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EXISTENCE, GLOBAL ATTRACTING SETS AND EXPONENTIAL DECAY OF SOLUTION TO STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS DRIVEN BY ROSENBLATT PROCESS

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Abstract. In this paper, we investigate a class of neutral stochastic functional integro-differential equations driven by Rosenblatt process in a Hilbert space. First, the existence and uniqueness of mild solution of an stochastic system driven by Rosenblatt process is established by combining some stochastic analysis techniques, resolvent operator theory, and stochastic integral inequalities. Further, the exponential decay in the p-th moment of the mild solution of the considered equations is investigated and the global attracting sets are identified. Finally, to illustrate our theoretical results an example is given.

1. Introduction

Recently, there has been increasing interest in the study of the existence, uniqueness and stability of mild solutions of stochastic partial functional differential equations due to their range of applications in various sciences such as physics, mechanical, engineering, control theory and economics, and many significant results have been obtained, see, for example [6, 7]. However, to the best of our knowledge, there exist only a few articles which dealt with the existence of attracting and invariant sets of stochastic partial functional differential equations. For deterministic differential systems with or without delays, see [4, 13, 17, 31]. For partial differential systems, see [29] and stochastic or random systems, see [10, 18].

Defined during the 60s and 70s (see [23, 26]) due to their appearance in the Non-Central Limit Theorem, the systematic analysis of Rosenblatt processes has only been developed during the last ten years, motivated by their nice properties.

We point out that the Rosenblatt have similar properties as fractional Brownian motion, such as self-similarity, memory property with $H > 1/2$ and stationary increments. The large utilization of the fractional Brownian motion in practice is due to its self-similarity, stationarity of increments and long-range dependence;

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one prefers in general fBm than other processes because it is Gaussian and the calculus for it is easier; but in concrete situations when the processes of model is not Gaussian, one can use for example the Rosenblatt process.

The fractional Brownian motion (fBm) basic properties have been studied by Tudor in [28]. During the last decades, there has been a considerable interest in the study of stochastic calculus with respect to fBm. We refer the reader to two monographs [5, 21] and the references therein for a more complete presentation of fBm. Note that a fractional Brownian motion (fBm) of Hurst index $H \in (0,1)$ is a centered Gaussian process $B^H = \left\{B^H(t), t \geq 0\right\}$ with the covariance function

$$R_H(t,s) = \mathbf{E}\left(B^H(t)B^H(s)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The fBm corresponds to the standard Brownian motion when $H = 1/2$, and the fBm B^H neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm $B^H, H > 1/2$, is a long-memory process and presents an aggregation behavior. Moreover, fBm belongs to the family of Hermite processes which are obtained by using non-central limit theorems established by Dobrushin and Major in [9]. They admit the following representation, for all $d \geq 1$:

$$Y_t^{H,d} = c(H_0)\int_{\mathbb{R}} ... \int_{\mathbb{R}} \left(\int_0^t \prod_{j=1}^{d} (s-x_j)_+^{H_0-1}\, ds\right) dBx_1 ... dBx_d, \quad \forall t > 0,$$

where $\{B_{x_i} : x_i \in \mathbb{R}, i = 1, ..., d\}$ are some two-sided Brownian motions, $c(H_0)$ is a normalizing constant such that

$$\mathbf{E}|Y_1^{H,d}|^2 = 1, \quad H_0 = \frac{1}{2} + \frac{H-1}{d}, \quad H \in (\frac{1}{2}, 1)$$

The above integral is a multiple Wiener-Itô stochastic integral with respect to a Brownian motion $(B_{x_i})_{x_i \in \mathbb{R}}$. The process $Z_t^{H,d}, t > 0$ is called the Hermite process and it is H-selfsimilar in the sense that for any $c > 0, Z_{ct}^{H,d} = c^H Z_t^{H,d}$ where "$=^{(d)}$" means equivalence of all finite dimensional distributions, and it has stationary increments. when $d = 1$ the process is nothing else that the fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $d \geq 2$ the process is not Gaussian. If $d = 2$ then the process is known as the Rosenblatt process (it has actually called in this way in [25].

On the other hand, some authors have studied attracting sets of stochastic dynamical systems. Among others, we would like to mention that Xu et al. [30] investigated the attracting sets of nonautonomous neural networks with delays. Zhao et al. [32] determined the attracting sets for bidirectional associated memory neural networks of neutral-type with time-varying and infinite distributed delays. Therefore, it is very interesting to investigate this problem driven by Rosenblatt process.

Therefore, inspired by the previous authors, we will use the resolvent operator and some analysis inequalities to estimate the mild solution of our equation and investigate the global attracting set and p-exponential stability. The results obtained extend, improve and complement many other important works in the field. In this article, we will study the following stochastic functional integro-differential equations driven by Rosenblatt with delay:

$$\begin{cases} d\big[\vartheta(t)+\mathrm{h}(t,\vartheta(t-u(t))\big] = \Big[\mathrm{A}\big[\vartheta(t)+\mathrm{h}(t,\vartheta(t-u(t))\big] + \int_0^t \Upsilon(t-s)\big[\vartheta(s) \\ \quad +\mathrm{h}(s,\vartheta(s-u(s))\big] + \mathrm{f}(t,\vartheta(t-\rho(t))\Big]dt + \tilde{\beta}(t)dZ_Q^H(t), \quad t\in[0,T] \\ \vartheta(t)=\varphi(\cdot)\in C([-r,0];\mathbf{X}), \quad t\in[-r,0] \end{cases} \tag{1}$$

where $\mathrm{A} : D(\mathrm{A}) \subset \mathbf{X} \to \mathbf{X}$ is infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ in a Hilbert space $\mathbf{X}$ with inner product $\langle .,. \rangle$ and norm $\|.\|$, $Z_Q^H(t)$ is a Rosenblatt process with parameter $H \in (1/2, 1)$ in a real and separable Hilbert space $(\mathbf{Y}, \|.\|_{\mathbf{Y}}, \langle .,. \rangle_{\mathbf{Y}})$, $r > 0$ and $\Upsilon(t)$ a closed linear operator on $\mathbf{X}$ with $D(\mathrm{A}) \subset D(\Upsilon)$ which is independent of t, $t \geq 0$. $C([-r, 0]; \mathbf{X})$ stand for space of all continuous $\mathbf{X}$-valued functions ξ from $[-r, 0]$ to $\mathbf{X}$ with the norm $\|\xi\|_C = \sup_{-r\leq t\leq 0} \|\xi(t)\|$, $u, \rho : [0, +\infty) \longrightarrow [0, r](r > 0)$ are continuous, $\mathrm{h}, \mathrm{f} : [0, +\infty) \times \mathbf{X} \longrightarrow \mathbf{X}$ are two given measurable mappings, and $\tilde{\beta} : [0, T] \longrightarrow L_Q^0(\mathbf{Y}, \mathbf{X})$ is a given function to be specified later.

Initially, the goal of this work is to investigate the existence of mild solution to Eq.(1) by using the resolvent operator, some analysis methods and integral inequalities. Therefore, the second goal is achieved by establishing sufficient conditions to ensure the exponential decay in p-th moment. Further, the global attracting sets of our mild solution is obtained. Our paper expands the usefulness of stochastic integro-differential equations, since the literature shows results for existence and exponential decay for such equations when $\Upsilon(t) = 0$ under semigroup theory.

The rest of this paper is organized as follows: In Section 2, we recall briefly the notations, concepts, and basic results about the Rosenblatt process, deterministic integro-differential equations which are used throughout this paper, as well as the definition of the mild solution for stochastic system(1) is also given. In Section 3, the existence of mild solution is investigated. Thus, the conditions to ensure the global attracting sets and the exponential decay in p-th moment of mild solution of Eq.(1) are established. In the last section, an application is given to illustrate our theoretical results.

2. Preliminaries

In this section, we provide some preliminaries needed to establish our main results. Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and for $t \geq 0$, $\mathcal{F}_t$ denote the σ-field generated by $\{Z_H(s), s \in [0, t]\}$ and the $\mathbb{P}$-null sets. The notation $L^2(\Omega, \mathbf{X})$ stands for the space of all $\mathbf{X}$-valued random variables x such that $\mathbf{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty$. For $x \in L^2(\Omega, \mathbf{X})$, let $\|x\|_2 = \big(\int_\Omega \|x\|^2 d\mathbb{P}\big)^{\frac{1}{2}}$. It is easy to check that $L^2(\Omega, \mathbf{X})$ is a Hilbert space equipped with the norm $\|.\|_2$. Let $L(\mathbf{Y}, \mathbf{X})$ denotes the space of all bounded linear operators from $\mathbf{Y}$ to $\mathbf{X}$, we abbreviate this notation to $L(\mathbf{Y})$ whenever $\mathbf{Y} = \mathbf{X}$ and $Q \in L(\mathbf{Y})$ represents a non-negative self-adjoint operator.

Let $\mathbf{Y}$ be a separable Hilbert space and $L_2^0 = L^2(\mathbf{Y}, \mathbf{X})$ be a separable Hilbert space with respect to the Hilbert-Schmidt norm $\|.\|_{L_2^0}$. Let $L_Q^0(\mathbf{Y}, \mathbf{X})$ be the space of all $\psi \in L(\mathbf{Y}, \mathbf{X})$ such that $\psi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by $\|\psi\|^2_{L_Q^0} = \left\|\psi Q^{\frac{1}{2}}\right\| = tr(\psi Q\psi^*)$. Then ψ is called a Q-Hilbert-Schmidt operator from $\mathbf{Y}$ to $\mathbf{X}$. In the sequel, $L_2^0(\Omega, \mathbf{X})$ denotes the space of $\mathcal{F}_0$-measurable, $\mathbf{X}$-valued and square integrable stochastic processes.

2.1. **Rosenblatt process.** In this subsection, we recall some basic knowledge on the Rosenblatt process as well as the Wiener integral with respect to it.
Consider $(\xi_n)_{n\in Z}$ a stationary Gaussian sequence with mean zero and variance 1 such that its correlation function satisfies that $R(n) := \mathbb{E}(\xi_0\xi_n) = n^{\frac{2H-2}{k}}L(n)$, with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Let g be a function of Hermite rank k, that is, if g admits the following expansion in Hermite polynomials

$$g(x) = \sum_{j\geq 0} c_j H_j(x), \quad c_j = \frac{1}{j!}\mathbb{E}(g(\xi_0 H_j(\xi_0))),$$

then $k = \min\{j|c_j \neq 0\} \geq 1$, where $H_j(x)$ is the Hermite polynomial of degree j given by $H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$. Then, the Non-Central Limit Theorem (see, for example, [9]) says $\frac{1}{n^H}\sum_{j=1}^{[nt]} g(\xi_j)$ converges as $n \to \infty$, in the sense of finite dimensional distributions, to the process

$$Z_H^k(t) = c(H,k)\int_{\mathbb{R}^k}\int_0^t \left(\prod_{j=1}^k (s-y_j)_+^{\left(-\frac{1}{2}+\frac{1-H}{k}\right)}\right) ds dB(y_1)\cdots dB(y_k), \qquad (2)$$

where the above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ and $c(H,k)$ is a positive normalization constant depending only on H and k. The process $(Z_H^k(t))_{t\geq 0}$ is called as the Hermite process and it is H self-similar in the sense that for any $c > 0$, $(Z_H^k(ct)) \stackrel{d}{=} (c^H Z_H^k(t))$ and it has stationary increments.

The most studied Hermite process is of course the fractional Brownian motion (which is obtained in (2) for $k = 1$) due to its large range of applications. When $k = 2$, the process given by (2) is known as the Rosenblatt process (it has actually been named in this way in [25]). This process is of interest in practical applications because of its self-similarity, stationarity of increments and long range dependence (see [27]). However, it is not Gaussian. Actually the very large utilization of the fractional Brownian motion in practice (hydrology, telecommunications) are due to these properties (self-similarity, long range dependence). One prefers in general fractional Brownian motion before other processes because it is Gaussian and the calculus for it is easier. However in concrete situations when the Gaussianity is not plausible for the model, one can use for example the Rosenblatt process. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt process. For example, extremal properties of the Rosenblatt distribution have been studied by author in [2]. The rate of convergence to the Rosenblatt process in the Non-Central Limit Theorem has been given in [14]. [1] gave the wavelet-type expansion of the Rosenblatt process. Author in [28] established its representation as a Wiener-Itô multiple integral with respect to the Brownian motion on a finite interval and developed the stochastic calculus with respect to it by using both pathwise type calculus and Malliavin calculus (see also [19]). For more details for Rosenblatt process, we refer the reader to [20, 21], [22] and the references therein.

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0,T]\}$ be a one-dimensional Rosenblatt process with parameter $H \in (\frac{1}{2}, 1)$. The Rosenblatt process with parameter (see Tudor in [28]) $H > \frac{1}{2}$ can be written as

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \quad (3)$$

where $K^H(t,s)$ is given by

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du \text{ for } t > s,$$

with

$$c_H = \sqrt{\frac{H(2H-1)}{\Gamma(2-2H, H-\frac{1}{2})}},$$

$\Gamma(.,.)$ represents the Beta function, $K^H(t,s) = 0$ when $t \leq s$, $(B(t), t \in [0,T])$ is a Brownian motion, $H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{Z_H(t), t \in [0,T]\}$ satisfies that $\mathbb{E}(Z_H(t) Z_H(s)) = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)$.
The covariance structure of the Rosenblatt process allows to construct Wiener integral with respect to it. We refer to [19] for the definition of Wiener integral with respect to general Hermite processes and to [12] for a more general context (see also [28]).

One note that $Z_H(t) = \int_0^T \int_0^T I(1_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2)$, where the operator I is defined on the set of functions $\tilde{\mathrm{f}} : [0,T] \to \mathbb{R}$, which takes its values in the set of functions $g : [0,T]^2 \to \mathbb{R}^2$ and is given by

$$I(\tilde{\mathrm{f}})(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T \tilde{\mathrm{f}}(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Let $\tilde{\mathrm{f}}$ be an element of the set $\mathcal{E}$ of step functions on $[0,T]$ of the form

$$\tilde{\mathrm{f}} = \sum_{i=0}^{n-1} a_i 1_{(t_i, t_{i+1}]}, \ t_i \in [0,T].$$

Then, it is natural to define its Wiener integral with respect to Z_H as

$$\int_0^T \tilde{\mathrm{f}}(u) dZ_H(u) := \sum_{i=0}^{n-1} a_i (Z_H(t_{i+1}) - Z_H(t_i)) = \int_0^T \int_0^T I(\tilde{\mathrm{f}})(y_1, y_2) dB(y_1) dB(y_2).$$

Let $\mathcal{H}$ be the set of functions $\tilde{\mathrm{f}}$ such that

$$\|\tilde{\mathrm{f}}\|_{\mathcal{H}}^2 := 2 \int_0^T \int_0^T (I(\tilde{\mathrm{f}})(y_1, y_2))^2 dy_1 dy_2 < \infty.$$

It follows that (see [28])
$\|\tilde{\mathrm{f}}\|_{\mathcal{H}}^2 = H(2H-1) \int_0^T \int_0^T \tilde{\mathrm{f}}(u) \tilde{\mathrm{f}}(v) |u-v|^{2H-2} du dv.$
It has been proved in [19] that the mapping

$$\tilde{\mathrm{f}} \to \int_0^T \tilde{\mathrm{f}}(u) dZ_H(u)$$

defines an isometry from $\mathcal{E}$ to $L^2(\Omega)$ and it can be extended continuously to an isometry from $\mathcal{H}$ to $L^2(\Omega)$ because $\mathcal{E}$ is dense in $\mathcal{H}$. We call this extension as the Wiener integral of $\tilde{\mathrm{f}} \in \mathcal{H}$ with respect to Z_H. Notice that the space $\mathcal{H}$ contains not only functions but its elements should be also distributions. Therefore it is suitable to know subspaces $|\mathcal{H}|$ of $\mathcal{H}$: $|\mathcal{H}| = \left\{\tilde{\mathrm{f}} : [0,T] \to \mathbb{R} | \int_0^T \int_0^T |\tilde{\mathrm{f}}(u)||\tilde{\mathrm{f}}(v)||u-v|^{2H-2} dudv < \infty\right\}$. The space $|\mathcal{H}|$ is not complete with respect to the norm $\|.\|_{\mathcal{H}}$ but it is a Banach space with respect to the norm

$$\|\tilde{\mathrm{f}}\|^2_{|H|} = H(2H-1)\int_0^T \int_0^T |\tilde{\mathrm{f}}(u)||\tilde{\mathrm{f}}(v)||u-v|^{2H-2} dudv.$$

As a consequence, we have

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

For any $\tilde{\mathrm{f}} \in L^2([0,T])$, we have

$$\|\tilde{\mathrm{f}}\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\tilde{\mathrm{f}}(s)|^2 ds$$

and

$$\|\tilde{\mathrm{f}}\|^2_{|\mathcal{H}|} \leq C(H)\|\tilde{\mathrm{f}}\|^2_{L^{1/H}([0,T])}, \tag{4}$$

for some constant $C(H) > 0$. For simplicity throughout this paper we let $C(H) > 0$ stand for a positive constant depending only on $\mathcal{H}$ and its value may be different in different appearances.

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^*\tilde{\mathrm{f}})(y_1,y_2) = \int_{y1\vee y_2}^T \tilde{\mathrm{f}}(t)\frac{\partial \mathcal{K}}{\partial t}(t,y_1,y_2)dt,$$

where $\mathcal{K}$ is the kernel of Rosenblatt process in representation (3)

$$\mathcal{K}(t,y_1,y_2) = 1_{[0,t]}(y_1)1_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

Notice that $(K_H^* 1_{[0,t]})(y_1,y_2) = \mathcal{K}(t,y_1,y_2)1_{[0,t]}(y_1)1_{[0,t]}(y_2)$. The operator K_H^* is an isometry between $\mathcal{E}$ to $L^2([0,T])$, which could be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\begin{aligned}
\left\langle K_H^* 1_{[0,t]}, K_H^* 1_{[0,s]}\right\rangle_{L^2([0,T])} &= \left\langle \mathcal{K}(t,.,.)1_{[0,t]}, \mathcal{K}(s,.,.)1_{[0,s]}\right\rangle_{L^2([0,T])} \\
&= \int_0^{t\wedge s}\int_0^{t\wedge s} \mathcal{K}(t,y_1,y_2)\mathcal{K}(s,y_1,y_2)dy_1dy_2 \\
&= H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}dudv \\
&= \left\langle 1_{[0,t]}, 1_{[0,s]}\right\rangle_{\mathcal{H}}.
\end{aligned}$$

Moreover, for $\tilde{\mathrm{f}} \in \mathcal{H}$, we have

$$Z_H(\tilde{\mathrm{f}}) = \int_0^T\int_0^T (K_H^*\tilde{\mathrm{f}})(y_1,y_2)dB(y_1)dB(y_2).$$

Let $\{Z_n(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. We consider a $\mathbf{Y}$-valued stochastic process

$Z_Q^H(t)$ given by the following series:

$$Z_Q^H(t) = \sum_{n=1}^{\infty} z_n(t) Q^{1/2} e_n, \quad t \geq 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space $\mathbf{Y}$, that is, it holds that $Z_Q^H(t) \in L^2(\Omega, \mathbf{Y})$. Then, we say that the above $Z_Q^H(t)$ is a $\mathbf{Y}$-valued Q- Rosenblatt process with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in $\mathbf{Y}$, then the stochastic process

$$Z_Q^H(t) = \sum_{n=1}^{\infty} z_n(t) Q^{1/2} e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n} z_n(t) e_n, \quad t \geq 0,$$

is well-defined as a $\mathbf{Y}$-valued Q- Rosenblatt process.

Definition 1 ([28]). Let $\varphi : [0,T] \to L_Q^0(\mathbf{Y}, \mathbf{X})$ such that $\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{1/2} e_n)\|_{L^2([0,T];\mathcal{H})} < \infty$. Then, its stochastic integral with respect to the Rosenblatt process $Z_Q^H(t)$ is defined, for $t \geq 0$, as follows :

$$\begin{aligned} \int_0^t \varphi(s) dZ_Q^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{1/2} e_n dz_n(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \int_0^t (K_H^*(\varphi Q^{1/2} e_n))(y_1, y_2) dB(y_1) dB(y_2). \end{aligned} \tag{5}$$

Lemma 1 For $\psi : [0,T] \to L_Q^0(\mathbf{Y}, \mathbf{X})$ such that $\sum_{n=1}^{\infty} \|\psi Q^{1/2} e_n\|_{L^{1/H}([0,T];\mathbf{X})} < \infty$ holds, and for any $a, b \in [0,T]$ with $b > a$, we have

$$\mathbf{E} \left\| \int_a^b \psi(s) dZ_Q(s) \right\|^2 \leq c(H)(b-a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \|\psi(s) Q^{1/2} e_n\|^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\psi(t) Q^{1/2} e_n\| \text{ is uniformly convergent for } t \in [0,T],$$

then, it holds that

$$\mathbf{E} \left\| \int_a^b \psi(s) dZ_Q^H(s) \right\|^2 = C(H)(b-a)^{2H-1} \int_a^b \|\psi(s)\|^2_{L_Q^0(K,X)} ds.$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthonormal basis of K introduced above.

Applying (3) and Hölder inequality, we have

$$\begin{aligned}
&\mathbf{E}\left\|\int_a^b \psi(s)dZ_Q^H(s)\right\|^2 \\
&= \mathbf{E}\left\|\sum_{n=1}^{\infty}\int_a^b \psi(s)Q^{1/2}e_n dz_n(s)\right\|^2 \\
&= \sum_{n=1}^{\infty}\mathbf{E}\left\|\int_a^b \psi(s)Q^{1/2}e_n dz_n(s)\right\|^2 \\
&= \sum_{n=1}^{\infty} H(2H-1)\int_a^b\int_a^b \|\psi(s)Q^{1/2}e_n\|\|\psi(t)Q^{1/2}e_n\||t-s|^{2H-2}dsdt \\
&\leq C(H)\sum_{n=1}^{\infty}\left(\int_a^b \|\psi(s)Q^{1/2}e_n\|^{1/H}ds\right)^{2H} \\
&\leq C(H)(b-a)^{2H-1}\sum_{n=1}^{\infty}\int_a^b \|\psi(s)Q^{1/2}e_n\|^2 ds.
\end{aligned}$$

2.2. **Partial integro-differential equations in Banach spaces.** In this section, we recall some fundamental results needed to establish our results. Regarding the theory of resolvent operators we refer the reader to [11]. Throughout the paper, $\mathbf{X}$ is a Banach space, A and $\Upsilon(t)$ are closed linear operators on $\mathbf{X}$. $\mathbf{Y}$ represents the Banach space $D(\mathrm{A})$ equipped with the graph norm defined by

$$|y|_{\mathbf{Y}} := |\mathrm{A}y| + |y| \quad \text{for } \ y \in \mathbf{Y}.$$

The notations $C([0,+\infty);\mathbf{Y}), \mathcal{B}(\mathbf{Y},\mathbf{X})$ stand for the space of all continuous functions from $[0,+\infty)$ into $\mathbf{Y}$, the set of all bounded linear operators from $\mathbf{Y}$ into $\mathbf{X}$, respectively. We consider the following Cauchy problem

$$\begin{cases} v'(t) &= \mathrm{A}v(t) + \displaystyle\int_0^t \Upsilon(t-s)v(s)ds, \quad \text{for} \quad t \geq 0, \\ v(0) &= v_0 \in \mathbf{X}. \end{cases} \tag{6}$$

Definition 2 ([11]). A resolvent operator for Eq.(6) is a bounded linear operator valued function $\mathrm{R}(t) \in L(\mathbf{X})$ for $t \geq 0$, satisfying the following properties:

(i) $\mathrm{R}(0) = I$ and $|\mathrm{R}(t)| \leq Ne^{\beta t}$ for some constants N and β.
(ii) For each $x \in \mathbf{X}$, $\mathrm{R}(t)x$ is strongly continuous for $t \geq 0$.
(iii) $\mathrm{R}(t) \in \mathcal{L}(\mathbf{Y})$ for $t \geq 0$. For $y \in \mathbf{Y}$, $\mathrm{R}(\cdot)y \in C^1([0,+\infty);\mathbf{X}) \cap C([0,+\infty);\mathbf{Y})$ and

$$\begin{aligned}
\mathrm{R}'(t)x &= \mathrm{AR}(t)x + \int_0^t \Upsilon(t-s)\mathrm{R}(s)xds \\
&= \mathrm{R}(t)Ax + \int_0^t \mathrm{R}(t-s)\Upsilon(s)xds \quad \text{for } t \geq 0.
\end{aligned}$$

The resolvent operators play an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (6) has a resolvent operator. For more details on resolvent operators, we refer the reader to [11]. The following theorem gives a satisfactory answer to this problem and it will be used in this work to develop our main results.

In what follows we suppose the following assumptions :

(H1) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbf{X}$.

(H2) For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from $D(\mathrm{A})$ to $\mathbf{X}$, and $\Upsilon(t) \in \mathcal{B}(\mathbf{Y}, \mathbf{X})$. For any $y \in \mathbf{Y}$, the map $t \to \Upsilon(t)y$ is bounded, differentiable and the derivative $t \to \Upsilon'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 1 ([11, Theorem 3.7]). Assume that **(H1)-(H2)** hold. Then there exists a unique resolvent operator for the Cauchy problem (6).

2.3. **Definitions of mild solution of Eq.**(1) **and assumptions.** Now, we present the definition of the mild solution for system (1).

Definition 3 An $\mathbf{X}$-valued process $\{\vartheta(t),\ t \in [-\tau, T]\}$, is called a mild solution of Eq.(1) if

(1) $\vartheta(t) \in \mathcal{C}([-\tau, T], L^2(\Omega, \mathbf{X}))$,
(2) $\vartheta(t) = \varphi(t)$ for $t \in [-\tau, 0]$,
(3) for each $t \in [0, T]$, $\vartheta(t)$ satisfies the following integral equation

$$\begin{aligned}\vartheta(t) &= \mathrm{R}(t)\left[\varphi(0) - \mathrm{h}(0, \varphi(-r(0)))\right] - \mathrm{h}(t, \vartheta(t - r(t))) \\ &\quad + \int_0^t \mathrm{R}(t-s)\mathrm{f}(s, \vartheta(s - \delta(s)))ds \\ &\quad + \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s) \quad \mathbb{P}-\text{a.s.}\end{aligned} \tag{7}$$

In the sequel, we will work under the following assumptions.

(H3) The resolvent operator $(\mathrm{R}(t))_{t\geq 0}$ satisfies the further condition: There exist a constant $\tilde{N} > 0$ and a real number $\mu > 0$ such that $\|\mathrm{R}(t)\| \leq \tilde{N}e^{-\mu t}, \forall t \geq 0$. In other words the resolvent operator $(\mathrm{R}(t))_{t\geq 0}$ is exponentially stable.

(H4) The function f $: [0, +\infty[\times \mathbf{X} \to \mathbf{X}$ satisfies the following conditions: there exist positive constant $L_{\mathrm{f}} > 0$ such that, for all $t \in [0, T]$ and $y_1, y_2 \in \mathbf{X}$

$$\|\mathrm{f}(t, y_1) - \mathrm{f}(t, y_2)\| \leq L_{\mathrm{f}} \|y_1 - y_2\|,\ \|\mathrm{f}(t, y_1)\|^2 \leq L_{\mathrm{f}}(1 + \|y_1\|^2)$$

(H5) The function h $: [0, +\infty[\times \mathbf{X} \to \mathbf{X}$ satisfies the following conditions:

(i) There exist positive constant $L_{\mathrm{h}} > 0$ such that, for all $t \in [0, T]$ and $y_1, y_2 \in \mathbf{X}$

$$\|\mathrm{h}(t, y_1) - \mathrm{h}(t, y_2)\| \leq L_{\mathrm{h}} \|y_1 - y_2\|,\ \mathrm{g}(t, 0) = 0$$

(ii) For all $\nu \in \mathcal{C}([0, T], L^2(\Omega, \mathbf{X}))$, $\lim_{t\to s} \mathbf{E}\|\mathrm{h}(t, \nu(t)) - \mathrm{h}(s, \nu(s))\|^2 = 0$.

(H6) The function $\tilde{\beta}\ :\ [0, +\infty[\to L_Q^0(\mathbf{Y}, \mathbf{X})$ satisfies that

$$\int_0^t \left\|\tilde{\beta}(s)\right\|_{L_Q^0}^2 ds < \infty,\ \ \forall\, t > 0.$$

3. Main results

This section is devoted to our mains results. Firstly we show the existence and uniqueness of mild solution to system (1). Secondly we identify the global attracting sets and finally we provide the exponential decay in p-th moment of the mild solution of (1).

3.1. **Existence and uniqueness of the mild solution.**

We have:

Theorem 2 Suppose that for every, $\varphi \in \mathcal{C}([-\tau, 0], L^2(\Omega, \mathbf{X}))$ $(T > 0)$ assumptions **(H1)-(H7)** and the following inequality hold

$$2L_{\mathrm{h}}^2 + 2TM_T^2 L_{\mathrm{f}}^2 < 1 \tag{8}$$

where $M_T = \sup_{[-r,T]} \|R(t)\|$.
Then the Eq.(1) has unique mild solution ϑ.

Proof. Assume that $T > 0$ is a fixed time and let $C_T := \mathcal{C}([-\tau, T], L^2(\Omega, \mathbf{X}))$ be the Banach space of all continuous functions from $[-\tau, T]$ into $L^2(\Omega, \mathbf{X})$ equipped with the supremum norm $\|\zeta\|_{C_T} = \sup_{z\in[-\tau,T]}(\mathbf{E}\,\|\zeta(z)\|^2)^{1/2}$ and let us consider the set

$$S_T(\varphi) := \{\vartheta \in \mathcal{C}([-\tau, T], L^2(\Omega, \mathbf{X})) : \vartheta(s) = \varphi(s), \text{ for } \quad s \in [-\tau, 0]\}.$$

$S_T(\varphi)$ is a closed subset of C_T provided with the norm $\|.\|_{C_T}$.
Define the operator Ψ on $S_T(\varphi)$ by $\Psi(\vartheta)(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and for $t \in [0, T]$ by

$$\begin{aligned} \Psi(\vartheta)(t) \quad &= \quad \mathrm{R}(t)\,[\varphi(0) - \mathrm{h}(0, \varphi(-r(0)))] - \mathrm{h}(t, \vartheta(t - r(t))) \\ &\quad + \int_0^t \mathrm{R}(t-s)\mathrm{f}(s, \vartheta(s - \delta(s)))ds + \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s). \end{aligned} \tag{9}$$

Then it is clear that, proving the existence of mild solutions to Eq. (1) is equivalent to find a fixed point for the operator Ψ.
Next we will show, by using the Banach fixed point theorem that Ψ has a fixed point. We split the proof into two steps.

Step 1: For arbitrary $\vartheta \in S_T(\varphi)$, let us prove that $t \to \Psi(\vartheta)(t)$ is continuous on the intervall $[0, T]$ in the $L^2(\Omega, \mathbf{X})$-sense. Let $0 < t < T$ and $|\ell|$ be sufficiently small. Then, for any fixed $\vartheta \in S_T(\varphi)$, we have

$$
\begin{aligned}
&\mathbf{E}\left\|(\Psi(\vartheta)(t+\ell)-(\Psi(\vartheta)(t)\right\|^2\\
&\leq 6\Big\{\mathbf{E}\left\|(\mathrm{R}(t+\ell)-\mathrm{R}(t))\left[\varphi(0)-\mathrm{h}(0,\varphi(-r(0)))\right]\right\|^2\\
&+\mathbf{E}\left\|\mathrm{h}(t+\ell,\vartheta(t+\ell-r(t+h)))-\mathrm{h}(t,\vartheta(t-r(t)))\right\|^2\\
&+\mathbf{E}\left\|\int_0^t\left[\mathrm{R}(t+\ell-s)-\mathrm{R}(t-s)\right]\mathrm{f}(s,\vartheta(s-\delta(s)))ds\right\|^2\\
&+\mathbf{E}\left\|\int_t^{t+\ell}\mathrm{R}(t+\ell-s)\mathrm{f}(s,\vartheta(s-\delta(s)))ds\right\|\\
&+\mathbf{E}\left\|\int_0^t\left[\mathrm{R}(t+\ell-s)-\mathrm{R}(t-s)\right]\tilde{\beta}(s)dZ_Q^H(s)\right\|^2\\
&+\mathbf{E}\left\|\int_t^{t+\ell}\mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\right\|^2\Big\}\\
&:=6\sum_{1=1}^{6}\mathfrak{I}_i(\ell).
\end{aligned}
$$

From the strong continuity of the resolvent operator $\mathrm{R}(t)$, $t \geq 0$, we can easily obtain that $\mathfrak{I}_1(\ell)=0$ as $\ell \to 0$.
From assumption **(H5)**-(ii) , we conclude

$$\lim_{\ell\to 0}\mathfrak{I}_2(\ell)=0.$$

For $\mathfrak{I}_3$, by **(H4)** and the property of Lebesgue integral, we obtain

$$\lim_{\ell\to 0}\mathfrak{I}_3(\ell)=0.$$

For $\mathfrak{I}_4$, we can get by direct calculations that

$$
\begin{aligned}
\mathfrak{I}_4(\ell) &\leq \ell N^2 e^{2\beta T}\mathbf{E}\int_0^T(\|\mathrm{f}(s,\vartheta(s-\delta(s)))\|^2)ds\\
&\leq \ell N^2 e^{2\beta T}\int_0^T L_{\mathrm{f}}(1+\|\vartheta\|^2)ds \to 0 \text{ as } \ell \to 0.
\end{aligned}
$$

Easily, we have by Lemma 1,

$$\mathfrak{I}_5(\ell) \leq 2C(H)HT^{2H-1}\int_0^t\left\|(\mathrm{R}(t+\ell-s)-\mathrm{R}(t-s))\tilde{\beta}(s)\right\|^2_{L_Q^0}.$$

Since, by Definition 2-(i)

$$\lim_{\ell\to 0}(\mathrm{R}(t+\ell-s)-\mathrm{R}(t-s))\tilde{\beta}(s)=0$$

and

$$\left\|(\mathrm{R}(t+\ell-s)-\mathrm{R}(t-s))\tilde{\beta}(s)\right\|_{L_Q^0} \leq 2N^2[e^{2\beta(t+\ell)}+e^{2\beta t}]\left\|\tilde{\beta}(s)\right\|^2_{L_Q^0},$$

thank's to Lebesgue majorant Theorem, we obtain

$$\lim_{\ell\to 0}\mathfrak{I}_5(\ell)=0.$$

Using similar arguments to $\mathfrak{I}_4(\ell)$ combined with Lemma 1, we obtain

$$\lim_{\ell\to 0}\mathfrak{I}_6(\ell)=0.$$

As a result, $\lim_{\ell\to 0}\mathbf{E}\,\|\Psi(\vartheta)(t+\ell)-\Psi(\vartheta)(t))\|^2=0$. Hence, we conclude that the function $t\to\Psi(\vartheta)(t)$ is continuous on $[0,T]$ in the L^2-sense.

Step 2: Next, we show that the operator $\Psi: S_T(\varphi)\to S_T(\varphi)$ is a contracting mapping by using some usual inequalities.
For every $\nu,\ \vartheta\in S_T(\varphi)$ and $t\in[0,T]$, we have

$$\begin{aligned}
&\mathbf{E}\,\|(\Psi\vartheta)(t)-(\Psi\nu)(t)\|^2\\
&\le 2\mathbf{E}\,\|\mathrm{h}(t,\vartheta(t-r(t)))-\mathrm{h}(t,\nu(t-r(t)))\|^2\\
&\quad+2\mathbf{E}\left\|\int_0^t \mathrm{R}(t-s)(\mathrm{f}(s,\vartheta(s-\delta(s)))-\mathrm{f}(s,\nu(s-\delta(s))))ds\right\|^2
\end{aligned}$$

Owing to the Lipschitz properties of f and g combined with Hölder's inequality, we obtain

$$\begin{aligned}
\mathbf{E}\,\|(\Psi\vartheta)(t)-(\Psi\nu)(t)\|^2 \quad\le\quad & 2L_{\mathrm{h}}^2\mathbf{E}\,\|\vartheta(t-r(t))-\nu(t-r(t))\|^2\\
&+M_T^2tL_{\mathrm{f}}^2\int_0^t\mathbf{E}\,\|\vartheta(s-\delta(s))-\nu(s-\delta(s))\|^2\,ds
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,t]}\mathbf{E}\,\|(\Psi\vartheta)(s)-(\Psi\nu)(s)\|^2\le\alpha(t)\sup_{s\in[-\tau,t]}\mathbf{E}\,\|\vartheta(s)-\nu(s)\|^2$$

where $\alpha(t)=2L_{\mathrm{h}}^2+2M_T^2L_{\mathrm{f}}^2t$.
Then there exists $0<T_1\le T$ such that $0<\alpha(T_1)<1$ and Ψ is a contraction mapping on $S_{T_1}(\varphi)$ and therefore has a unique fixed point, which is a mild solution of Eq. (1) on $[-\tau,T_1]$. This procedure can be repeated a finite number of times in order to extend the solution to the entire interval $[-\tau,T]$. This completes the proof.

Remark 1 Notice that we can extend the solution for $t\ge T$. Indeed, if we assume that the constants L_f, L_h which appear in assumptions **(H4)–(H5)** are independent of $T>0$, then the mild solution is defined for all $t\in[-r,T]$, for each $T>0$. This will play a key role in our stability analysis. We will therefore assume that the solutions are described globally in time in the next section (for example, under the previous assumptions).

3.2. **Global attracting sets of Eq.**(1)**.** In this section, we shall get the global attracting sets of Eq.(1).We begin with the concept of the global attracting sets and the following lemmas :

Definition 4 The set $\Sigma\subset\mathbf{X}$ is called an global attracting set of Eq.(1) if for any initial value $\varphi\in C([-r,0];L^2(\Omega;\mathbf{X}))$, the solution process $\vartheta(t,\varphi)$ converges to Σ as $t\longrightarrow\infty$, i.e.,

$$\mathrm{dist}(\vartheta(t,\varphi),\Sigma)\longrightarrow 0,\ \text{as } t\longrightarrow\infty,\ \text{where } \mathrm{dist}(x,\Sigma)=\inf_{y\in\Sigma}\mathbf{E}\|x-y\|.$$

Lemma 2 Let $\tilde{\beta}:[0,\infty)\longrightarrow L_Q^0(\mathbf{Y},\mathbf{X})$ and assume that **(H3)**,**(H7)** and $\sup_{t\geq 0}\|\tilde{\beta}(t)\|_{L_Q^0}<\infty$ hold. Then, for any $p\geq 2, t>0$, we have

$$\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p \leq C\sup_{t\geq 0}\|\tilde{\beta}(t)\|_{L_Q^0}^p,$$

where $C>0$ is a constant depending only on H, M, p, and α.

Proof. By Kahane-Khintchine inequality, there exists a constant $C_p>0$ such that

$$\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p \leq C_p\left(\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2\right)^{p/2},$$

Using equality (5) of Definition 1, we have

$$\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2 = \sum_{n=1}^{\infty}\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)e_n dZ_n^H(s)\Big\|^2,$$

where $\{Z_n^H(s),\ \ n\in\mathbb{N}\}$ is a sequence of independent, real-valued Rosenblatt processes each with the same paramete $H\in(1/2,1)$ and $\{e_n,\ \ n\in\mathbb{N}\}$ is a complete orthonormal basis in $\mathbf{Y}$.
By Itô isometry and **(H3)**, we have

$$\begin{aligned}
&\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2\\
&=\sum_{n=1}^{\infty}\int_0^t\int_0^t\langle \mathrm{R}(t-s)\tilde{\beta}(s)e_n, \mathrm{R}(t-\tau)\tilde{\beta}(\tau)e_n\rangle|s-\tau|^{2H-2}dsd\tau,\\
&\leq H(2H-1)\sum_{n=1}^{\infty}\int_0^t\|\mathrm{R}(t-s)\tilde{\beta}(\tau)\|\left(\int_0^t\|\mathrm{R}(t-s)\tilde{\beta}(s)\||s-\tau|^{2H-2}ds\right)d\tau\\
&\leq H(2H-1)\tilde{N}^2\int_0^t e^{-\mu(t-s)}\|\tilde{\beta}(\tau)\|_{L_Q^0}\left(\int_0^t e^{-\mu(t-s)}\|\tilde{\beta}(s)\|_{L_Q^0}|s-\tau|^{2H-2}ds\right)d\tau.
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2 \leq & H(2H-1)\tilde{N}^2\left(\sup_{t\geq 0}\|\tilde{\beta}(t)\|_{L_Q^0}\right)^2\\
&\times\int_{-\infty}^t e^{-\mu(t-s)}\left(\int_{-\infty}^t e^{-\mu(t-s)}|s-\tau|^{2H-2}ds\right)d\tau.
\end{aligned}$$

We set $u = t - \tau$ and $v = t - s$, thus we can rewrite that

$$\begin{aligned}
&\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2 \\
&\le H(2H-1)\tilde{N}^2 \left(\sup_{t\ge 0}\|\tilde{\beta}(t)\|_{L_Q^0}\right)^2 \int_0^\infty e^{-\mu v}\left(\int_0^\infty e^{-\mu u}|u-v|^{2H-2}du\right)dv \\
&= H(2H-1)\tilde{N}^2 \left(\sup_{t\ge 0}\|\tilde{\beta}(t)\|_{L_Q^0}\right)^2 \left(\int_0^\infty e^{-\mu v}\left(\int_v^\infty e^{-\mu u}(u-v)^{2H-2}du\right)dv\right. \\
&\qquad \left.+ \int_0^\infty e^{-\mu v}\left(\int_0^v e^{-\mu u}(u-v)^{2H-2}du\right)dv\right) \\
&=: H(2H-1)\tilde{N}^2 \left(\sup_{t\ge 0}\|\tilde{\beta}(t)\|_{L_Q^0}\right)^2 (G_1+G_2).
\end{aligned}$$

We reset that $w = u - v$, therefore we obtain

$$G_1 = \int_0^\infty e^{-2\mu v}\left(\int_0^\infty e^{-\mu w}w^{2H-2}dw\right)dv = \frac{1}{2}\mu^{-2H}\Gamma(2H-1)$$

where $\Gamma(.)$ stand for Gamma function.
For the term G_2, we get that

$$\begin{aligned}
G_2 &\le \int_0^\infty e^{-\mu v}\left(\int_0^v (v-u)^{2H-2}du\right)dv \\
&= \frac{1}{2H-1}\int_0^\infty e^{-\mu v}v^{2H-1}dv \\
&= \frac{1}{2H-1}\mu^{-2H}\Gamma(2H-1).
\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
&\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p \\
&\le C_p\left(H(2H-1)\left(\frac{1}{2H-1}+\frac{1}{2}\right)\Gamma(2H-1)\mu^{-2H}\tilde{N}^2\right)^{p/2}\sup_{t\ge 0}\|\tilde{\beta}(t)\|_{L_Q^0}^p.
\end{aligned}$$

The proof is complete.

Theorem 3 Assume that assumptions **(H1)-(H7)** hold. Then the set

$$\Xi = \left\{\varphi \in U : \|\varphi\|^p \le (1-\theta)^{-1}\bar{\alpha}\right\}$$

is a global attracting set of Eq.(1) provided that the following conditions

$$\begin{aligned}
\theta :=& 4^{p-1}\tilde{N}^p L_{\mathrm{h}}^p + 8^{p-1}L_{\mathrm{f}}^p\tilde{N}^p\mu^{1-p} \\
<& 1
\end{aligned} \tag{10}$$

and

$$C := \sup_{t\ge 0}\|\tilde{\beta}(s)\|_{L_Q^0}^p < \infty \tag{11}$$

hold for $p \ge 2$, and

$$\bar{\alpha} = 8^{p-1}L_{\mathrm{f}}^p\tilde{N}^p\mu^{1-p} + 4^{p-1}C_p\sup_{t\ge 0}\|\tilde{\beta}(s)\|_{L_Q^0}^p. \tag{12}$$

Proof. First, due to [24], Theorem 3.1, under assumptions **(H1)-(H7)** and (10), Eq.(1) has a unique mild solution.
From (7), we have

$$\begin{aligned}\mathbf{E}\|\vartheta(t)\|^p =&\mathbf{E}\Big\|\mathrm{R}(t)\left[\varphi(0)-\mathrm{h}(0,\varphi(-u(0)))\right]-\mathrm{h}(t,\vartheta(t-u(t)))\\&+\int_0^t \mathrm{R}(t-s)\mathrm{f}(s,\vartheta(s-\rho(s)))ds+\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p\\ \leq&4^{p-1}\mathbf{E}\|\mathrm{R}(t)\left[\varphi(0)+\mathrm{h}(0,\varphi(-u(0)))\right]\|^p+4^{p-1}\mathbf{E}\|\mathrm{h}(t,\vartheta(t-u(t)))\|^p\\&+4^{p-1}\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\mathrm{f}(s,\vartheta(s-\rho(s)))ds\Big\|^p\\&+4^{p-1}\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p\\ =:&4^{p-1}\sum_{i=1}^4\mathfrak{H}_i. \end{aligned}\tag{13}$$

It follows from **(H3)** and **(H5)** that

$$\begin{aligned}\mathfrak{H}_1 =&\mathbf{E}\|\mathrm{R}(t)\left[\varphi(0)-\mathrm{h}(0,\varphi(-u(0)))\right]\|^p\\ \leq&\tilde{N}^pe^{-p\mu t}\mathbf{E}\left[\|\varphi(0)\|+\|\mathrm{h}(0,\varphi(-u(0)))\|\right]^p\\ \leq&\tilde{N}^pe^{-p\mu t}\mathbf{E}\left[\|\varphi(0)\|+L_{\mathrm{h}}\|\varphi(-u(0))\|\right]^p\\ \leq&N^*\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\varphi(\theta)\|^pe^{-\lambda t}\end{aligned}\tag{14}$$

where $N^*\geq 1$ is an appropriate constant.
From **(H5)** , we can obtain

$$\begin{aligned}\mathfrak{H}_2 =&\mathbf{E}\|\mathrm{h}(t,\vartheta(t-u(t)))\|^P\\ \leq&L_{\mathrm{h}}^p\mathbf{E}\sup_{-r\leq\theta\leq 0}\|\vartheta(t+\theta)\|^p.\end{aligned}\tag{15}$$

From **(H3)**, **(H4)** and Hölder's inequality, we get

$$\begin{aligned}\mathfrak{H}_3 =&\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\mathrm{f}(s,\vartheta(s-\rho(s)))ds\Big\|^p\\ \leq&\mathbf{E}\left(\int_0^t\tilde{N}e^{-\mu(t-s)}L_{\mathrm{f}}(1+\|\vartheta(s-\rho(s))\|)ds\right)^p\\ \leq&2^{p-1}\tilde{N}^p\mu^{1-p}L_{\mathrm{f}}^p\left(\int_0^t e^{-\mu(t-s)}\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\vartheta(s-\theta)\|^pds+1\right).\end{aligned}\tag{16}$$

By Lemma 2 and (11), we have

$$\mathfrak{H}_4=\mathbf{E}\Big\|\int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p\leq C_p\sup_{t\geq 0}\|\tilde{\beta}(s)\|_{L_Q^0}^p.\tag{17}$$

Substituting (14)-(17) into (13), we obtain

$$\mathbf{E}\|\vartheta(t)\|^p \leq 4^{p-1}N^* \sup_{-r\leq\theta\leq 0} \mathbf{E}\|\varphi(\theta)\|^p e^{-\mu t} + 4^{p-1}L_{\mathrm{h}}^p \sup_{-r\leq\theta\leq 0} \mathbf{E}\|\vartheta(t+\theta)\|^p$$
$$+ 8^{p-1}L_{\mathrm{f}}^p \tilde{N}^p \mu^{1-p} \int_0^t e^{-\mu(t-s)} \sup_{-r\leq\theta\leq 0} \mathbf{E}\|\vartheta(s+\theta)\|^p ds + 8^{p-1}L_{\mathrm{f}}^p \tilde{N}^p \mu^{1-p}$$
$$+ 4^{p-1}C_p \sup_{t\geq 0} \|\tilde{\beta}(s)\|^p_{L_Q^0}.$$

Now, put

$$b_1 := 4^{p-1}L_{\mathrm{h}}^p,$$

$$b_2 := 8^{p-1}L_{\mathrm{f}}^p \tilde{N}^p \mu^{1-p},$$

and put $\bar{\alpha}$ as in (12). By using (10), we easily get (21). Since $\varphi \in C([-r,0];X)$, there exist constants $\widehat{P} \geq 0$, $\widehat{L} \geq 0$ and $\mu_1 \in (0,\mu)$ such that

$$5N^*\|\varphi\|_C \leq \widehat{P}$$

$$\theta_{\mu_1} := b_1 e^{\mu_1} + \frac{b_2 e^{\mu_1 r}}{\mu - \mu_1} \leq 1, \quad \frac{(\mu-\mu_1)\left[\widehat{P} - \frac{b_2\bar{\alpha}}{\mu(1-\theta)}\right]}{b_2 e^{\mu_1 r}} \leq \widehat{L}. \tag{18}$$

Thus, using Lemma 3, we get

$$\mathbf{E}\|\vartheta(t)\|^p \leq \widehat{L}e^{-\mu t} + \frac{\bar{\alpha}}{1-\theta}. \tag{19}$$

Therefore,

$$\lim_{t\longrightarrow\infty} \mathbf{E}\|\vartheta(t)\|^p \leq \frac{\bar{\alpha}}{1-\theta}.$$

Thus, by Definition 4, we see that Ξ is an global attracting set of the mild solution of Eq.(1) .
The proof is complete.

3.3. **Exponential decay.** This section is devoted to investigate the exponential decay in p-th moment of mild solution to Eq.(1). The concept of this notion is given by the following definition:

Definition 5 The mild solution of Eq.(1) is said to have exponential decay in the p-th moment , if there exists constants $\beta > 0$ and $\bar{N} \geq 0$ such that for any initial data $\varphi \in C([-r,0];\mathbf{X})$, we have

$$\mathbf{E}\|\vartheta(t,\varphi)\| \leq \bar{N}e^{-\beta t},\ t \geq 0.$$

When $p = 2$, the mild solution of Eq.(1) is said to have exponential decay in in mean square.

We recall the following result which allows us to establish the exponential decay in the p-th moment of system (1):

Lemma 3 ([15]). Let $\tilde{z} : [-r, \infty) \longrightarrow [0, \infty)$ be a Borel measurable function. If $\tilde{z}(.)$ is a solution of the following delay integral inequality:

$$\tilde{z}(t) \leq \begin{cases} \|\chi\|_C e^{-\gamma t} + b_1 \|\tilde{z}_t\|_C + b_2 \displaystyle\int_0^t e{-}\gamma(t-s) \|\tilde{z}_s\|_C ds + \beta, & t \geq 0, \\ \chi(t), & t \in [-r, 0], \end{cases} \tag{20}$$

where $\chi \in C([-r, 0]; [0, \infty)$, $\gamma > 0$, b_1, b_2 and β are nonnegative constants, $\|\chi\|_C \geq K$ for some constant $K > 0$, and

$$b_1 + \frac{b_2}{\gamma} =: \rho < 1, \tag{21}$$

then there are constants $\lambda \in (0, \gamma)$ and $N \geq K$ such that

$$\tilde{z}(t) \leq N e^{-\lambda t} + \frac{\beta}{1 - \rho}, \quad t \geq 0,$$

where λ and N satisfy

$$\rho_\lambda := b_1 e^{\lambda r} + \frac{b_2 e^{\lambda r}}{\gamma - \lambda} < 1, \quad N \geq \frac{K}{1 - \rho_\lambda},$$

or $b_2 \neq 0$ and

$$\rho_\lambda \leq 1, \quad N \geq \frac{(\gamma - \lambda)[K - \frac{b_2 \beta}{\gamma(1-\rho)}]}{b_2 e^{\lambda r}}.$$

To establish the exponential decay in the p-th moment of system (1), we assume that

(H4') There exists a positive constant L_{f} such that

$$\|\mathrm{f}(t, x_1) - \mathrm{f}(t, x_2)\| \leq L_{\mathrm{f}} \|x_1 - x_2\|, \quad \|\mathrm{f}(t, x_1)\| \leq L_{\mathrm{f}} \left(\gamma(t) + \|x_1\|\right) \quad x_1, x_2 \in \mathbf{X}, t \geq 0,$$

where $\gamma : [0, \infty) \longrightarrow [0, \infty)$ is a local integrable continuous function.

Theorem 4 Assume that hypothesis **(H4)**, **(H4')**,**(H7)** and the following inequalities hold:

$$\sup_{t \geq 0} \left\{ \int_0^t e^{\mu s} \|\tilde{\beta}(s)\|^2_{L^0_Q} ds \right\} < \infty, \tag{22}$$

$$\delta_1 := \sup_{t \geq 0} \left\{ \int_0^t e^{\mu(1-p)t/p} e^{\mu s} \gamma(s) ds \right\} < \infty \tag{23}$$

and

$$\theta := 4L^p_{\mathrm{h}} + 8L^p_{\mathrm{f}} \mu^{-p} < 1. \tag{24}$$

hold for $p \geq 2$. Then the mild solution of Eq.(1) is said to have exponential decay in the p-th moment.

Proof. By using Hölder's inequality and **(H4')**, we have

$$
\begin{aligned}
\mathfrak{H}_3 =&\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\mathrm{f}(s,\vartheta(s-\rho(s)))ds\Big\|^p \\
\leq& L_{\mathrm{f}}^p \left(\int_0^t e^{-\mu(t-s)}\left(\gamma(s)+\mathbf{E}\|\vartheta(s-\rho(s))\|\right)ds\right)^p \\
\leq& 2L_{\mathrm{f}}^p \left(\int_0^t e^{-\mu(t-s)}\gamma(s)ds\right)^p + 2L_{\mathrm{f}}^p\left(\int_0^t e^{-\mu(t-s)}\mathbf{E}\|\vartheta(s-\rho(s))\|ds\right)^p \\
\leq& 2L_{\mathrm{f}}^p e^{-\mu t}\left(\int_0^t e^{\mu(1-p)t/p}e^{\mu s}\gamma(s)ds\right)^p + 2L_{\mathrm{f}}^p\mu^{1-p}\int_0^t e^{-\mu(t-s)}\mathbf{E}\|\vartheta(s-\rho(s))\|^p ds \\
\leq& 2L_{\mathrm{f}}^p e^{-\mu t}\delta_1^p + 2L_{\mathrm{f}}^p\mu^{1-p}\int_0^t e^{-\mu(t-s)}\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\vartheta(s+\theta)\|^p ds. \qquad (25)
\end{aligned}
$$

For every $t>0$, using the Kahane-Khintchine inequality there exists a constant C_p such that

$$
\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^p \leq C_p\left(\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2\right)^{p/2}.
$$

Choosing a suitable $\epsilon>0$ small enough such that $(\mu-\epsilon)p\geq 2r$ and $\delta:=\mu-\epsilon>0$ by (25) and [24], Lemma 2.2, we derive that

$$
\begin{aligned}
\mathbf{E}\Big\| \int_0^t \mathrm{R}(t-s)\tilde{\beta}(s)dZ_Q^H(s)\Big\|^2 \leq& \tilde{N}^2 c_H H(2H-1)t^{2H-1}\int_0^t e^{-2\mu(t-s)}\|\tilde{\beta}(s)|_{L_Q^0}^2 ds \\
\leq& \tilde{N}^2 c_H H(2H-1)t^{2H-1}\int_0^t e^{-\mu(t-s)}\|\tilde{\beta}(s)|_{L_Q^0}^2 ds \\
\leq& e^{-\delta t}\tilde{N}^2 c_H H(2H-1)t^{2H-1}e^{-\epsilon t}\int_0^t e^{\mu s}\|\tilde{\beta}(s)|_{L_Q^0}^2 ds. \qquad (26)
\end{aligned}
$$

Therefore, (22) ensures the existence of a positive constant K such that

$$
\tilde{N}^2 c_H H(2H-1)t^{2H-1}e^{-\epsilon t}\int_0^t e^{\mu s}\|\tilde{\beta}(s)|_{L_Q^0}^2 ds \leq K, \quad t\geq 0.
$$

Thus, by (26), we have

$$
\mathfrak{H}_4 \leq C_p K^{p/2} e^{-\mu t}. \qquad (27)
$$

Then, substituting (14)-(16), (25), and (27) into (13), we obtain

$$
\begin{aligned}
\mathbf{E}\|\vartheta(t)\|^p \leq& \left(5^{p-1}N^*\sup_{-r\leq\theta\leq 0}\|\varphi(\theta)\| + 5^{p-1}C_pK^{p/2} + 10^{p-1}L_{\mathrm{h}}^p\delta_1^p\right)e^{-\mu t} \\
&+ b_1\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\vartheta(t+\theta)\|^p + b_2\int_0^t e^{-\mu(t-s)}\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\vartheta(s+\theta)\|^p ds.
\end{aligned}
$$

By using (24), we obtain (21). Since $\varphi\in C([-r,0];\mathbf{X})$, there exist $\widehat{P}\geq 0,\ \widehat{L}>0$, and $\mu_1\in(0,\mu)$ such that

$$
5^{p-1}N^*\sup_{-r\leq\theta\leq 0}\mathbf{E}\|\varphi(\theta)\| + 5^{p-1}C_pK^{p/2} + 10^{p-1}L_{\mathrm{f}}^p\sigma_1^p \leq \widehat{P}
$$

and (18) holds. It follows from Lemma 3 that

$$
\mathbf{E}\|\vartheta(t)\|^p \leq \widehat{L}e^{-\mu t}.
$$

The proof is complete.

If $\gamma(t) \equiv 0$, we can easily get (23) and we can replace assumption **(H4')** by the following one
(H4") There exists a positive constant L_{f} such that for all $x_1, x_2 \in \mathbf{X}$ and $t \geq 0$,

$$\|\mathrm{f}(t, x_1) - \mathrm{f}(t, x_2)\| \leq L_{\mathrm{f}}\|x_1 - x_2\|, \ \|\mathrm{f}(t, 0)\| = 0.$$

Corollary 1 Assume that assumptions **(H1)**, **(H4")**, **(H5)**, and (22) are verified and the following inequality hold

$$\theta := 5L_{\mathrm{h}}^p + 10L_{\mathrm{f}}^p \mu_1^{-p} < 1, \tag{28}$$

for $p \geq 2$. Then the mild solution of Eq.(1) is exponential decay in p-th moment.

4. Example

In this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. Consider the following stochastic functional integro-differential equations driven by Rosenblatt process

$$\begin{cases} d\big[Z(t,\xi) + \eta_1 Z(t-r,\xi)\big] = \dfrac{\partial^2}{\partial Z^2}\big[Z(t,\xi) + \eta_1 Z(t-r,\xi)\big]dt \\ \qquad\qquad + \left[\displaystyle\int_0^t \mathrm{g}(t-s)\frac{\partial^2}{\partial \psi^2}\left[Z(s,\xi) + \eta_1 Z(s-r,\xi)\right]ds + \eta_2 Z(t-r,\xi)\right]dt \\ \qquad\qquad + \tilde{\beta}(t)dZ_Q^H(t), \ 0 \leq z \leq \pi, \ t \geq 0, \\ Z(t,0) = Z(t,\pi) = 0, \ \ t \geq 0, \\ Z(t,\xi) = \varphi(t) \in C([-r,0]; L^2([0,\pi])), \ \ -r \leq t \leq 0, \end{cases} \tag{29}$$

where $\eta_i > 0, i = 1,2$ are constants, Z_Q^H denotes a Q-Rosenblatt process.
Let $\mathbf{X} = \mathbf{Y} = L^2([0,\pi])$ and $e_n := \sqrt{\frac{2}{\pi}}\sin(nx), \ \ (n = 1,2,3,\cdots)$.
Then $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in $\mathbf{Y}$. In order to define the operator $Q : \mathbf{Y} \to \mathbf{Y}$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$ and set $Qe_n = \sigma_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty}\sqrt{\sigma_n} < \infty$. Define the process $Z_Q^H(t)$ by

$$Z_Q^H(t) = \sum_{n=1}^{\infty}\sqrt{\sigma_n}\gamma_n^H(t)e_n, \ t \geq 0$$

where $H \in (\frac{1}{2}, 1)$ and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional Rosenblatt processes being mutually independent.
Let define $\mathrm{A} : D(\mathrm{A}) \subset \mathbf{X} \to \mathbf{X}$ by $\mathrm{A} = \frac{\partial^2}{\partial z^2}$, with domain $D(\mathrm{A}) = H^2([0,\pi]) \cap H_0^1([0,\pi])$ and

$$\mathrm{A}z = -\sum_{n=1}^{\infty} n^2\langle z, e_n\rangle e_n, \ D(\mathrm{A}).$$

A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbf{X}$, which is given by $T(t)\phi = -\sum_{n=1}^{\infty} e^{-n^2 t} < \phi, e_n > e_n, \ \ \phi \in D(\mathrm{A})$.

Let $\Upsilon : D(\mathrm{A}) \subset \mathbf{X} \to \mathbf{X}$ be the operator defined by

$$\Upsilon(t)(z) = \mathrm{g}(t)\mathrm{A}z \ \text{ for } \ t \geq 0 \ \text{ and } \ z \in D(\mathrm{A}).$$

In order to rewrite Eq.(29) in an abstract form in $\mathbf{X}$, we introduce the following notations

$$\begin{cases} \vartheta(t) &= z(t,\tau) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \tau \in [0,\pi] \\ \varphi(t)(\tau) &= z_0(t,\tau) \quad \text{for} \quad t \in [-r,0] \quad \text{and} \quad \tau \in [0,\pi], \end{cases}$$

We define the operators $\mathrm{h}, \mathrm{f} : [0,\infty) \times \mathbf{X} \longrightarrow \mathbf{X}$ by

$$\mathrm{h}(t,\phi)(\xi) = \eta_1 \phi(t-r,\xi), \ \xi \in [0,\pi] \ t \geq 0,$$

$$\mathrm{f}(t,\phi)(\xi) = \eta_2 \phi(t-r,\xi), \ \xi \in [0,\pi] \ t \geq 0.$$

then Eq.(29) takes the following abstract form

$$\begin{cases} d\big[\vartheta(t) + \mathrm{h}(t,\vartheta(t-u(t))\big] = \Big[\mathrm{A}\big[\vartheta(t) + \mathrm{h}(t,\vartheta(t-u(t))\big] \\ + \displaystyle\int_0^t \Upsilon(t-s)\big[\vartheta(s) + \mathrm{h}(s,\vartheta(s-u(s))\big] + \mathrm{f}(t,\vartheta(t-\rho(t))\Big]dt \\ + \tilde{\beta}(t) dZ_Q^H(t), \ t \in [0,T] \\ \vartheta\,(t) = \varphi(\cdot) \in C([-r,0];\mathbf{X}), \ t \in [-r,0] \end{cases} \tag{30}$$

We suppose g is bounded and C^1 function such that g' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied and hence, by Theorem 1, Eq. (6) has a resolvent operator $(\mathrm{R}(t))_{t\geq 0}$ on $\mathbf{X}$. We assume moreover that there exists $\beta > a > 1$ and $\mathrm{g}(t) < \frac{1}{a}e^{-\beta t}$ for all $t \geq 0$. Thanks to lemma 5.2 in [8], we have the following estimates $\|\mathrm{R}(t)\| \leq e^{-\mu t}$ where $\mu = 1 - \frac{1}{a}$. Consequently, all the hypotheses of Theorem 2 are fulfilled. Therefore, Eq.(29) possesses a unique mild solution.

Let $p = 2$. Then we get

$$\begin{aligned} \widehat{\theta} :=& 4\tilde{N}^2 L_{\mathrm{h}}^2 + 8 L_{\mathrm{f}}^2 \tilde{N}^2 \mu_1^{-2} \\ :=& \widehat{\theta}_0, \end{aligned}$$

$$\widehat{\alpha} := L_{\mathrm{f}}^2 \tilde{N}^2 \mu_1^{-2} + 4C_2 \sup_{t\geq 0} \|\tilde{\beta}(t)\|^2_{L_Q^0}.$$

Therefore, by Theorem 3, we derive that the set given by $\Xi = \{\varphi \in \mathbf{X} \ \mid \ \|\varphi\|^2 \leq \Big(1 - \widehat{\theta}\Big)^{-1} \widehat{\alpha}\}$ is a global attracting set of Eq.(30) provided $\widehat{\theta}_0 < 1$ and $\sup_{t\geq 0} \|\tilde{\beta}(t)\|^2_{L_Q^0} < \infty$.

In addition, if

$$\sup_{t\geq 0} \left\{ \| \int_0^t e^{\mu s} \tilde{\beta}(s) \|^2_{L_Q^0} ds \right\} < \infty, \quad \widehat{\theta}_0 < 1,$$

thus, using Corollary 1, we conclude that, the mild solution of Eq.(30) is exponential decay in mean square.

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