ISSN 1541-308X, Physics of Wave Phenomena, 2022, Vol. 30, No. 6, pp. 387–396. © Allerton Press, Inc., 2022.

FUNDAMENTALS IN WAVE PROCESSES

Nonlocal Cubic-Quintic Nonlinear Schrödinger Equation: Symmetry Breaking Solitons and Its Trajectory Rotation

P. Sakthivinayagam*

Physics Department, PSG College of Arts and Science (Autonomous), Civil Aerodrome Post, Coimbatore, Tamilnadu, 641014 India

**e-mail: psvinayagam11@gmail.com*

Received May 11, 2022; revised July 11, 2022; accepted July 14, 2022

Abstract—Using two analytical methods, we derive exact and more general solutions of the nonlocal nonlinear Schrödinger equation with nonlocal cubic and nonlocal quintic terms. In the first method, equations are analyzed, and some of their mathematical and physical properties are inferred, which are then used to derive the exact stationary solutions. In the second method, we demonstrate the Darboux transformation method and construct exact and more general soliton solutions for the nonlocal NLS equation with nonlocal cubic and quintic terms. We reconsider the collisional dynamics of the nonlocal NLS equation and observe that apart from intensity redistribution in the interaction of bright and dark solitons, one also witnesses a rotation of the trajectories of the solitons. The angle of rotation can be varied by suitably manipulating the self-phase-modulation (SPM) or cross-phase-modulation (XPM) parameters and also spectral parameters. The angle of rotation of the solitons arises due to the excess energy that is injected into the dynamical system through SPM and XPM. We also notice the parallel traveling solitons due to the rotation in the soliton trajectories. These observations which exclude the quantum superposition for the field vectors may have wider ramifications in nonlinear optics, Bose–Einstein condensates, and left- and right-handed metamaterials.

Keywords: Darboux transformation, nonlocal NLSE, soliton solutions, cubic-quintic interaction, rotating solitons

DOI: 10.3103/S1541308X2206005X

1. INTRODUCTION

The exploration of the dynamical effects of the nonlinear Schrödinger (NLS) equation paves the way to understanding many nonlinear wave concepts [1–3] and studying their quantum phenomena [4–6]. The NLS equation is widely used in academia to research about waves in deep water [7] and are used to study the evolution of Bose–Einstein condensates with external governs [8]. It can also be used to describe the propagation of laser beams in optical media [9]. The introduction of the nonlocal NLS equation as a parity-time integrable system by Ablowitz and Musslimani [10] in 2013 has led to extensive work to investigate exact solutions and dynamics of nonlocal NLS equations in various forms. Soliton solutions and interactions under the influence of nonlocal interaction have been witnessed in [11]. Nonlocal nature from the integral point of view has been investigated using the inverse scattering method [12], and a discrete version of the same has also been investigated [13]. In this paper, we develop two analytic methods to study the general exact solutions of the nonlocal NLS with nonlocal cubic-quintic terms.

The nonlocal interaction or the PT-symmetric potential in NLS equation has got tremendous attention from theoretical physicists in recent years. Replacing $q^*(x, t)$ by $q^*(-x, t)$ in the NLS equation results to a self-induced potential of the form $V(x, t) = q(x, t)q^*(-x, t)$ and satisfies the PT-symmetric condition $V(x, t) = V^*(-x, t)$. Inclusion of this PT-symmetric term in classic NLS equation has led to the discovery of new dynamics which reveals many unknown results in the field of nonlinear fiber optics refinement and Bose–Einstein condensates (BEC). For example, it is well known that the focusing NLS equation admits only bright solitons and defocusing NLS equation admits dark solitons, but the introduction of nonlinear nonlocal interaction in the NLS equation will admit both bright and dark solitons without any change in its nonlinearity. This type of nonlinearity is observed in many modern experiments such as the diffusion of charge carriers, atoms, or molecules in atomic vapors [14, 15]. It is also observed in the study of BEC with long-range interaction. The BEC with magnetic dipole–dipole forces were reported in [16] and verified in the case of optical spatial solitons in a high nonlocal medium [17]. It has been verified that nonlocal PT-symmetric integrable can act as a new scope for researchers to work on the ramification of nonlocality in optics. In optics, the paraxial equation of diffraction is mathematically isomorphic to the Schrödinger equation in quantum mechanics [18–21]. This analogy allowed the observation of PT-symmetry in optical waveguide structures and lattices [22, 23]. The study of PT-symmetric concepts in optics could also pave the way to alternative classes of optical structures and devices such as Hermitian Bloch Oscillations [24], simultaneous lasing absorption [25, 26], and selective lasing [27]. Lately, PT-symmetric concepts have also been studied rigorously in plasmonics [28], optical metamaterial [29] and coherent atomic medium [30].

Motivated by the above research and its significant achievements and unique dynamical behavior, we investigate the PT-symmetric nonlocal NLS equation with nonlocal cubic-quintic interaction in continuous media. Our study and analysis indicate that the PT-symmetric cubic and quintic NLS equation could exhibit unique dynamics and features so that we can observe different kinds of solitons for $Q(x, t)$ and $Q^*(-x, t)$ in contrast to the classical NLS equation where we found same behavioral solitons for both field variable and its conjugate. In addition, one can also rotate the path of the bright bound state and dark bound state in a two-soliton solution by selectively fine-tuning the spectral parameter while the energy in each mode remains conserved, unlike the Manakov model.

The plan of the paper is as follows. In Section 2, we present the mathematical (integrable) model governing the dynamics of nonlocal PT-symmetric cubic and quintic nonlinear Schrödinger equation and its Lax pair. In Section 3, we present the family of all solitonic solutions generated from the general solution obtained in Section 2. In Section 4, we discuss the collisional dynamics of solitons and the mechanism which involves the rotation of bright, dark solitons. The results are then summarized in Section 5.

2. MODEL EQUATION AND SOLITON SOLUTIONS

Let us consider the PT symmetric nonlocal NLS equation in the following form [12]:

$$iq(x,t)_t + q(x,t)_{xx} + 2\sigma q(x,t)q^*(-x,t)q(x,t) = 0, \quad (1)$$

where $q(x, t)$ is the slowly varying pulse envelope of the field which evolve in space (x) and time (t) variables, respectively, and * denotes the complex conjugation. The $q(x, t)_t$ and $q(x, t)_{xx}$ represents the one and two space and time derivatives, respectively, σ is nonlocal interaction strength. Equation (1) is a non-Hermitian PT symmetric system in the sense that the self-induced potential $V(x, t) = q(x, t)\, q^*(-x, t)$ satisfies the PT symmetric condition $V(x, t) = V^*(-x, t)$. It is worth mentioning here that Eq. (1) is nonlocal, i.e., the evolution of the field variable $q(x, t)$ at the transverse coordinate x always requires information from the opposite point $-x$.

2.1. Modified NLS Equation and Phase Dependent Amplitude of Matter Wave Solitons

To introduce the new integrable model describing the impact of both cubic and quintic interactions on the matter wave solitons, we now consider an additional phase imprint on the order parameter $q(x, t)$ to generate a new order parameter $Q(x, t)$ as

$$\begin{aligned} Q(x,t) &= q(x,t)\, e^{2i\theta(x,t)}, \\ Q^*(-x,t) &= q^*(-x,t)\, e^{-2i\theta(x,t)}, \end{aligned} \quad (2)$$

where $\theta(x, t)$ is the phase imprint in the old order parameter $q(x, t)$. We now engineer the phase imprint $\theta(x, t)$ in accordance with the following conditions:

$$\theta_x = -\sqrt{\tau}|q(x,t)|^2, \quad (3)$$

$$\begin{aligned} \theta_t = i\sqrt{\tau}\big[q(x,t)\, q^*(-x,t)_x - q(x,t)_x\, q^*(-x,t)\big] \\ + 4\tau|q(x,t)|^4, \end{aligned} \quad (4)$$

so that the transformed order parameter $Q(x, t)$ obeys an evolution equation

$$\begin{aligned} iQ(x,t)_t - Q(x,t)_{xx} + Q(x,t)^2 Q^*(-x,t) \\ + 12\tau^2 Q(x,t)^3 Q^*(-x,t)^2 \\ - 8i\tau Q(x,t)_x\, Q(x,t)\, Q^*(-x,t) = 0, \end{aligned} \quad (5)$$

where τ is the arbitrary constant. The above nonlocal cubic and quintic NLS Eq. (5) has an initial eigenvalue problem (Lax pair), which will be presented in the Appendix.

3. SOLITON SOLUTIONS

Our prior aim is to achieve a solution that itself can generate all forms of soliton solutions already witnessed for our model given by Eq. (5) in addition with new results. Over the years numerous results have been published using bright soliton solution generated by vacuum seed for NLS equation with cubic and quintic interaction [32–34]. Notably, the following papers [35–43] explain how the instability arises in the dynamical system and how one can manage this instability due to the reinforcement of quintic nonlinearity in addition to cubic by means of Feshbach resonance management in various experimental conditions which includes spin-orbital (SO) coupling, spatiotemporal solitons, and even in discrete solitons. Keeping this fact in mind, we omitted the vacuum seed solution and bright solitons as points of convergence.

To achieve a more general soliton solution, we start constructing the soliton using a nontrivial seed. Constant wave background solutions are more effective than that vacuum soliton solutions. How? In a way, one can achieve the vacuum seed solution from a constant wave solution by carefully fine-tuning the parameters involved with the assumed seed, but the reverse is not possible means one cannot generate a constant wave solution from vacuum seed solutions in any way. So, keeping this context in mind, we worked on a more general one by feeding plane wave solution as seed.

We assumed the following form of constant plane wave seed, $q(x,t) = Ae^{2iA^2t}$, $q^*(-x,t) = Ae^{-2iA^2t}$ to the defocusing NLS equation ($\sigma = -1$). Inserting these seed solutions in the Lax pair Eqs. (15) will lead to the basic solutions in the following form:

$$\psi_1(x,t) = e^{iA^2t}\left(c_1e^{\chi_1 s_1} + c_2e^{-\chi_1 s_1}\right), \quad (6)$$

$$\phi_1(x,t) = e^{iA^2t}\left[-\frac{c_1}{A}(\lambda_1 + is_1)e^{\chi_1 s_1} - \frac{c_2}{A}(-\lambda_1 + is_1)e^{-\chi_1 s_1}\right], \quad (7)$$

$$\psi_2(x,t) = e^{iA^2t}\left(\bar{c}_1e^{\chi_1 s_1} + \bar{c}_2e^{-\chi_1 s_1}\right), \quad (8)$$

$$\phi_2(x,t) = e^{iA^2t}\left[-\frac{\bar{c}_1}{A}(\lambda_1 + is_1)e^{\chi_1 s_1} - \frac{\bar{c}_2}{A}(-\lambda_1 + is_1)e^{-\chi_1 s_1}\right], \quad (9)$$

where $\chi_1 = x + 2\lambda_1 t$, $\chi_2 = x + 2\lambda_2 t$, $s_1 = \sqrt{-\lambda_1^2 + A^2}$ and $s_2 = \sqrt{-\lambda_2^2 + A^2}$. Inserting these basic solutions into the Darboux transformation formula [31], and simplifying the resultant expressions with $\tau_1 = c_1/c_2$ and $\tau_2 = \bar{c}_1/c_2$, we have after tedious calculations that

$$Q(x,t) = Ae^{2iA^2t+\theta}\left[1 + 2(\lambda_2 - \lambda_1)\frac{N_1}{D_1}\right], \quad (10)$$

$$Q^*(-x,t) = Ae^{-2iA^2t+\theta}\left[1 - \frac{2}{A^2}(\lambda_2 - \lambda_1)\frac{N_2}{D_1}\right], \quad (11)$$

with

$$\begin{aligned} N_1 &= \tau_1\tau_2e^{s_1\chi_1+s_2\chi_2} + \tau_2e^{-s_1\chi_1+s_2\chi_2} \\ &\quad + \tau_1e^{s_1\chi_1-s_2\chi_2} + e^{-s_1\chi_1-s_2\chi_2}, \\ N_2 &= \tau_1\tau_2\rho_1\rho_2e^{s_1\chi_1+s_2\chi_2} + \tau_2\rho_2\rho_3e^{-s_1\chi_1+s_2\chi_2} \\ &\quad + \tau_1\rho_1\rho_4e^{s_1\chi_1-s_2\chi_2} + \rho_3\rho_4e^{-s_1\chi_1-s_2\chi_2}, \\ D_1 &= \tau_1\tau_2(\rho_1 - \rho_2)e^{s_1\chi_1+s_2\chi_2} + \tau_3(\rho_2 - \rho_3)e^{-s_1\chi_1+s_2\chi_2} \\ &\quad + \tau_1(\rho_4 - \rho_1)e^{s_1\chi_1-s_2\chi_2} - (\rho_4 - \rho_3)e^{-s_1\chi_1-s_2\chi_2}, \end{aligned} \quad (12)$$

where $\rho_1 = \lambda_1 + is_1$, $\rho_2 = \lambda_2 + is_2$, $\rho_3 = \lambda_1 - is_1$, $\rho_4 = \lambda_2 - is_2$. The above solution is more general in nature compared to all the other known solutions. Why? The reason is explained in the next section by creating all form of solitons states by fine tuning the parameters. While transforming this solution to model Eq. (5), the phase imprint parameter θ has the following form:

$$\theta(x,t) = -\frac{\sqrt{\tau}[D_1 + 2N_1(\lambda_2 - \lambda_1)][a_1a_2D_1 + 2N_2(\lambda_1 - \lambda_2)]}{D_1^2}. \quad (13)$$

In the next section, we have studied the dynamics of the above general solution for different parameter and displayed all types of soliton pairs using this solution.

4. FAMILY OF EXACT SOLUTIONS

The soliton solution given by Eqs. (10) and (11) is the more general one for the cubic quintic NLS type equations compared with all the other previously determined solutions in the literature. We call this equation symmetry-breaking soliton solutions since as we said the solution given by Eqs. (10) and (11) act as a host for all types of soliton pairs such as Dark–Dark (DD), Bright–Bright (BB), Dark–Bright (DB), Bright–Dark (BD) by a suitable combination of parameters involved in it. The mentioned soliton pairs DD, BB, DB, BD are displayed in Figs. 1–4, respectively. Why do we call it a symmetry-broken solution? Because if you look at the soliton solutions $Q(x, t)$ and $Q^*(-x, t)$, you see that they are independent and cannot be deduced from one to the other. Usually, when we derive soliton solutions for vector systems, we end up with more or less the same solitons for both components. But here if you look at the figures of soliton pairs $Q(x, t)$ and $Q^*(-x, t)$, you see that they are not the same and exhibit completely different behaviors. Let us discuss them one by one. In Fig. 1, there are the components one and two displayed DD and BB solitons (in plots a and b, respectively), and the final plot is the combination of the first two is again DD (c). In Fig. 2, the components one and two displayed BB and DD solitons (in plots (a) and (b), respectively), and the final plot is the combination of the first two is again DD (c). Similarly, in Figs. 3 and 4, the component one and two are not the same. This behavior is the one that we could never come across in the literature for cubic and quintic NLS equations. In the next section, we have discussed some interesting new features found in the dynamics of the model given by Eq. (5) called "Rotation of trajectory".

4.1. Rotation of the Trajectories of Solitons

In this subsection, we are going to display some new phenomena, for that we choose only one component in every pair, and we try to study the dynamics by fine-tuning the parameters related to it. While tuning the spectral parameter λ_2, we have seen that one mode of every soliton pair rotates by 360°, while the other component sticks in one place. The rotation of the single-mode of the soliton pair and its contour plots are shown in Fig. 5. From the contour plots (d–k) in Fig. 5, one can infer that one of the modes is sticky at one point and the other can rotate clockwise. It is interesting to see that in the final phase shown in plot 5i and its corresponding contour plot in 5l after a full rotation of one mode, the interaction between the modes is invisible, and they evolve like parallel solitons without any interaction between them, which is really an inter-

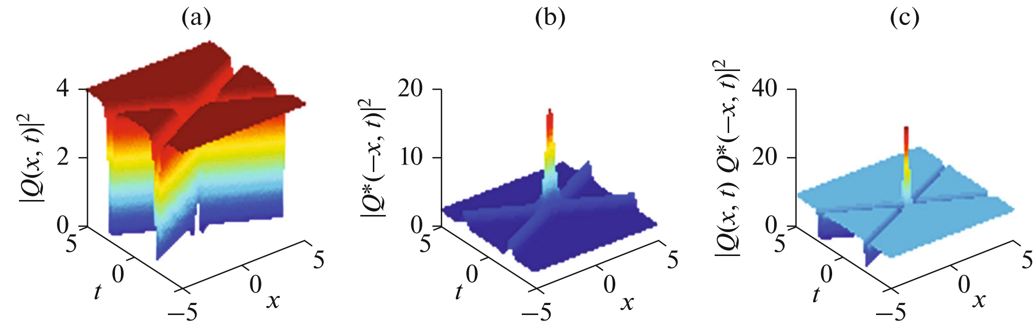

Fig. 1. (Color online) (a) Dark–Dark soliton for $q(x, t)$ mode, (b) Bright–Bright soliton for $q^*(x, t)$, and (c) multiplication for first two component $q(x, t)$ and $q^*(x, t)$ leads to Dark–Dark solitons for the choice of parameters $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = -1 - 0.7i$, $\tau_2 = -0.5 + 0.3i$.

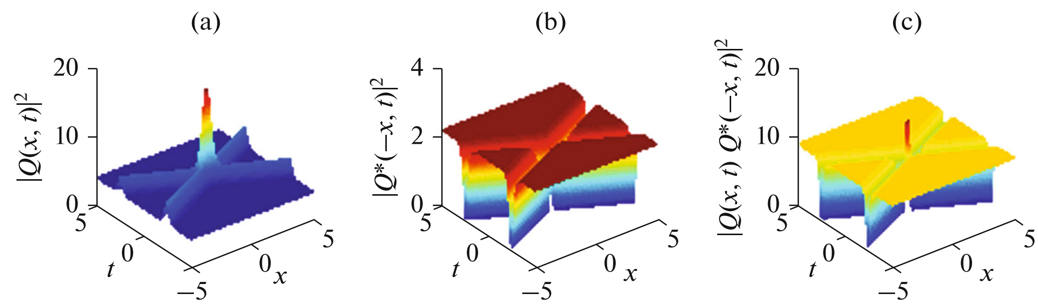

Fig. 2. (Color online) (a) Bright–Bright soliton for $q(x, t)$ mode, (b) Dark–Dark soliton for $q^*(x, t)$, and (c) multiplication for first two component $q(x, t)$ and $q^*(x, t)$ leads to Dark–Dark solitons for the choice of parameters $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 + 0.5i$, $\tau_2 = 0.5 + 0.3i$.

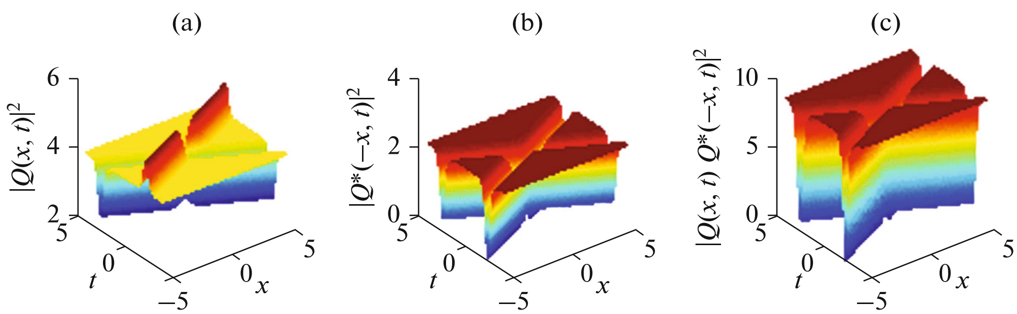

Fig. 3. (Color online) (a) Dark–Bright soliton for $q(x, t)$ mode, (b) Dark–Dark soliton for $q^*(x, t)$, and (c) multiplication for first two component $q(x, t)$ and $q^*(x, t)$ leads to Dark–Dark solitons for the choice of parameters $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 - 0.5i$, $\tau_2 = 1.5 - 0.3i$.

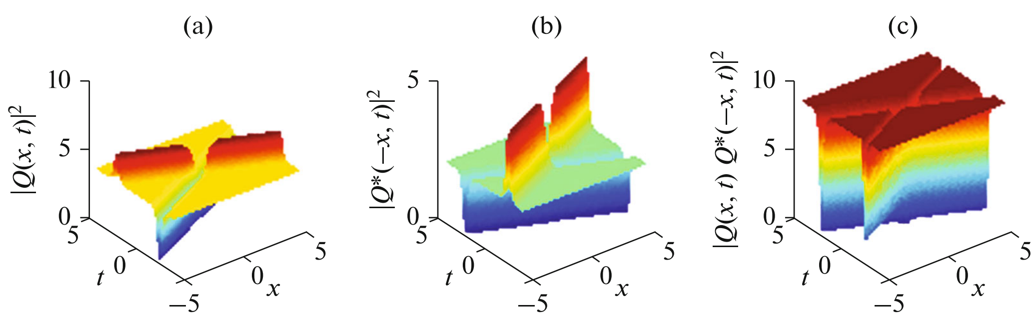

Fig. 4. (Color online) (a) Bright–Dark soliton for $q(x, t)$ mode, (b) Dark–Bright soliton for $q^*(x, t)$, and (c) multiplication for first two component $q(x, t)$ and $q^*(x, t)$ leads to Dark–Dark solitons for the choice of parameters $\lambda_1 = -1$, $\lambda_2 = 1.2$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 + 0.5i$, $\tau_2 = -0.5 + 0.3i$.

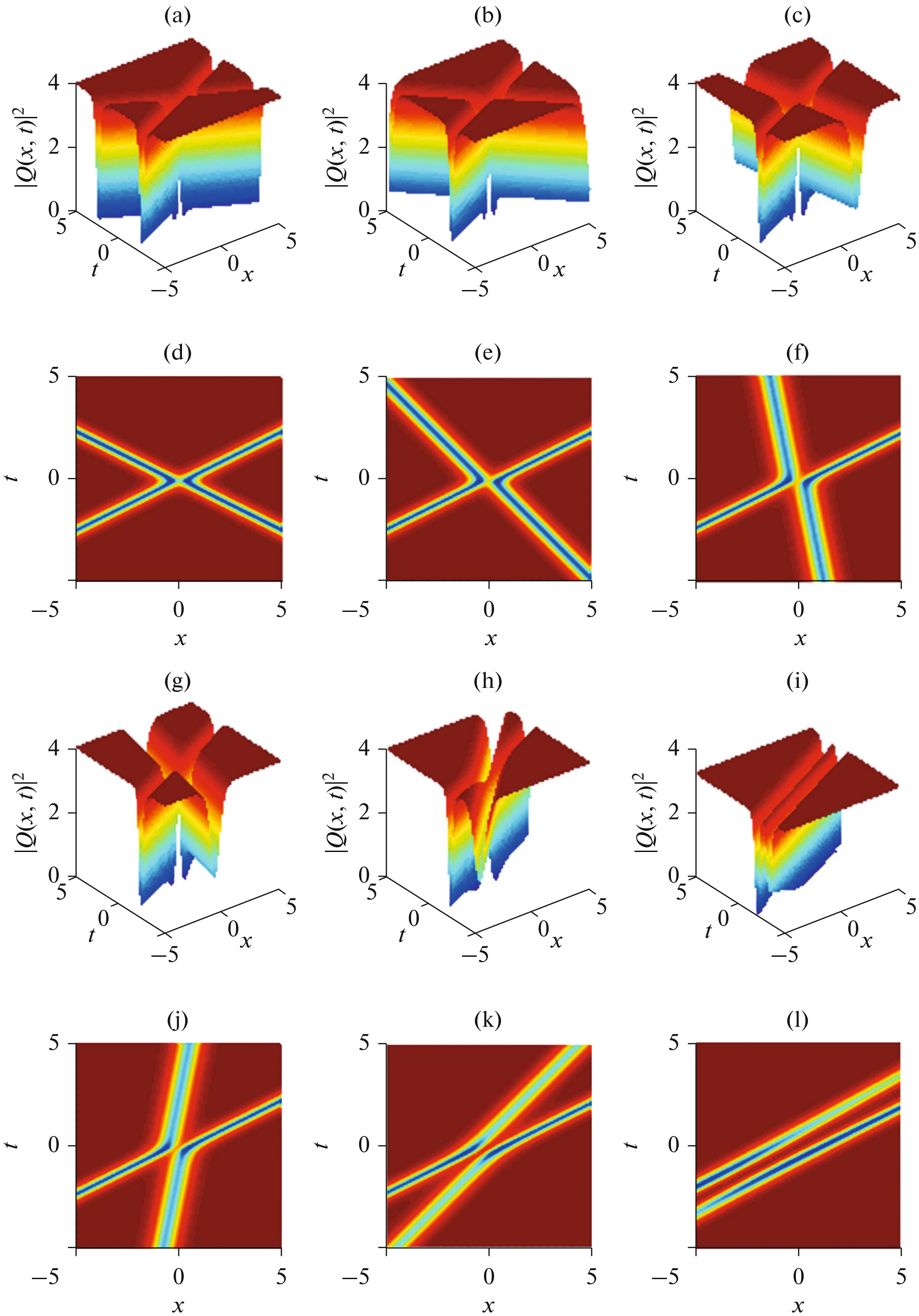

Fig. 5. (Color online) (a–c) and (g–i) Rotation of the trajectories of the Dark–Dark solitons and its corresponding contour plots (d–f) and (j–l) for the choice of parameter for the plots (a, d) $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = -1 - 0.7i$, $\tau_2 = -0.5 + 0.3i$; (b, e) $\lambda_1 = 0.5$, all other parameters are same as in the plots (a, d); (c, f) $\lambda_1 = 0.1$; (g, j) $\lambda_1 = -0.1$; (h, k) $\lambda_1 = -0.5$; (i, l) parallel travelling solitons $\lambda_1 = -0.9$.

esting note we witnessed. We have never before observed such parallel-evolving solitons. We call this phenomenon of rotation of the one mode in solitons pairs "Rotation of the trajectories of the solitons". We have tried to find the same behavior in all other pairs of solitons also. We succeeded in BB pair and showed the rotation of the trajectories of the soliton pair in Fig. 6 with its corresponding contour plots. We notice

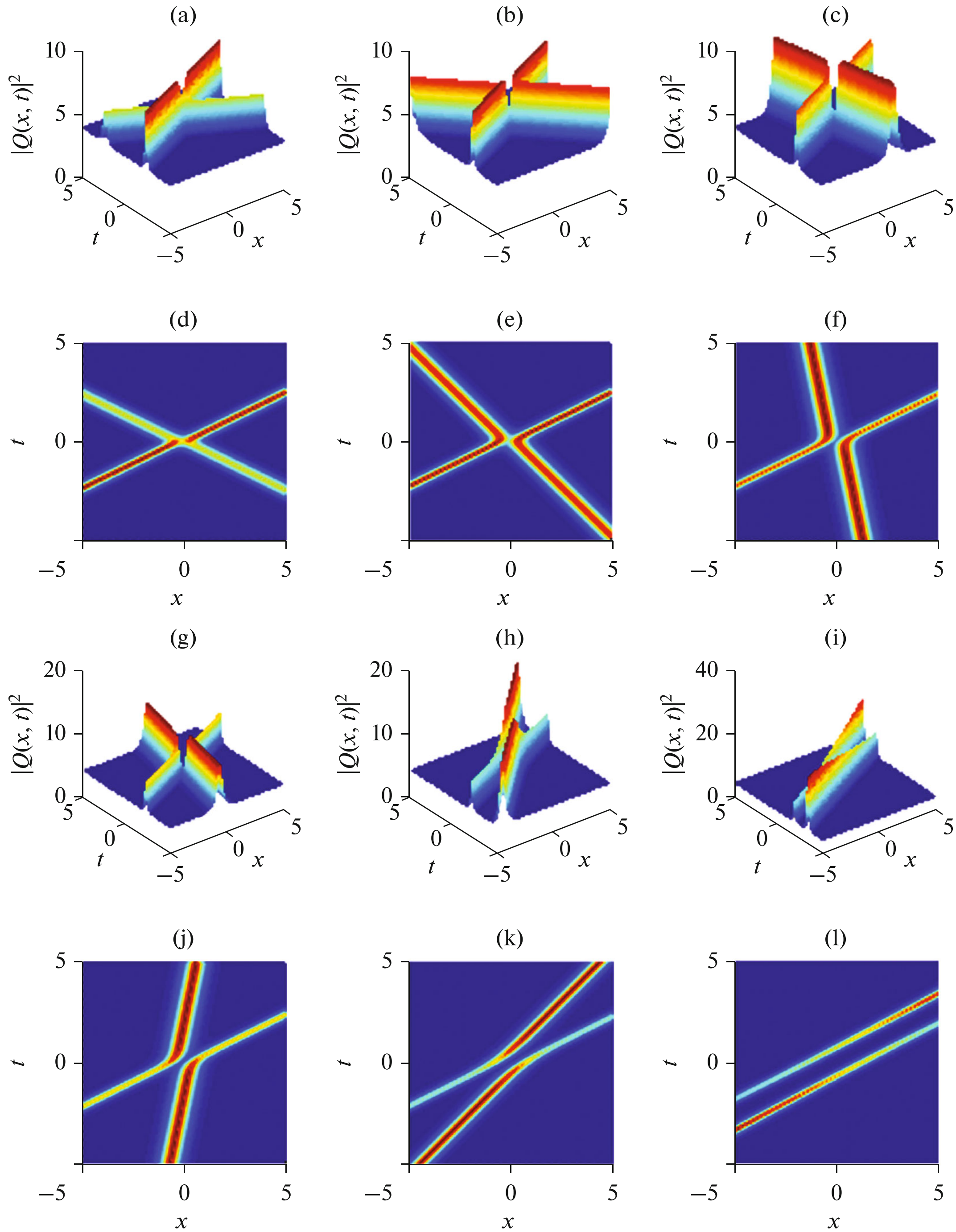

Fig. 6. (Color online) (a–c) and (g–i) Rotation of the trajectories of the Bright–Bright solitons and its corresponding contour plots (d–f) and (j–l) for the choice of parameter for plots (a, d) $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 - 0.5i$, $\tau_2 = -1.5 - 0.3i$; (b, e) $\lambda_1 = 0.5$, all other parameters are same as in plots (a, d); (c, f) $\lambda_1 = 0.1$; (g, j) $\lambda_1 = -0.1$; (h, k) $\lambda_1 = -0.5$; (i, l) parallel travelling solitons $\lambda_1 = -0.9$.

a small difference in the BB pair compared to the previous one in the final phase of the rotation shown in plot 6i and its corresponding contour in plot 6l. We achieved the same parallel travelling solitons without interaction, but here we notice even though there is no interaction, the energy sharing happens which was clearly depicted in plot 6i for better understanding. We have also checked the same behavior in a mixed state of the soliton pairs such as DB and BD; here in these two cases, we were able to rotate as we did before in BB and DD pairs, but not the whole rotation. Due to the nature of the solitons, we are not able to rotate one

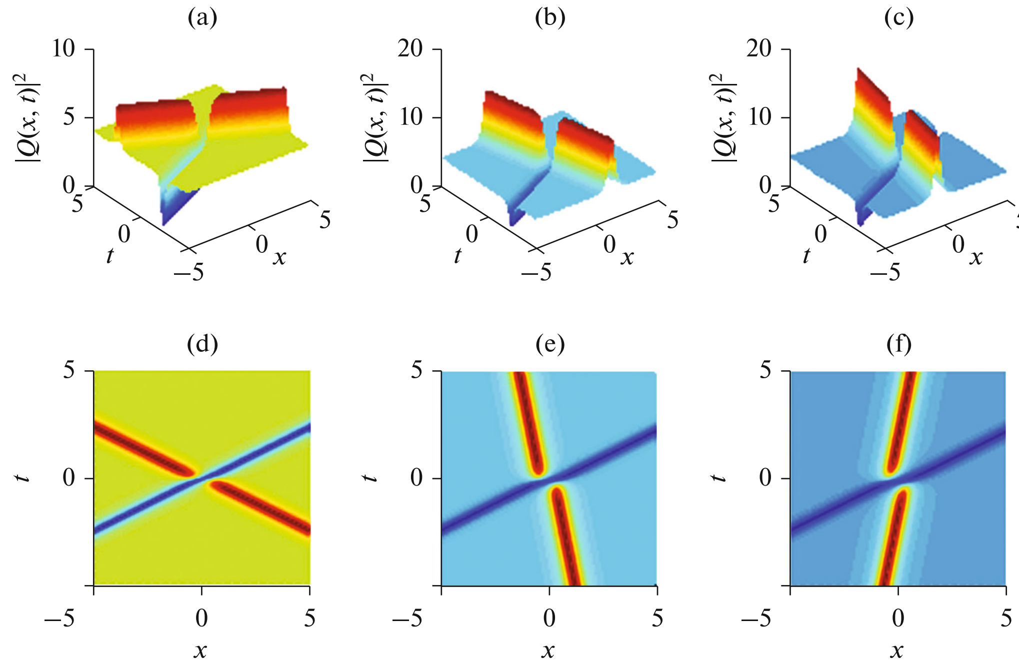

Fig. 7. (Color online) (a–c) Rotation of the trajectories of Dark–Bright solitons and its corresponding contour plots (d–f) for the choice of parameters: (a, d) $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 + 0.6\text{i}$, $\tau_2 = -0.5 + 0.3\text{i}$; (b, e) except $\lambda_1 = 0.1$, all other parameters are the same as in the plots (a, d); (c, f) $\lambda_1 = -0.1$, all other parameters are the same as in plots (c, f).

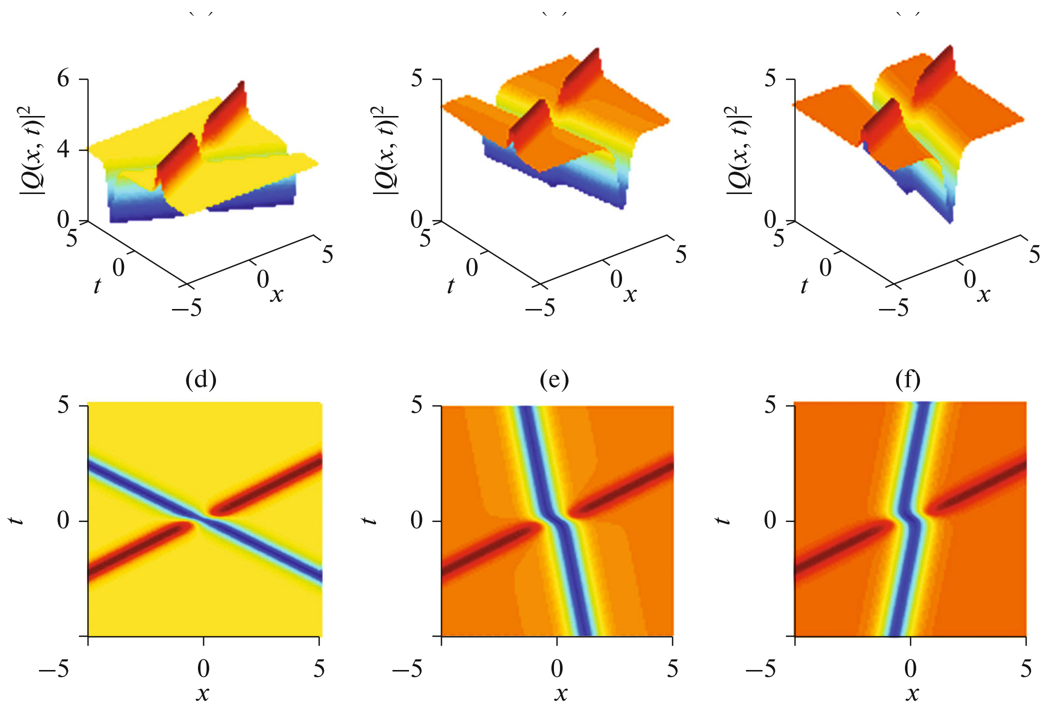

Fig. 8. (Color online) (a–c) Rotation of the trajectories of Bright–Dark solitons and its corresponding contour plots (d–f) for the choice of parameters: (a, d) $\lambda_1 = 1$, $\lambda_2 = -1$, $a_1 = 2$, $a_2 = 1.5$, $\tau_1 = 1 + 0.6\text{i}$, $\tau_2 = -0.5 + 0.3i$; (b, e) except $\lambda_1 = 0.1$, all other parameters are same as in plot (a, d); (c, f) $\lambda_1 = -0.1$, all other parameters are same as in plot (c, f).

mode after half of the circle like 180° and cannot move further, and the parallel travelling soliton solutions have also not been achievable in mixed states of the solitons as shown in Figs. 7 and 8, respectively.

5. CONCLUSIONS

In this paper, we investigated PT-symmetric cubic and quintic NLS equations employing Darboux transformation to generate a family of exact soliton solutions including combinations of bright dark solitons. The soliton solution we derived exhibits two different solitons for the two modes (field variable and its conjugate) which we called symmetry-broken solitons. In addition to that symmetry-broken property, we observed intensity redistribution and rotation of trajectories of solitons by suitably varying SPM, XPM parameter, and scattering length parameters. We have also shown the parallel traveling soliton which is noninteractive and interactive in nature when we increase the angle between the solitons by rotation. We believe that the above phenomena such as symmetry-broken solitons, rotation of the trajectories of solitons, and parallel traveling solitons occur due to the nonlocal nature of the dynamical system. A refinement of the present model would be incorporating PT-symmetry through cubic and quintic interaction terms is more realistic and can be verified experimentally, which we leave to future study.

APPENDIX

LAX PAIR FORMALISM

Equation (6) admits the following Lax pair:

$$\mathbf{\Phi}_x = \mathbf{U\Phi}, \tag{14}$$

$$\mathbf{\Phi}_t = \mathbf{V\Phi}. \tag{15}$$

Here **U** and **V** are called Lax pair matrices, which are functionals of the solutions of the model equations. The consistency condition of the linear system $\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}$ must be equivalent to the model equation under consideration,

$$\mathbf{\Phi}_x = \mathbf{U}_0\mathbf{\Phi} + \mathbf{U}_1\mathbf{\Phi\Lambda}, \tag{16}$$

$$\mathbf{\Phi}_t = \mathbf{V}_0\mathbf{\Phi} + \mathbf{V}_1\mathbf{\Phi\Lambda} + \mathbf{V}_2\mathbf{\Phi\Lambda}^2. \tag{17}$$

Here

$$\mathbf{\Phi} = \begin{pmatrix} \psi_1(x,t) & \psi_2(x,t) \\ \phi_1(x,t) & \phi_2(x,t) \end{pmatrix}, \tag{18}$$

$$\mathbf{U}_0 = \begin{pmatrix} 0 & iq(x,t) \\ -i\sigma q^*(-x,t) & 0 \end{pmatrix}, \tag{19}$$

$$\mathbf{U}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{20}$$

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{21}$$

$$\mathbf{V}_0 = \begin{pmatrix} -i\sigma q(x,t)\, q^*(-x,t) & q_x(x,t) \\ -\sigma q_x^*(-x,t) & -i\sigma q(x,t)\, q^*(-x,t) \end{pmatrix}, \tag{22}$$

$$\mathbf{V}_1 = 2\begin{pmatrix} 0 & iq(x,t) \\ i\sigma q^*(-x,t) & 0 \end{pmatrix}, \tag{23}$$

$$\mathbf{V}_2 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{24}$$

where $\lambda_{1,2}$ is the spectral parameter, $\mathbf{U}_{0,1}$, $\mathbf{V}_{0,1,2}$, and $\mathbf{\Phi}$ denotes the Lax pair matrix operators.

The consistency condition $\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}$ leads to $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$ which should generate the nonlocal NLS Eq. (1). Applying the transformation given by Eq. (2), keeping the constraints given by Eqs. (3), will generate the required nonlocal cubic and quintic NLS equation given by Eq. (5).

ACKNOWLEDGMENTS

PSV wholeheartedly thanks the Principal and Secretary, PSG CAS and management for their moral support and encouragement.

CONFLICT OF INTEREST

The author declares that he (she) has no conflicts of interest.

REFERENCES

1. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wiemann, and E. A. Cornell, "Observation of Bose–Einstein condensation in a dilute atomic vapor," *Science* **269**, 198–201 (1995). https://doi.org/10.1126/science.269.5221.198
2. F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, "Theory of Bose–Einstein condensation in trapped gases," *Rev. Mod. Phys.* **71** (3), 463–512 (1999). https://doi.org/10.1103/RevModPhys.71.463
3. C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation* (Springer, New York, 1999).
4. G. Theocharis, P. Schmelcher, P. G. Kevrekidis, and D. J. Frantzeskakis, "Matter-wave solitons of collisionally inhomogeneous condensates," *Phys. Rev. A* **72** (3), 033614 (2005). https://doi.org/10.1103/PhysRevA.72.033614
5. D. S. Wang, D. J. Zhang, and J. Yang, "Integrable properties of the general coupled nonlinear Schrödinger equations," *J. Math. Phys.* **51** (2), 023510 (2010). https://doi.org/10.1063/1.3290736
6. D.-S. Wang, X.-H. Hu, J. Hu, and W. M. Liu, "Quantized quasi-two-dimensional Bose–Einstein condensates with spatially modulated nonlinearity," *Phys. Rev. A* **81** (2), 025604 (2010). https://doi.org/10.1103/PhysRevA.81.025604
7. V. E. Zakharov, "Stability of periodic waves of finite amplitude on the surface of a deep fluid," *J. Appl. Mech. Tech. Phys.* **9**, 190–194 (1968). https://doi.org/10.1007/BF00913182
8. L. P. Pitaevskii and S. Stringari, *Bose–Einstein Condensation* (Oxford Univ. Press, Oxford, 2003).
9. Y. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, New York, 2003).
10. M. J. Ablowitz and Z. H. Musslimani, "Integrable nonlocal nonlinear Schrödinger equation," *Phys. Rev. Lett.*

110 (6), 064105 (2013).
https://doi.org/10.1103/PhysRevLett.110.064105

11. M. Li and T. Xu, Dark and antidark soliton interactions in the nonlocal nonlinear Schrödinger equation with the self-induced parity-time-symmetric potential." *Phys. Rev. E* **91** (3), 033202 (2015). https://doi.org/10.1103/PhysRevE.91.033202
12. M. J. Ablowitz and Z. H. Musslimani, "Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation," *Nonlinearity* **29** (3), 915–946 (2016). https://doi.org/10.1088/0951- 7715/29/3/915
13. M. J. Ablowitz and Z. H. Musslimani, "Integrable discrete PT symmetric model," *Phys. Rev. E* **90** (3), 032912 (2014). https://doi.org/10.1103/PhysRevE.90.032912
14. A. C. Tam and W. Happer, "Long-range interactions between cw self-focused laser beams in an atomic vapor," *Phys Rev Lett*. **38** (6), 278–281 (1977). https://doi.org/10.1103/PhysRevLett.38.278
15. D. Suter and T. Blasberg, "Stabilization of transverse solitary waves by a nonlocal response of the nonlinear medium," *Phys Rev A* **48** (6), 4583–4587 (1993). https://doi.org/10.1103/PhysRevA.48.4583
16. K. Góral, K. Rzążewski, and T. Pfau, "Bose–Einstein condensation with magnetic dipole–dipole forces," *Phys Rev A* **61** (5), 051601(R) (2000). https://doi.org/10.1103/PhysRevA.61.051601
17. C. Conti, M. Peccianti, and G. Assanto, "Observation of optical spatial solitons in a highly nonlocal medium," *Phys Rev Lett*. **92** (11), 113902 (2004). https://doi.org/10.1103/PhysRevLett.92.113902
18. R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, "Theory of coupled optical PT-symmetric structures," *Opt. Lett*. **32** (17), 2632–2634 (2007). https://doi.org/10.1364/OL.32.002632
19. K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, "Beam dynamics in PT symmetric optical lattices," *Phys Rev Lett*. **100** (10), 103904 (2008). https://doi.org/10.1103/PhysRevLett.100.103904
20. C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, "Observation of parity-time symmetry in optics," *Nat. Phys*. **6**, 192 (2010). https://doi.org/10.1038/nphys1515
21. A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, "Parity-time synthetic photonic lattices," *Nature* **488**, 167–171 (2012). https://doi.org/10.1038/nature11298
22. A. Regensburger, M.-A. Miri, C. Bersch, J. Näger, G. Onishchukov, D. N. Christodoulides, and U. Peschel, "Observation of defect states in PT-symmetric optical lattices," *Phys. Rev. Lett*. **110** (22), 223902 (2013). https://doi.org/10.1103/PhysRevLett.110.223902
23. A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, "Observation of defect states in PT-symmetric optical lattices," *Phys. Rev. Lett*. **103** (9), 093902 (2009). https://doi.org/10.1103/PhysRevLett.103.093902
24. S. Longhi, "Bloch oscillations in complex crystals with PT symmetry," *Phys. Rev. Lett*. **103** (12), 123601 (2009). https://doi.org/10.1103/PhysRevLett.103.123601
25. Y. D. Chong, L. Ge, and A. D. Stone, "PT-symmetry breaking and laser-absorber modes in optical scattering systems," *Phys. Rev. Lett*. **106** (9), 093902 (2011). https://doi.org/10.1103/PhysRevLett.106.093902
26. S. Longhi, "PT-symmetric laser absorber," *Phys. Rev. A* **82** (3), 031801 (2010). https://doi.org/10.1103/PhysRevA.82.031801
27. H. Benisty, A. Degiron, A. Lupu, A. De Lustrac, S. Chénais, S. Forget, M. Besbes, G. Barbillon, A. Bruyant, S. Blaize, and G. Lérondel, "Implementation of PT symmetric devices using plasmonics: Principle and applications," *Opt Express* **19** (19), 18004–18019 (2011). https://doi.org/10.1364/OE.19.018004
28. M. Kulishov, J. M. Laniel, N. Bélanger, J. Azaña, and D. V. Plant, "Nonreciprocal waveguide Bragg gratings," *Opt Express* **13** (8), 3068–3078 (2005). https://doi.org/10.1364/opex.13.003068
29. Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, "Unidirectional invisibility induced by PT-symmetric periodic structures," *Phys. Rev. Lett*. **106** (21), 213901 (2011). https://doi.org/10.1103/PhysRevLett.106.213901
30. J. Sheng, M.-A. Miri, D. N. Christodoulides, and M. Xiao, "PT-symmetric optical potentials in a coherent atomic medium," *Phys. Rev. A* **88** (4), 041803(R) (2013). https://doi.org/10.1103/PhysRevA.88.041803
31. V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).
32. V. Ramesh Kumar, R. Radha and M. Wadati, "Phase engineering and solitons of Bose–Einstein condensates with two- and three-body interactions," *J. Phys. Soc. Jpn*. **79** (7), 074005 (2010). https://doi.org/10.1143/JPSJ.79.074005
33. S. Loomba, R. Pal, and C. N. Kumar, "Bright solitons of the nonautonomous cubic-quintic nonlinear Schrödinger equation with sign-reversal nonlinearity," *Phys. Rev. A* **92** (3), 033811 (2015). https://doi.org/10.1103/PhysRevA.92.033811
34. U. Al Khawaja and H. Bahlouli, "Integrability conditions and solitonic solutions of the nonlinear Schrödinger equation with generalized dual-power nonlinearities, PT-symmetric potentials, and space- and time-dependent coefficients," *Commun. Nonlinear Sci. Numer. Simul*. **69**, 248–260 (2019). https://doi.org/10.1016/j.cnsns.2018.07.015
35. R. Radha and P. S. Vinayagam, "Stabilization of matter wave solitons in weakly coupled atomic condensates," *Phys. Lett. A* **376** (8–9), 944–949 (2012). https://doi.org/10.1016/j.physleta.2012.01.029
36. R. Radha, P. S. Vinayagam, and K. Porsezian, "Soliton dynamics of spatially coupled vector BECs," *Rom. Rep. Phys*. **66** (2), 427–442 (2014). http://rrp.infim.ro/2014_66_2/A15.pdf
37. R. Radha and P. S.Vinayagam, "An analytical window into the world of ultracold atoms," *Rom. Rep. Phys*. **67** (1), 89–142 (2015). http://www.rrp.infim.ro/2015_67_1/A5.pdf

38. U. Al. Khawaja, P. S. Vinayagam, and S. M. Al-Marzoug, "Enhanced mobility of discrete solitons in anisotropic two-dimensional waveguide arrays with modulated separations," *Phys. Rev. A* **97** (2), 023820 (2018). https://doi.org/10.1103/PhysRevA.97.023820
39. P. S. Vinayagam, A. Javed, U. and Al Khawaja, "Stable discrete soliton molecules in two-dimensional waveguide arrays," *Phys. Rev. A* **98** (6), 063839 (2018). https://doi.org/10.1103/PhysRevA.98.063839
40. R. Radha, P. S. Vinayagam, H. J. Shin, and K. Porsezian, "Spatiotemporal binary interaction and designer quasi-particle condensates," *Chin. Phys. B* **23** (3), 034214 (2014). https://doi.org/10.1088/1674-1056/23/3/034214
41. P. S. Vinayagam, R. Radha, S. Bhuvaneswari, R. Ravisankar, and P. Muruganandam, "Bright soliton dynamics in spin orbit-Rabi coupled Bose–Einstein condensates," *Commun. Nonlinear Sci. Numer. Simul.* **50**, 68–76 (2017). https://doi.org/10.1016/j.cnsns.2017.02.012
42. H. Chaachoua Sameut, Sakthivinayagam Pattu, U. Al Khawaja, M. Benarous, and H. Belkroukra, "Peregrine soliton management of breathers in two coupled Gross–Pitaevskii equations with external potential," *Phys. Wave. Phenom*. **28** (3), 305–312 (2020). https://doi.org/10.3103/S1541308X20030036
43. P. Sakthivinayagam and J. Chen, "PT symmetric cubic-quintic nonlinear Schrödinger equation with dual power nonlinearities and its solitonic solutions," *Optik* **217**, 164665 (2020). https://doi.org/10.1016/j.ijleo.2020.164665