$$

From this inequality we can establish that ,

$$E\|P_5(t)\|^2 \leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds, \qquad (15)$$

where $\lambda' = \lambda \wedge \gamma$. Indeed, if $\lambda < \gamma$, then $\lambda' = \lambda$ and we have

$$\begin{aligned}
E\|P_5(t)\|^2 &\leq 2M^2 H t^{2H-1} e^{-2\lambda t} \int_0^t e^{2\lambda s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds \\
&\leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds.
\end{aligned}$$

If $\gamma < \lambda$, then $\lambda' = \gamma$ and we have

$$\begin{aligned}
E\|P_5(t)\|^2 &\leq 2M^2 H t^{2H-1} e^{-2\gamma t} \int_0^t e^{-2(\lambda-\gamma)(t-s)} e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds. \\
&\leq 2M^2 H t^{2H-1} e^{-2\lambda' t} \int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathscr{L}_2^0} ds.
\end{aligned}$$

We know that $\sup_{t\geq 0}(t^{2H-1}e^{-\lambda' t}) < \infty$, and using the inequality 15, gives us

$$E\|P_5(t)\|^2 \leq M_5^* e^{-\lambda' t}. \qquad (16)$$

Using assumptions (H_7), (H_8) and Burholder's inequality, we found that

$$\begin{aligned}
E\|P_6(t)\|^2 &= E\bigg\| \int_0^t \int_Z S(t-s)h(s,x_s,z)\tilde{N}(ds,dz)\bigg\|^2 \\
&\leq c\bigg\{ \int_0^t \int_Z E\|S(t-s)h(s,x_s,z)\|^2 v(dz)ds \\
&+ E\bigg(\int_0^t \int_z \|S(t-s)h(s,x_s,z)\|^4 v(dz)ds\bigg)^{\frac{1}{2}}\bigg\} \\
&\leq c\bigg\{ \int_0^t E\|S(t-s)\|ds \int_0^t \int_z S(t-s)\|h(s,x_s,z)\|^2 v(dz)ds \\
&+ E\int_0^t \|S(t-s)\|^2 ds \bigg(\int_0^t \int_z \bigg(S(t-s))^2\|h(s,x_s,z)\|^4 v(dz)ds\bigg)^{\frac{1}{2}}\bigg\} \\
&\leq c\bigg\{ 2M^2\lambda^{-1}\bigg(\int_0^t \int_z e^{-\lambda(t-s)}\|h(s,x_s,z)-h(s,0,z)\|^2 v(dz)ds \\
&+ \int_0^t \int_z e^{-\lambda(t-s)}\|h(s,0,z)\|^2 v(dz)ds\bigg) \\
&+ M^2\lambda^{-2}\bigg(M^2\int_0^t e^{-2\lambda(t-s)}\bigg)^{\frac{1}{2}} \int_0^t \|\|x(s)\|^2 ds\bigg\} \\
&\leq c\bigg\{ 2M^2\lambda^{-1}\int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|^2)e^{-as}ds \\
&+ 2M^2L_3\lambda^{-2} + M^3\lambda^{-2}\int_0^t e^{-\lambda(t-s)}(M^* + E\|\phi\|^2)e^{-as}ds\bigg\}
\end{aligned}$$

After reckoning we found the following.

$$\begin{aligned}
E\|P_6(t)\|^2 &\leq c\bigg\{ 2M^2\lambda^{-1}(\lambda-a)^{-1}(M^* + E\|\phi\|^2)e^{-at} + 2M^2L_3\lambda^{-2}e^{-\lambda_1 t} \\
&+ M^3\lambda^{-2}\|(M^* + E\|\phi\|^2)(\lambda-a)^{-1}e^{-at}\bigg\} \leq M_6^* e^{-(a+\lambda_1)t}. \qquad (17)
\end{aligned}$$

From (H_5) and Holder's inequality, we get the following estimate for $P_6(t)$

$$\begin{aligned}
E\|P_7(t)\|^2 &= E\bigg\| \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\bigg\|^2 \\
&\leq E\bigg(\sum_{0<t_k<t} \|S(t-t_k)\|\|I_k(x(t_k)) - I_k(0)\|\bigg)^2 \\
&\leq M^2E\bigg(\sum_{0<t_k<t} e^{-\lambda(t-t_k)}q_k\|x(t_k)\|\bigg)^2 \\
&\leq M^2 \sum_{0<t_k<t} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}E\|x(t_k)\|^2 \\
&\leq M^2 \sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{-2\lambda(t-t_k)}M^* e^{-at_k} \\
&\leq M^2M^*e^{-at} \sum_{k=1}^{\infty} q_k \sum_{0<t_k<t} q_k e^{(a-2\lambda)(t-t_k)}
\end{aligned}$$

$$\leq M^2 M^* e^{-at}\left(\sum_{k=1}^{\infty} q_k\right)^2 \leq M_6^* e^{-at}. \tag{18}$$

Combining 10 - 13 and 16 - 18 we found there exist $\overline{M}^* > 0$ and $\overline{a} > 0$ such that

$$E\|(\Phi x)(t)\|^2 \leq \overline{M}^* e^{-\overline{a}t}, t \geq 0. \tag{19}$$

It is easy to check that $(\Phi x)(t)$ satisfies the conditions (i), (ii) in Definition 1. Hence, we can conclude that $\Phi(\mathbb{S}) \subset \mathbb{S}$.

Step 2 We now show that Φ is a contraction mapping. For any $x, y \in \mathbb{S}$, we have

$$E\|(\Phi x)(t) - (\Phi y)(t)\|^2 \leq 4\sum_{i=1}^{4} Q_i. \tag{20}$$

Since $x(t) = y(t) = \phi(t), t \in (-\tau, 0]$, this implies that

$$E\|x_t - y_t\|^2 \leq \sup_{t\geq 0} E\|x(t) - y(t)\|^2. \tag{21}$$

Using assumption (H_3), we get the following result.

$$\begin{aligned} Q_1 = E\|g(t,x_t) - g(t,y_t)\|^2 &\leq L_2\|(-A)^{-\beta}\|^2 E\|x_t - y_t\|^2 \\ &\leq L_2\|(-A)^{-\beta}\|^2 \sup_{t\geq 0} E\|x(t) - y(t)\|^2. \end{aligned}$$

and

$$\begin{aligned} Q_2 &= E\left\|\int_0^t AS(t-s)[g(s,x_s) - g(s,y_s)]ds\right\|^2 \\ &\leq M_{1-\beta}^2 L_2 \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} ds \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E\|x_s - y_s\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma(\beta)}{\lambda^\beta} \int_0^t (t-s)^{\beta-1} e^{-\lambda(t-s)} E\|x_s - y_s\|^2 ds \\ &\leq M_{1-\beta}^2 L_2 \frac{\Gamma^2(\beta)}{\lambda^{2\beta}} \sup_{t\geq 0} E\|x(t) - y(t)\|^2. \end{aligned}$$

By assumption (H_2)

$$\begin{aligned} Q_3 &= E\left\|\int_0^t S(t-s)[f(s,x_s) - f(s,y_s)]ds\right\|^2 \\ &\leq M^2 L_1 \int_0^t e^{-\lambda(t-s)} ds \int_0^t e^{-\lambda(t-s)} E\|x_s - y_s\|^2 ds \\ &\leq M^2 L_1 \lambda^{-1} \int_0^t e^{-\lambda(t-s)} E\|x_s - y_s\|^2 ds \leq M^2 L_1 \lambda^{-2} \sup_{t\geq 0} E\|x(t) - y(t)\|^2. \end{aligned}$$

By assumption (H_7)

$$\begin{aligned} Q_4 &= E\Big\|\int_0^t\int_Z S(t-s)[h(s,x_s,z)-h(s,y_s,z)]\tilde{N}(ds,dz)\Big\|^2 \\ &\leq \int_0^t S(t-s)ds\int_0^t\int_z S(t-s)E\|h(s,x_s,z)-h(s,y_s,z)\|^2\tilde{N}(ds,dz) \\ &\leq M^2\int_0^t e^{-\lambda(t-s)}\int_0^t e^{-\lambda(t-s)}\|x_s-y_s\|^2 ds \\ &\leq M^2\lambda^{-1}\int_0^t e^{-\lambda(t-s)}\|x_s-y_s\|^2 ds \leq M^2\lambda^{-2}\sup_{t\geq 0}E\|x(t)-y(t)\|^2. \end{aligned}$$

By assumption (H_5)

$$\begin{aligned} Q_5 &= E\Big\|\sum_{0<t_k<t} s(t-t_k)[I_k(x(t_k))-I_k(y(t_k))]\Big\|^2 \\ &\leq M^2\Big(\sum_{0<t_k<t} e^{-\lambda(t-t_k)}q_k E\|x(t_k)-y(t_k)\|\Big)^2 \\ &\leq M^2\Big(\sum_{k=1}^{\infty} q_k\Big)^2\sup_{t\geq 0}E\|x(t)-y(t)\|^2. \end{aligned}$$

Thus

$$\begin{aligned} E\|(\Phi x)(t)-(\Phi y)(t)\|^2 \leq 4\Big(& L_2\|(-A)^{-\beta}\|^2 + M_{1-\beta}^2 L_2\Gamma^2(\beta)\lambda^{-2\beta} \\ &+ M^2L_1\lambda^{-2} + M^2\lambda^{-2} + M^2\Big(\sum_{k=1}^{\infty} q_k\Big)^2\Big)\sup_{t\geq 0}E\|x(t)-y(t)\|^2. \end{aligned}$$

By the condition (7), we claim that Φ is contractive. So, applying the Banach fixed point principle, the proof is complete.

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