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Controllability and qualitative property results for Ambartsumian equation via Ξ- Hilfer generalized proportional fractional derivative on time scales

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Abstract. In this study, we discuss the existence, uniqueness, and controllability results for an Ambartsumian equation with impulses using the Ξ-Hilfer generalized proportional fractional derivative (PFD). Common fixed point theorems are employed in our analysis. We achieve a few suitable conditions for existence, uniqueness, and controllability.

1 Prerequisites

In this section, we introduce the Ξ-Hilfer generalized PFD. For that we begin by recalling the notion of time scales as discussed in [3, 4, 17].

Let $0 \leq a < b < \infty$, $J = [a,b]$ be a finite interval and ϑ be a parameter such that $n-1 \leq \vartheta < n$. $C[a,b]$ be the space of the continuous functions $\mathbb{Q}$ on J with the norm defined by

$$\|\mathbb{Q}\|_{C[a,b]} = \max_{t \in J} |\mathbb{Q}(t)|.$$

The weighted space $C^n_{\vartheta,\Xi}[a,b]$ of functions $\mathbb{Q}$ on $[a,b]$ is defined by

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$$C^n_{\vartheta,\Xi}[a,b] = \left\{\mathbb{Q} : [a,b] \to \mathbb{R}; \mathbb{Q}(t) \in C^{n-1}[a,b]; \mathbb{Q}^n(t) \in C_{\vartheta,\Xi}[a,b]\right\},$$

with the norm defined by,

$$\|\mathbb{Q}\|_{C^n_{\vartheta,\Xi}[a,b]} = \sum_{k=0}^{n-1} \|\mathbb{Q}^k\|_{C[a,b]} + \|\mathbb{Q}^n\|_{C_{\vartheta,\Xi}[a,b]}.$$

Definition 1.1. [12, 16] A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, one defines the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s > t\}.$$

If $\max \mathbb{T}$ is finite and there exists a finite $\min \mathbb{T}$ in addition, we put $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ and $\rho(\min \mathbb{T}) = \min \mathbb{T}$.

If $\sigma(t) > t$ then we say that t is right-scattered, while $\rho(t) < t$ then we say that t is left-scattered, and also if $t < \max \mathbb{T}$ and $\sigma(t) = t$ then t is called right-dense, and if $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called right-dense.

Definition 1.2. [12, 16](Delta derivative) Suppose that $\mathbb{Q} : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$. The delta derivative defined by

$$\mathbb{Q}^{\triangle}(t) = \lim_{s \to t} \frac{\mathbb{Q}(\sigma(s)) - \mathbb{Q}(t)}{\sigma(s) - t}, \quad t \neq \sigma(s).$$

Definition 1.3. [12, 16] Let $[a,b]$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a,b] \to \mathbb{R}$ is called an delta anti-derivative of function $f : [a,b) \to \mathbb{R}$ provided F is continuous on $[a,b]$, delta differentiable on $[a,b)$ and $F^{\triangle}(t) = f(t)$ for all $t \in [a,b)$. Then we defined the $\triangle$-integral by

$$\int_a^b f(t)\triangle(t) = F(b) - F(a).$$

Proposition 1.1. [12, 16] Suppose $a, b \in \mathbb{T}, a < b$ and $\mathbb{Q}(t)$ is continuous on $[a,b]$. Then,

$$\int_a^b \mathbb{Q}(t)\triangle t = [\sigma(a) - a]\mathbb{Q}(a) + \int_{\sigma(a)}^b \mathbb{Q}(t)\triangle t.$$

Proposition 1.2. [12, 16] Let $\mathbb{T}$ is a time scale and $\mathbb{Q}$ is an increasing continuous function on $[a,b]$. If Q is the extension of $\mathbb{Q}$ to the real interval $[a,b]$ given by

$$Q(s) = \begin{cases} \mathbb{Q}(s) & s \in \mathbb{T}, \\ \mathbb{Q}(t) & s \in (t, \sigma(t)) \not\subset \mathbb{T}, \end{cases}$$

then,

$$\int_a^b \mathbb{Q}(t)\triangle t \leq \int_a^b Q(t).$$

Definition 1.4. [12, 16] (Ξ-Hilfer generalized PFD) Let $J=[a,b]$, where $-\infty \leq a < b \leq \infty$ be an interval and $\mathbb{Q},\Xi \in C^n[a,b]$ be two functions such that Ξ is positive strictly increasing and $\Xi'(t) \neq 0, \forall t \in [a,b]$. The Ξ-Hilfer generalized PFD of order p and type q of $\mathbb{Q}$ with respect to the another function Ξ are defined by

$$\left(\mathcal{D}_{a^{\pm}}^{p,q,\rho,\Xi}\mathbb{Q}\right)(t)=\left(I_{a^{\pm}}^{q(n-p),\rho,\Xi}\left(\mathcal{D}^{n,\rho,\Xi}\right)\mathcal{D}_{a^{\pm}}^{(1-q)(n-p),\rho,\Xi}\mathbb{Q}\right)(t), \tag{1.1}$$

where $n-1<p<n, 0 \leq q \leq 1$ with $n \in \mathbb{N}$ and $\rho \in (0,1]$. Also $\mathcal{D}^{\rho,\Xi}\mathbb{Q}(t)=(1-\rho)\mathbb{Q}(t)+\rho\frac{\mathbb{Q}'(t)}{\Xi'(t)}$ and $I_{a^{\pm}}^{q(n-p),\rho,\Xi}(.)$ is the generalized proportional fractional integral.

For more details about fractional calculus, one can refer to [1, 5, 10, 14, 15, 18].

2 Main results

In this section, we introduce the Ξ-Hilfer generalized PFD on time scales. The subject of fractional calculus on time scales is very rich and under strong current research [2, 6, 7, 8, 9, 11, 13].

Definition 2.1. Let $\mathbb{T}$ is a time scale and let $J=[a,b]$, where $-\infty \leq a < b \leq \infty$ be an interval and $\mathbb{Q},\Xi \in C^n[a,b]$ be two functions such that Ξ is positive strictly increasing and $\Xi^{\triangle}(t) \neq 0, \forall t \in [a,b]$. Then the Ξ-Hilfer generalized proportional fractional integral on time scales of order $p(0<p<1)$ of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by,

$$\left({}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\right)(t)=\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}(s)\triangle s, \quad t>a. \tag{2.1}$$

Definition 2.2. Let $\mathbb{T}$ is a time scale and let $J=[a,b]$, where $-\infty \leq a < b \leq \infty$ be an interval and $\mathbb{Q},\Xi \in C^n[a,b]$ be two functions such that Ξ is positive strictly increasing and $\Xi^{\triangle}(t) \neq 0, \forall t \in [a,b]$. Then the Ξ-Hilfer generalized PFD on time scales of order $p(0<p<1)$ of the function $\mathbb{Q}$ with respect to the another function Ξ is defined by,

$$\left({}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}\right)(t)=\left({}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi}\left({}^{\mathbb{T}}\triangle^{n,\rho,\Xi}\right){}^{\mathbb{T}}\triangle_{a^+}^{(1-q)(n-p),\rho,\Xi}\mathbb{Q}\right)(t), \tag{2.2}$$

where $n-1<p<n, 0 \leq q \leq 1$ with $n \in \mathbb{N}$ and $\rho \in (0,1]$.

Remark 2.1. From the Definition 2.2, we can obtain the Riemann-Liouville and Caputo fractional derivative as follows

$$\left({}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}\right)=\begin{cases}{}^{\mathbb{T}}\triangle^{n,\rho,\Xi\mathbb{T}}I_{a^+}^{(n-p),\rho,\Xi}\mathbb{Q}(t) & q=0,\\ {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi}\left({}^{\mathbb{T}}\triangle^{n,\rho,\Xi}\right) & q=1.\end{cases}$$

Remark 2.2. The Ξ-Hilfer generalized PFD on time scales ${}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}$ is equivalent to

$$\begin{aligned}{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t)&=\left({}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi}\left({}^{\mathbb{T}}\triangle^{n,\rho,\Xi}\right){}^{\mathbb{T}}\triangle_{a^+}^{(1-q)(n-p),\rho,\Xi}\mathbb{Q}\right)(t)\\ &={}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi}\left({}^{\mathbb{T}}\triangle^{n,\rho,\Xi}\right)\mathbb{Q}(t),\end{aligned}$$

where $\vartheta=p+q(n-p)$.

Proposition 2.1. For any integrable function $\mathbb{Q}$ on $[a,b]$, $\rho \in (0,1], Re(p) > 0$ and $Re(q) > 0$ the Ξ-Hilfer generalized proportional fractional integral on time scales satisfies

$$^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\mathbb{Q}\right)(t) = ^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\right)(t) = \left(^{\mathbb{T}}I_{a^+}^{p+q,\rho,\Xi}\mathbb{Q}\right)(t).$$

Proof.

$$\begin{aligned}
^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\right)\mathbb{Q}(t) &= \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\mathbb{Q}(s)\right)\triangle s\\
&= \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\\
&\left(\frac{1}{\rho^q\Gamma(q)}\int_{a^+}^{s} e^{\frac{\rho-1}{\rho}(\Xi(\tau)-\Xi(s))}(\Xi(\tau)-\Xi(s))^{q-1}\Xi^{\triangle}(\tau)\mathbb{Q}(\tau)\right)\triangle s\\
&= \frac{1}{\rho^p\rho^q\Gamma(p)\Gamma(q)}\int_{a^+}^{t}\int_{a^+}^{s} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\\
&e^{\frac{\rho-1}{\rho}(\Xi(\tau)-\Xi(s))}(\Xi(\tau)-\Xi(s))^{q-1}\Xi^{\triangle}(\tau)\mathbb{Q}(\tau)\triangle s.
\end{aligned}$$

Now, we interchange the order of integration, then we get

$$\begin{aligned}
^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\right)\mathbb{Q}(t) &= \frac{1}{\rho^{p+q}\Gamma(p)\Gamma(q)}\int_{a^+}^{t}\left(\int_{a^+}^{s} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\right.\\
&\left. e^{\frac{\rho-1}{\rho}(\Xi(\tau)-\Xi(s))}(\Xi(\tau)-\Xi(s))^{q-1}\Xi^{\triangle}(\tau)\triangle s\right)\mathbb{Q}(\tau)\triangle\tau.
\end{aligned}$$

Now we making the change of variable $r = \frac{\Xi(s)-\Xi(\tau)}{\Xi(t)-\Xi(\tau)}$, $r \in \mathbb{R}$ and we have $dr = \Xi^{\triangle}(s)\triangle s$ and when $s \to \tau$ gives $r \to 0$ and if $s \to t$ gives $r \to 1$. Hence we get,

$$\begin{aligned}
^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\right)\mathbb{Q}(t) &= \frac{1}{\rho^{p+q}\Gamma(p)\Gamma(q)}\int_{a^+}^{t}\left(\int_{\tau}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}\Xi^{\triangle}(s)\left(1-\frac{\Xi(s)-\Xi(\tau)}{\Xi(t)-\Xi(\tau)}\right)^{p-1}\right.\\
&\left.(\Xi(t)-\Xi(\tau))^{p-1}.e^{\frac{\rho-1}{\rho}(\Xi(s)-\Xi(\tau))}(\Xi(s)-\Xi(\tau))^{q-1}\Xi^{\triangle}(\tau)\triangle s\right)\mathbb{Q}(\tau)\triangle\tau,\\
&= \frac{\beta(p,q)}{\rho^{p+q}\Gamma(p)\Gamma(q)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(\tau))}(\Xi(t)-\Xi(\tau))^{p+q-1}\Xi^{\triangle}(\tau)\mathbb{Q}(\tau)\triangle\tau,\\
&= \frac{1}{\rho^{p+q}\Gamma(p+q)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(\tau))}(\Xi(t)-\Xi(\tau))^{p+q-1}\Xi^{\triangle}(\tau)\mathbb{Q}(\tau)\triangle\tau.
\end{aligned}$$

$$^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\mathbb{Q}\right)(t) = ^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi}\left(^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\right)(t) = \left(^{\mathbb{T}}I_{a^+}^{p+q,\rho,\Xi}\mathbb{Q}\right)(t).$$

Lemma 2.1. Let $n-1 \leq \vartheta < n, n-1 < p < n$ with $n \in \mathbb{N}, \rho \in (0,1]$. If $\mathbb{Q} \in C_{\vartheta}[a,b]$ then

$$^{\mathbb{T}}I_{a^+}^{p,q,\Xi}\mathbb{Q}(a) = \lim_{t\to a^+} {}^{\mathbb{T}}I_{a^+}^{p,q,\Xi}\mathbb{Q}(t) = 0, \quad n-1 \leq \vartheta < p. \tag{2.3}$$

Proof. Considering $\mathbb{Q} \in C_{\vartheta}[a,b]$, then $(\Xi(t)-\Xi(a))^{\vartheta}\mathbb{Q}(t)$ is continuous on $[a,b]$ and there exists M such that $M > 0$ for which

$$\left|(\Xi(t)-\Xi(a))^{\vartheta}\mathbb{Q}(t)\right| < M \implies |\mathbb{Q}(t)| < \left|(\Xi(t)-\Xi(a))^{-\vartheta}\right|M, \tag{2.4}$$

where, $t \in [a,b]$ and $M > 0$ is a constant.
Now applying ${}^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}$ on both sides of Eq.(2.4), we get

$$\left|{}^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}\mathbb{Q}(t)\right| < \left|{}^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{-\vartheta}\right| M$$
$$\left|{}^{\mathbb{T}}I_{a^+}^{q,\rho,\Xi} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}\mathbb{Q}(t)\right| < M\left[\frac{\Gamma(n-\vartheta)}{\rho^p\Gamma(p+n-\vartheta)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{-\vartheta}\right],$$

if $p > \vartheta$,

$$M\left[\frac{\Gamma(n-\vartheta)}{\rho^p\Gamma(p+n-\vartheta)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{-\vartheta}\right] \to 0 \quad as \quad t \to a^+.$$

Lemma 2.2. Let $n-1<p<n, \vartheta \in (0,1], 0 \le q \le 1$ with $n \in \mathbb{N}$ and $\vartheta = p+q(n-p)$. If $\mathbb{Q} \in C^{\vartheta}_{n-\vartheta}[a,b]$ then

$$\begin{aligned}
{}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi}\mathbb{Q} &= {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q} \quad and \\
{}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} &= {}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}.
\end{aligned}$$

Proof.

$$\begin{aligned}
{}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi}\mathbb{Q} &= {}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{-q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}\right) \\
&= {}^{\mathbb{T}}I_{a^+}^{p+q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{-q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}.
\end{aligned}$$

$$\begin{aligned}
{}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} &= {}^{\mathbb{T}}\triangle^{n,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}\triangle^{n,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{n-q(n-p),\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}.
\end{aligned}$$

Lemma 2.3. Let $\mathbb{Q} \in \mathcal{L}_1(a,b)$. If there exists ${}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}$ in $\mathcal{L}_1(a,b)$, then

$${}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} = {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}.$$

Proof.

$$\begin{aligned}
{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} &= {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi}\left({}^{\mathbb{T}}\triangle_{a^+}^{n,\rho,\Xi}{}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\right){}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{n,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{n-q(n-p),\rho,\Xi}\mathbb{Q} \\
&= {}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}.
\end{aligned}$$

Lemma 2.4. Assume $n-1<p<n$ for $n \in \mathbb{N}, \rho \in (0,1], 0 \le q \le 1$ and $\vartheta = p+q(n-p)$. If $\mathbb{Q} \in C^n_{n-\vartheta}[a,b]$, then ${}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}$ exists in $(a,b]$ and

$${}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}(t) = \mathbb{Q}(t), \quad t \in (a,b].$$

Proof.

$$
\begin{aligned}
{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi} \quad {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}(t) &= \left({}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{q(n-p),\rho,\Xi}\mathbb{Q}\right)(t) \\
&= \mathbb{Q}(t) - \sum_{k=1}^{n} \frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{q(n-p)-k}}{\rho^{q(n-p)-k}\Gamma(q(n-p)-k+1)} I_{a^+.}^{k-q(n-p),\rho,\Xi}(a) \\
&= \mathbb{Q}(t).
\end{aligned}
$$

Theorem 2.1. Assume $n-1<p<n$ for $n\in\mathbb{N},\rho\in(0,1],0\leq q\leq 1$ and $\vartheta=p+q(n-p)$. If $\mathbb{Q}$ $C^n[a,b]$, then

$$
{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) = {}^{\mathbb{T}}\triangle_{a^+}^{n-q(n-p),\rho,\Xi}\left[{}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\rho^k k!}{}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi}\mathbb{Q}(a)\right],
$$

where, $\vartheta=p+q(k-p)$.

Proof. Let $f(t)={}^{\mathbb{T}}I_{a^+}^{(1-q)(n-p),\rho,\Xi}\mathbb{Q}(t)$ and $\zeta=n-q(n-p)$,

$$
\begin{aligned}
{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) &= {}^{\mathbb{T}}_{C}\triangle_{a^+}^{\zeta,\rho,\Xi}f(t) \\
&= {}^{\mathbb{T}}\triangle_{a^+}^{\zeta,\rho,\Xi}\left[f(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\rho^k k!}{}^{\mathbb{T}}\triangle_{a^+}^{\zeta,\rho,\Xi}f(a)\right] \\
&= {}^{\mathbb{T}}\triangle_{a^+}^{\zeta,\rho,\Xi}\left[{}^{\mathbb{T}}I_{a^+}^{(1-q)(n-p),\rho,\Xi}f(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\rho^k k!}\right. \\
&\quad \left.\times\left\{{}^{\mathbb{T}}\triangle_{a^+}^{\zeta,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{(1-q)(n-p),\rho,\Xi}\right)f(a)\right\}\right] \\
&= {}^{\mathbb{T}}\triangle_{a^+}^{n-q(n-p),\rho,\Xi}\left[{}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^k}{\rho^k k!}{}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi}\mathbb{Q}(a)\right].
\end{aligned}
$$

Lemma 2.5. Assume $n-1<p<n$ for $n\in\mathbb{N},\rho\in(0,1],0\leq q\leq 1$ and $\vartheta=p+q(n-p)$ such tha $n-1<\vartheta<n$. If $\mathbb{Q}\in C_{\vartheta}^n[a,b]$ and ${}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\in C_{\vartheta,\Xi}^n[a,b]$ then

$$
{}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) = \mathbb{Q}(t) - \sum_{k=0}^{n-1}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-k}}{\rho^{\vartheta-k}\Gamma(\vartheta-k+1)}{}^{\mathbb{T}}I_{a^+}^{k-\vartheta,\rho,\Xi}\mathbb{Q}(a). \qquad (2.5
$$

Proof.

$$
\begin{aligned}
{}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) &= {}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{q(n-p),\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}\right)(t) \\
&= {}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi} \quad {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t).
\end{aligned}
$$

$$\begin{aligned}
{}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi}\ {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) &= \frac{1}{\rho^{\vartheta}\Gamma(\vartheta)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\Xi^{\triangle}(s)\cdot{}^{\mathbb{T}}\triangle_{a^+}^{\vartheta,\rho,\Xi}\mathbb{Q}(s)\triangle s \\
&= \frac{1}{\rho^{\vartheta}\Gamma(\vartheta)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\Xi^{\triangle}(s) \\
&\quad \left\{{}^{\mathbb{T}}\triangle_{a^+}^{n,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(s)\right\}\triangle s \\
&= \frac{1}{\rho^{\vartheta}\Gamma(\vartheta)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\Xi^{\triangle}(s)\quad {}^{\mathbb{T}}\triangle_{a^+}^{1,\rho,\Xi} \\
&\quad \left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(s)\right\}\triangle s.
\end{aligned}$$

By using integration by parts, we obtain

$$\begin{aligned}
&{}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi}\ {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) \\
&= -\frac{1}{\rho^{\vartheta-1}\Gamma(\vartheta)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{\vartheta-1}\left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(a)\right\} \\
&+ \frac{1}{\rho^{\vartheta-1}\Gamma(\vartheta-1)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi()s)^{\vartheta-2}\Xi^{\triangle}(s)\quad {}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi} \\
&\left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(s)\right\}\triangle s.
\end{aligned}$$

If we proceed the above steps $(n-1)$ times, we get

$$\begin{aligned}
&{}^{\mathbb{T}}I_{a^+}^{\vartheta,\rho,\Xi}\ {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathbb{Q}(t) \\
&= \frac{1}{\rho^{\vartheta-n}\Gamma(\vartheta-n)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi()s)^{\vartheta-(n-1)}\Xi^{\triangle}(s)\quad {}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi} \\
&\left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-1,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(s)\right\}\triangle s \\
&-\sum_{k=1}^{n}\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-k}}{\rho^{\vartheta-k}\Gamma(\vartheta-k+1)}\left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-k,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(a)\right\} \\
&= \mathbb{Q}(t)-\sum_{k=1}^{n}\frac{1}{\rho^{\vartheta-k}\Gamma(\vartheta-k+1)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-k} \\
&\left\{{}^{\mathbb{T}}\triangle_{a^+}^{n-k,\rho,\Xi}\left({}^{\mathbb{T}}I_{a^+}^{n-\vartheta,\rho,\Xi}\mathbb{Q}\right)(a)\right\}.
\end{aligned}$$

2.1 Equivalence relation between the generalized Cauchy problem and the Volterra integral equation

Let us consider the non-linear Ξ-Hilfer generalized proportional fractional differential equation

$$ {}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathcal{A}(t) = \mathbb{Q}\left(t,\mathcal{A}(t),\mathcal{A}\left(\frac{t}{\eta}\right)\right),\quad t\in J,\quad \eta>1,\quad b>a\geq 0. \tag{2.6}$$

$$ {}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(t) = \sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i).\quad \mu_i\in\mathbb{R},\quad \tau_i\in J. \tag{2.7}$$

where, $\mathbb{Q}\left(t,\mathcal{A}(t),\mathcal{A}\left(\frac{t}{\eta}\right)\right)=\frac{1}{\eta}\mathcal{A}\left(\frac{t}{\eta}\right)-\mathcal{A}(t)$.
$0<p<1, 0\leq q\leq 1$ and $\mathbb{Q}: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Moreover, we will add a operator B and the control function u into the problem 2.6-2.7

$$\begin{cases}{}^{\mathbb{T}}\triangle_{a^+}^{p,q,\rho,\Xi}\mathcal{A}(t)=\mathbb{Q}\left(t,\mathcal{A}(t),\mathcal{A}\left(\frac{t}{\eta}\right)\right)+Bu(t)\\ {}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(t)=\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i).\end{cases}\tag{2.8}$$

Now, we have to show that the equivalence relation between the Cauchy problem 2.8 and the Volterra integral equation,

$$\begin{aligned}\mathcal{A}(t)&=\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\&\quad\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\&\quad+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s,\end{aligned}\tag{2.9}$$

where,

$$\wedge=\frac{1}{\rho^{\vartheta-1}\Gamma(\vartheta)-\sum_{i=1}^{m}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}}.$$

Lemma 2.6. Let $0<p<1, 0\leq q\leq 1, \vartheta=p+q(1-p)$ and assume that $\mathbb{Q}\left(.,\mathcal{A}(.),\mathcal{A}\left(\frac{\cdot}{\eta}\right)\right)\in C_{1-\vartheta}[a,b]$ for any $\mathcal{A}\in C_{1-\vartheta}[a,b]$ where $\mathbb{Q}:(a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function. If $\mathcal{A}\in C_{1-\vartheta}^{\vartheta}[a,b]$ then $\mathcal{A}$ satisfies our proposed problem Eq.(2.6)-Eq.(2.7) if and only if $\mathcal{A}$ satisfies Eq.(2.9).

Proof. Let us consider $\mathcal{A}\in C_{1-\vartheta}^{\vartheta}[a,b]$ is a solution of our proposed problem Eq.(2.6)-Eq.(2.7).

Now we have to prove that $\mathcal{A}$ is a solution of (2.9). Now,

$$\left({}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\,{}^{\mathbb{T}}\mathcal{D}_{a^+}^{p,q,\rho,\Xi}\mathcal{A}\right)(t)=\mathcal{A}(t)-\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\rho^{\vartheta-1}\Gamma(\vartheta)}\left({}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}\right)(a),$$

which gives,

$$\begin{aligned}\mathcal{A}(t)&=\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}}{\rho^{\vartheta-1}\Gamma(\vartheta)}\left({}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}\right)(a)\\&\quad+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s.\end{aligned}\tag{2.10}$$

Let us take $t=\tau_i$ and multiplying μ_i on both sides of (2.10), we get

$$\begin{aligned}\mu_i\mathcal{A}(\tau_i)&=\frac{\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}}{\rho^{\vartheta-1}\Gamma(\vartheta)}\left({}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}\right)(a)\\&\quad+\frac{1}{\rho^p\Gamma(p)}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s,\end{aligned}$$

which implies that

$$\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i)=\frac{1}{\rho^{\vartheta-1}\Gamma(\vartheta)}\sum_{i=1}^{m}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}\left({}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}\right)(a)$$

$$+\frac{1}{\rho^{p}\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s, \tag{2.11}$$

where $\tau_i > a$.
Hence by (2.7), we get

$${}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(a)=$$

$$\frac{\rho^{\vartheta-1}\Gamma(\vartheta)}{\rho^{p}\Gamma(p)}\wedge\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s. \tag{2.12}$$

After substituting Eq.(2.12) in Eq.(2.10), we get the required result.
Hence $\mathcal{A}(t)$ satisfies Eq(2.9).
Conversely, suppose that $\mathcal{A}\in C_{1-\vartheta}^{\vartheta}[a,b]$ satisfies Eq.(2.9), we have to show that $\mathcal{A}(t)$ also satisfies Eq.(2.6)-Eq.(2.7). Now applying the operator ${}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}$ on both sides of the equation Eq.(2.9) we get,

$${}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\mathcal{A}(t)={}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\left(\frac{\wedge}{\rho^{p}\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\right.$$
$$\left.\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\right)$$

$$+{}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\left(\frac{1}{\rho^{p}\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\right). \tag{2.13}$$

By using the Proposition (1.1) , we get

$${}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\mathcal{A}(t)={}^{\mathbb{T}}\mathcal{D}_{a^+}^{q(1-p),\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t).$$

Since $\mathcal{A}\in C_{1-\vartheta}^{\vartheta}[a,b]$ and by the definition of weighted space $C_{1-\vartheta}^{\vartheta}[a,b]$ we get ${}^{\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\mathcal{A}\in C_{1-\vartheta}[a,b]$ and by the Eq.(2.13)we get

$${}^{\mathbb{T}}\mathcal{D}_{a^+}^{q(1-p),\rho,\Xi}\mathbb{Q}={}^{\mathbb{T}}\mathcal{D}^{1,\rho,\Xi\,\mathbb{T}}I^{1-q(1-p),\rho,\Xi}\mathbb{Q}\in C_{1-\vartheta,\Xi}[a,b].$$

And for $\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t)\in C_{1-\vartheta}[a,b]$, it gives that

$${}^{\mathbb{T}}I_{a^+}^{1-q(1-p),\rho,\Xi}\mathbb{Q}\in C_{1-\vartheta,\Xi}[a,b],$$

and from the definition of $C_{1-\vartheta,\Xi}^{n}[a,b]$, that

$$^{\mathbb{T}}I_{a^+}^{1-q(1-p),\rho,\Xi}\mathbb{Q}\in C^1_{1-\vartheta,\Xi}[a,b].$$

Now applying the operator $^{\mathbb{T}}I_{a^+}^{q(1-p),\rho,\Xi}$ on both sides of Eq.(2.13) ,we get

$$^{\mathbb{T}}I_{a^+}^{q(1-p),\rho,\Xi\,\mathbb{T}}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\mathcal{A}(t) = {}^{\mathbb{T}}I_{a^+}^{q(1-p),\rho,\Xi\,\mathbb{T}}\mathcal{D}_{a^+}^{q(1-p),\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t)$$

$$= \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right) - \frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}\,{}^{\mathbb{T}}I_{a^+}^{1-q(1-p),\rho,\Xi}\mathbb{Q}(a)}{\rho^{q(1-p)-1}\Gamma(q(1-p))}(\Xi(t)-\Xi(s))^{q(1-p)-1} \quad (2.14)$$

$$= \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right).$$

Hence,

$$I_{a^+}^{q(1-p),\rho,\Xi}\mathcal{D}_{a^+}^{\vartheta,\rho,\Xi}\mathcal{A}(t) = \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right).$$

Now, we have to prove that Eq.(2.17) holds. To prove the initial condition Eq.(2.17) holds, we hav to applying the operator $I_{a^+}^{1-\vartheta,\rho,\Xi}$ on the both sides of Eq.(2.9), we get

$$\begin{aligned}
^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(t) = {}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}&\left(\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\right.\\
&\left.\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\right)\\
&+{}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\\
&\left(\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\right).
\end{aligned}$$

And then using Proposition 1.1, we get

$$^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(t) = \frac{\rho^{\vartheta-1}\Gamma(\vartheta)}{\rho^p\Gamma(p)}\wedge e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}$$

$$(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s + {}^{\mathbb{T}}I_{a^+}^{1-q(1-p),\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)(t). \quad (2.15)$$

Since $1-\vartheta < q(1-p)$, so taking the limit as $t\to a^+$ in the Eq.(2.15), we get

$$^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(a^+)$$

$$= \frac{\rho^{\vartheta-1}\Gamma(\vartheta)}{\rho^p\Gamma(p)}\wedge\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\,\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s. \quad (2.16)$$

Next, substituting $t=\tau_i$ and multiply μ_i on both sides of Eq.(2.9), we get

$$\mu_i\mathcal{A}(\tau_i) = \frac{\wedge}{\rho^p\Gamma(p)}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}\left(\Xi(\tau_i)-\Xi(a)\right)^{\vartheta-1}$$
$$\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s$$
$$+\frac{\mu_i}{\rho^p\Gamma(p)}\int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s,$$

which gives that,

$$\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i) = \wedge\sum_{i=1}^{m}\mu_i\left({}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)\sum_{i=1}^{m}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}\left(\Xi(\tau_i)-\Xi(a)\right)^{\vartheta-1}$$
$$+\sum_{i=1}^{m}\mu_i\left({}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)$$
$$=\sum_{i=1}^{m}\mu_i\left({}^{\mathbb{T}}I_{a^+}^{p,\rho,\Xi}\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right)(\tau_i)\left(1+\wedge\sum_{i=1}^{m}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}\left(\Xi(\tau_i)-\Xi(a)\right)^{\vartheta-1}\right).$$

Thus,

$$\sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i) = \frac{\rho^{\vartheta-1}\Gamma(\vartheta)}{\rho^p\Gamma(p)}\wedge\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s. \tag{2.17}$$

Hence from Eq.(2.16) and Eq.(2.17), we get

$${}^{\mathbb{T}}I_{a^+}^{1-\vartheta,\rho,\Xi}\mathcal{A}(a^+) = \sum_{i=1}^{m}\mu_i\mathcal{A}(\tau_i), \tag{2.18}$$

this completes the proof.

3 Existence of solution

Now in our next theorem, we prove the existence and uniqueness of solution of (2.6)-(2.7) in the weighted space $C_{1-\vartheta,\Xi}^{p,q}$ by the concepts of Krasnoselskii's fixed point theorem.

Theorem 3.1. [12, 16](Krasnoselskii's fixed point theorem) Let B be a non empty bounded closed convex subset of a Banach space X. Let $N, M : B \to X$ be two continuos operators satisfying:

- $Nx + My \in B$ whenever $x, y \in B$,
- N is compact and continuous,
- M is contraction mapping,

then, there exist $u \in B$ such that $u = Nu + Mu$.

Let us consider the following hypotheses:
(H_1) : Let $\mathbb{Q} : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{Q} \in C_{1-\vartheta,\Xi}^{q(1-p)}[a,b]$ for any $\mathcal{A} \in C_{1-\vartheta,\Xi}^{\vartheta}[a,b]$.
(H_2) : There exists a constant $k > 0$ such that

$$|\mathbb{Q}(t,x,\omega) - \mathbb{Q}(t,y,\overline{\omega})| \leq k|x-y|\,|\omega,\overline{\omega}|, \quad \forall\omega-\overline{\omega} \in \mathbb{R} \quad \text{and} \quad t \in J.$$

(H_3) : Let us assume that

$$k\phi < 1,$$

where,

$$\phi = \frac{\beta(\vartheta,p)}{\rho^p\Gamma(p)}\left(|\wedge|\sum_{i=1}^{m}\mu_i\left(\Xi(\tau_i)-\Xi(a)\right)^{p+\vartheta-1}+\left(\Xi(b)-\Xi(a)\right)^p\right), \tag{3.1}$$

and

$$\beta(\vartheta,p)=\int_0^1 t^{\vartheta-1}(1-t)^{p-1}dt, \quad Re(\vartheta), \quad Re(p)>0,$$

is the Beta function.
(H_4): Also let

$$k\lambda < 1,$$

where,

$$\lambda = \frac{\beta(\vartheta,p)}{\rho^p\Gamma(p)}|\wedge|\sum_{i=1}^{m}\mu_i\left(\Xi(\tau_i)-\Xi(a)\right)^{p+\vartheta-1}. \tag{3.2}$$

(H_5): The linear operator $Wu : \mathcal{L}^2(J,\mathbb{R}) \to \mathbb{R}$ is defined by

$$\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}\left(\Xi(t)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\,(Bu)\,(s)\triangle s,$$

has bounded invertible operators $(W_\alpha u)^{-1}$, which takes the values in $\mathcal{L}^2(J,\mathbb{R})$ and there exists a positive constant M_W such that $\left\|(W_\alpha u)^{-1}\right\| \le M_W$.

Theorem 3.2. Let $0 < p < 1, 0 \le q \le 1$ and $\vartheta = p+q(1-p)$. Suppose that the assumptions $(H_1),(H_2),(H_4)$ hold. Then the problem (2.6)-(2.7) has at least one solution in the space $C^{\vartheta}_{1-\vartheta}[a,b]$.

Proof. Given that $\|X\|_{C_{1-\vartheta,\Xi}[a,b]} = \sup_{t\in J}\left|\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}X(t)\right|$ and choose $k \ge M\|X\|_{C_{1-\vartheta,\Xi}[a,b]}$, where

$$M = \frac{\beta(\vartheta,p)}{\rho^p\Gamma(p)}\left(|\wedge|\sum_{i=1}^{m}\mu_i\left(\Xi(\tau_i)-\Xi(a)\right)^{p+\vartheta-1}+\left(\Xi(b)-\Xi(a)\right)^p\right), \tag{3.3}$$

also consider $B_k = \left\{\mathcal{A}\in C[a,b] : \|\mathcal{A}\|_{C_{1-\vartheta}[a,b]} \le k\right\}$. Thus $\forall t\in[a,b]$ consider the operators $\mathcal{G}$ and $\mathcal{H}$ defined on B_k by

$$(\mathcal{H}\mathcal{A})(t) = \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}\left(\Xi(t)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s,$$

$$(\mathcal{G}\mathcal{A})(t) = \frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}\left(\Xi(t)-\Xi(a)\right)^{\vartheta-1}$$
$$\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s.$$

Step1. For all $\mathcal{A},\overline{\mathcal{A}}\in B_k$, yields

$$\begin{aligned}
&\left|\left(\mathcal{H}\mathcal{A}(t)+\mathcal{G}\overline{\mathcal{A}}(t)\right)\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}\right| \\
&\leq \frac{\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}}{\rho^p\Gamma(p)}\int_{a^+}^{t}\left(\Xi(t)-\Xi(s)\right)^{p-1}\left(\Xi(s)-\Xi(a)\right)^{\vartheta-1}\Xi^{\triangle}(s) \\
&\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\left(\Xi(s)-\Xi(a)\right)^{\vartheta-1}\right|\triangle s \\
&+\frac{\wedge}{\rho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\frac{\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}}{\rho^p\Gamma(p)}\int_{a^+}^{\tau_i}\left(\Xi(t)-\Xi(s)\right)^{p-1}\left(\Xi(s)-\Xi(a)\right)^{\vartheta-1} \\
&\left|\mathbb{Q}\left(s,\overline{\mathcal{A}}(s),\overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right)\left(\Xi(\tau_i)-\Xi(a)\right)^{\vartheta-1}\right|\triangle s \\
&\leq \|X\|\left[\frac{\beta(\vartheta,p)}{\rho^p\Gamma(p)}\left(|\wedge|\sum_{i=1}^{m}\mu_i\left(\Xi(\tau_i)-\Xi(a)\right)^{p+\vartheta-1}+\left(\Xi(b)-\Xi(a)\right)^p\right)\right] \\
&\leq \|X\|M \\
&\leq \rho<\infty,
\end{aligned}$$

this implies that $\mathcal{H}\mathcal{A}+\mathcal{H}\overline{\mathcal{A}}\in B_k$.

Step2. We show that M is a contraction. Let $\mathcal{A},\overline{\mathcal{A}}\in C_{1-\vartheta}[a,b]$ and $t\in J$ then

$$\begin{aligned}
&\left|\left(\mathcal{G}\mathcal{A}(t)-\mathcal{G}\overline{\mathcal{A}}(t)\right)\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}\right| \\
&=\left|\wedge e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}\sum_{i=1}^{m}\mu_i I_{a^+}^{1-q(1-p),\rho,\Xi}\left(\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\overline{\mathcal{A}}(s),\overline{\mathcal{A}}\left(\frac{s}{\eta}\right)\right)\right)\right|\triangle s \\
&\leq \frac{2k\wedge}{\rho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}\left(\Xi(\tau_i)-\Xi(s)\right)^{p-1}\left(\Xi(s)-\Xi(a)\right)^{\vartheta-1}\Xi^{\triangle}(s)\left|\mathcal{A}(s)-\overline{\mathcal{A}}(s)\right|\triangle s \\
&\leq \left[\frac{2k\wedge}{\rho^p\Gamma(p)}\beta(\vartheta,p)\sum_{i=1}^{m}\mu_i\left(\Xi(\tau_i)-\Xi(s)\right)^{p+\vartheta-1}\right]\left\|\mathcal{A}-\overline{\mathcal{A}}\right\|_{C_{1-\vartheta,\Xi}[a,b]}
\end{aligned}$$

$$\left|\left(\mathcal{G}\mathcal{A}(t)-\mathcal{G}\overline{\mathcal{A}}(t)\right)\left(\Xi(t)-\Xi(a)\right)^{1-\vartheta}\right|\leq k\triangle\left\|\mathcal{A}-\overline{\mathcal{A}}\right\|_{C_{1-\vartheta,\Xi}[a,b]}. \tag{3.4}$$

Hence by (H_4) and Eq.(3.4), we can say M is a contraction.

Step3. Now, we have to verify that the operator N is continuous and compact.
Since the function $\mathbb{Q}$ is continuous, so the operator N is also continuous.
Hence, for any $\mathcal{A}\in C_{1-\vartheta}[a,b]$, we get

$$\left\|\mathcal{H}\mathcal{A}\right\|\leq\|X\|\frac{\beta(\vartheta,p)}{\rho^p\Gamma(p)}\left(\Xi(b)-\Xi(a)\right)^p<\infty.$$

This shows that $\mathcal{H}$ is uniformly bounded on B_k. Therefore it remains to prove that the operator $\mathcal{H}$ is compact. Denoting $\sup_{(t,\mathcal{A})\in J\times B_k}\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|=\delta<\infty$ and for any $a<\tau_1<\tau_2<b$,

$$\left|\left(\Xi(\tau_2)-\Xi(a)\right)^{1-\vartheta}\left(\mathcal{H}\mathcal{A}(\tau_2)\right)+\left(\Xi(\tau_1)-\Xi(a)\right)^{1-\vartheta}\left(\mathcal{H}\mathcal{A}(\tau_1)\right)\right|$$

$$= \left| \frac{(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{\tau_2} e^{\frac{\rho-1}{\rho}(\Xi(\tau_2)-\Xi(s))}(\Xi(\tau_2)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right.$$

$$\left. + \frac{(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{\tau_1} e^{\frac{\rho-1}{\rho}(\Xi(\tau_1)-\Xi(s))}(\Xi(\tau_1)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right|$$

$$\leq \frac{1}{\vartheta^p\Gamma(p)} \int_{\tau_1}^{\tau_2} \left[(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}(\Xi(\tau_2)-\Xi(s))^{p-1}(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\Xi(\tau_1)-\Xi(s))^{p-1}\right]$$

$$\times \Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|\triangle s$$

$$+ \frac{1}{\vartheta^p\Gamma(p)} \int_{\tau_1}^{\tau_2} (\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\Xi(\tau_1)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|\triangle s$$

$$\left|(\Xi(\tau_2)-\Xi(a))^{1-\vartheta}(\mathcal{H}\mathcal{A}(\tau_2))+(\Xi(\tau_1)-\Xi(a))^{1-\vartheta}(\mathcal{H}\mathcal{A}(\tau_1))\right| \to 0 \quad as \quad \tau_2 \to \tau_1.$$

As a consequence of Arzela-Ascoli theorem $\mathcal{H}$ is compact on B_k. Thus, as a result of our propose problem (2.6)-(2.7) has at least one solution.

4 Uniqueness of solution

Theorem 4.1. Let $0 < p < 1$, $0 \leq q \leq 1$ and $\vartheta = p + q(1-p)$. Suppose that the assumptions (H_2) - (H_3) hold, then the problem (2.6)-(2.7) has a unique solution in the space $C^{\vartheta}_{1-\vartheta,\Xi}[a,b]$.

Proof. Consider the fractional operator $\mathcal{T} : C_{1-\vartheta,\Xi}[a,b] \to C_{1-\vartheta,\Xi}[a,b]$ defined by:

$$(\mathcal{T}\mathcal{A})(t) = \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}$$

$$\times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s$$

$$+ \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s. \quad (4.1$$

Clearly the operator $\mathcal{T}$ is well defined. Now for any $\mathcal{A}_1, \mathcal{A}_2 \in C_{1-\vartheta}[a,b]$, $t \in J$ and $\left|e^{\frac{\rho-1}{\rho}\Xi(t)}\right| < 1$ gives

$$\left|((\mathcal{T}\mathcal{A}_1)(t)-(\mathcal{T}\mathcal{A}_2)(t))(\Xi(t)-\Xi(a))^{1-\vartheta}\right|$$

$$\leq \frac{\wedge}{\rho^p\Gamma(p)}\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}_1(s),\mathcal{A}_1\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}_2(s),\mathcal{A}_2\left(\frac{s}{\eta}\right)\right)\right|\triangle s$$
$$+\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)}\int_{a^+}^{t}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}_1(s),\mathcal{A}_1\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}_2(s),\mathcal{A}_2\left(\frac{s}{\eta}\right)\right)\right|\triangle s$$
$$\leq \frac{k\wedge}{\rho^p\Gamma(p)}\left(\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}(\Xi(\tau_i)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{\vartheta-1}\Xi^{\triangle}(s)\triangle s\right)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$$
$$+\frac{k(\Xi(t)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)}\left(\int_{a^+}^{t}(\Xi(t)-\Xi(s))^{p-1}(\Xi(s)-\Xi(a))^{1-\vartheta}\Xi^{\triangle}(s)\right)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}\cdot$$
$$\leq \frac{k\wedge}{\rho^p\Gamma(p)}\beta(\vartheta,p)\sum_{i=1}^{m}\mu_i(\Xi(\tau_i)-\Xi(s))^{p+\vartheta-1}\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$$
$$+\frac{k(\Xi(b)-\Xi(a))^{p}}{\rho^p\Gamma(p)}\beta(\vartheta,p)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}\cdot$$

Hence,
$\|\mathcal{T}\mathcal{A}_1-\mathcal{T}\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$

$$\leq \frac{k}{\rho^p\Gamma(p)}\beta(\vartheta,p)\left(|\wedge|\sum_{i=1}^{m}\mu_i(\Xi(\tau_i)-\Xi(s))^{p+\vartheta-1}+(\Xi(b)-\Xi(a))^{p}\right)\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}$$

$$\|\mathcal{T}\mathcal{A}_1-\mathcal{T}\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}\leq k\phi\|\mathcal{A}_1-\mathcal{A}_2\|_{C_{1-\vartheta,\Xi}[a,b]}\cdot \tag{4.2}$$

Thus, from (H_3) and (4.2), $\mathcal{T}$ is a contraction map. Hence by the theorem, our proposed problem (2.6)-(2.7) has a unique solution.

5 Controllability

In this part, we investigate the controllability of our proposed problem.

Definition 5.1. Suppose that for any given initial state $\mathcal{A}_0$ and any given final state $\overline{\mathcal{A}}$, there exists a piecewise right-dense continuous function $u\in\mathcal{L}^2(J,U)$ such that the solution $\mathcal{A}$ of satisfies $\mathcal{A}(t)=\overline{\mathcal{A}}$ then we can say that is controllable on J.

Definition 5.2. A function $\mathcal{A}\in C(J,\mathbb{R})$ is a solution of system if and only if this function is a solution of the following integral equation

$$\mathcal{A}(t)=\begin{cases}\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\ \times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s\\ +\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s\end{cases} \tag{5.1}$$

where,

$$\wedge=\frac{1}{\rho^{\vartheta-1}\Gamma(\vartheta)-\sum_{i=1}^{m}\mu_i e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(a))}(\Xi(\tau_i)-\Xi(a))^{\vartheta-1}},$$

$t\in[a,t]=J\subset\mathbb{T}$.

Lemma 5.1. Suppose that the assumptions $(H_1)-(H_5)$ are satisfied and $\mathcal{A}(t_0)$ be an arbitrary poin
Then the solution $\mathcal{A}(t)$ of a system (2.6)-(2.7) on $[a,t]$ is defined by the control function

$$u(t) = (W_\alpha)^{-1}\Bigg[\mathcal{A}_1 - \frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1} \times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s + \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\Bigg],$$

$t\in[t_0,t_0+a]$. The control function $u(t)$ has an estimate $\|u(t)\|\leq M_u^o$ with

$$M_u^o = |\mathcal{A}_1| + \frac{\wedge e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(t_0))}}{\rho^p\Gamma(p)}(\Xi(t)-\Xi(t_0))^{\vartheta-1} + \frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(t_0))}}{\rho^p\Gamma(p+1)}(\Xi(t)-\Xi(t_0))^p.$$

Proof. Consider the solution $\mathcal{A}(t)$ of system on $t\in[a,t]\subset\mathbb{T}$ defined by Eq.(5.1). Then

$$\begin{aligned}\mathcal{A}(t) &= \frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\ &\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s\\ &+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s\\ &= \frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\ &\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\end{aligned}$$

$$
\begin{aligned}
&+\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\
&\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)B(W_\alpha)^{-1}\\
&\Bigg[\mathcal{A}_1-\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\
&\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\Bigg]\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)B(W_\alpha)^{-1}\\
&\Bigg[\mathcal{A}_1-\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\
&\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\Bigg]\\
&=\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\
&\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+W(\alpha)(W_\alpha)^{-1}\Bigg[\mathcal{A}_1-\frac{\wedge}{\rho^p\Gamma(p)}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}\\
&\times\sum_{i=1}^{m}\mu_i\int_{a^+}^{\tau_i}e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\\
&+\frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t}e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\Bigg]\\
&=\mathcal{A}_1.
\end{aligned}
$$

Now ,we find the control estimate as follows

$$
\begin{aligned}
|u(t)| &= \Bigg|(W_\alpha)^{-1}\Bigg[\mathcal{A}_1 - \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1} \\
&\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \\
&+ \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s\Bigg]\Bigg| \\
&\leq \left|(W_\alpha)^{-1}\right|\Bigg[|\mathcal{A}_1| - \frac{|\wedge|}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1} \\
&\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|\triangle s \\
&+ \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\left|\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right|\triangle s\Bigg] \\
&\leq (W_\alpha)^{-1}\Bigg(|\mathcal{A}_1| + \frac{\wedge e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(t_0))}}{\rho^p\Gamma(p)}(\Xi(t)-\Xi(t_0))^{\vartheta-1} \\
&+ \frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(t_0))}}{\rho^p\Gamma(p+1)}(\Xi(t)-\Xi(t_0))^{p}\Bigg) \\
&\leq M_u^o.
\end{aligned}
$$

The proof is complete.

Theorem 5.1. Suppose that the hypotheses $(H_1)-(H_5)$ hold, then the controllable on J provided

$$
M\frac{e^{\frac{\rho-1}{\rho}(\Xi(t_0+a)-\Xi(t_0))}(\Xi(t_0+a)-\Xi(t_0))^{1-q(1-p)}}{\rho^p\Gamma(p+1)} < 1. \tag{5.2}
$$

Proof. Consider the subset $D_{\Xi,\delta} \subseteq C_{1-\vartheta,\Xi}(J,\mathbb{R})$ as follows

$$
D_{\Xi,\delta} = \left\{x \in C_{1-\vartheta,\Xi}(J,\mathbb{R}) : \|x\|_{C_{1-\vartheta,\Xi}} \leq \delta\right\}.
$$

Now define the operator $K : D_{\Xi,\delta} \to D_{\Xi,\delta}$ as

$$
\begin{aligned}
K\mathcal{A}(t) &= \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1} \\
&\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s \\
&+ \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)+(Bu)(s)\right)\triangle s.
\end{aligned}
$$

The operator K is well defined and the fixed points of K are solutions of Eq.(2.8) . Indeed $x \in D_{\Xi,\delta}$ is a solution of Eq.(2.8) if and only if, it is a solution of the operator equation $x = Kx$. Hence the existence of a solution of Eq.(2.8) is equivalent to determine a positive constant δ such that K has a fixed point on $D_{\Xi,\delta}$. Consider the operators K_1, K_2 defined on $D_{\Xi,\delta}$ by

$$K_1\mathcal{A}(t) = \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))} (\Xi(t)-\Xi(a))^{\vartheta-1}$$
$$\times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1} \Xi^{\triangle}(s) (Bu)(s) \triangle s$$
$$= \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi^{\triangle}(s) (Bu)(s) \triangle s,$$

and

$$K_2\mathcal{A}(t) = \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))} (\Xi(t)-\Xi(a))^{\vartheta-1}$$
$$\times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1} \Xi^{\triangle}(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \triangle s$$
$$+ \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi^{\triangle}(s) \mathbb{Q}\left(s, \mathcal{A}(s), \mathcal{A}\left(\frac{s}{\eta}\right)\right) \triangle s.$$

Step 1: The operator K_1 maps $D_{\Xi,\delta}$ into itself, it follows from Lemma 5.3 that,
$\left\|(\Xi(t)-\Xi(t_0))^{1-\vartheta}(K_1\mathcal{A})(t)\right\|$

$$= \left\| \frac{\wedge(\Xi(t)-\Xi(t_0))^{1-\vartheta}}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))} (\Xi(t)-\Xi(a))^{\vartheta-1}\right.$$
$$\times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1} \Xi^{\triangle}(s) (Bu)(s) \triangle s$$
$$\left.= \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi^{\triangle}(s) (Bu)(s) \triangle s\right\|$$
$$\leq \frac{\wedge(\Xi(t)-\Xi(t_0))^{1-\vartheta}}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))} (\Xi(t)-\Xi(a))^{\vartheta-1}$$
$$\times \sum_{i=1}^{m} \mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))} (\Xi(\tau_i)-\Xi(s))^{p-1} \Xi^{\triangle}(s) \|B\| \|(u)(s)\| \triangle s$$
$$= \frac{1}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1} \Xi^{\triangle}(s) \|B\| \|(u)(s)\| \triangle s$$
$$\leq \delta.$$

Thus K_1 maps $D_{\Xi,\delta}$ into itself.
Step 2: The operator K_2 is continuous. Let $\{\mathcal{A}_n\}$ be a sequence in $D_{\Xi,\delta}$ satisfying $\mathcal{A}_n \to \mathcal{A}$ as $n \to \infty$. Then,
$\left\|(\Xi(t)-\Xi(t_0))^{1-\vartheta}((K_2\mathcal{A}_n)(t)-(K_2\mathcal{A})(t))\right\|_C$

$$= \frac{\wedge(\Xi(t)-\Xi(t_0))^{1-\vartheta}}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{\vartheta-1}$$
$$\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)$$
$$\left\|\mathbb{Q}\left(s,\mathcal{A}_n(s),\mathcal{A}_n\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right\|\triangle s$$
$$= \frac{1}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)$$
$$\left\|\mathbb{Q}\left(s,\mathcal{A}_n(s),\mathcal{A}_n\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right\|\triangle s$$
$$\leq \left\|\mathbb{Q}\left(s,\mathcal{A}_n(s),\mathcal{A}_n\left(\frac{s}{\eta}\right)\right)-\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\right\|_C$$
$$\frac{(\Xi(t)-\Xi(a))^{1-q(1-p)}}{\rho^p\Gamma(p+1)}.$$

By the Lebesgue dominated convergence theorem, we know that $\|((K_2\mathcal{A}_n)(t)-(K_2\mathcal{A})(t))\|_C \to 0 \quad as \quad n\to\infty$. Hence K_2 is continuous.

Step 3: Now we show that $K_2(D_{\Xi,\delta})$. We prove this by contradiction, suppose that there exists a function $G(.)\in D_{\Xi,\delta}$ such that $\|K_2\mathcal{A}\|_C > \delta$.

$$\delta \leq \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}$$
$$\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,G(s),G\left(\frac{s}{\eta}\right)\right)\triangle s$$
$$+\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,G(s),G\left(\frac{s}{\eta}\right)\right)\triangle s$$
$$\leq \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}$$
$$\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,G(s),G\left(\frac{s}{\eta}\right)\right)ds$$
$$+\frac{(\Xi(t)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)}\int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))}(\Xi(t)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,G(s),G\left(\frac{s}{\eta}\right)\right)ds$$
$$\leq \frac{\wedge}{\rho^p\Gamma(p)} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}$$
$$\times \sum_{i=1}^{m}\mu_i \int_{a^+}^{\tau_i} e^{\frac{\rho-1}{\rho}(\Xi(\tau_i)-\Xi(s))}(\Xi(\tau_i)-\Xi(s))^{p-1}\Xi'(s)\mathbb{Q}\left(s,G(s),G\left(\frac{s}{\eta}\right)\right)ds$$
$$+M\frac{(\Xi(t)-\Xi(a))^{1-q(1-p)}}{\rho^p\Gamma(p+1)}\delta.$$

Dividing both sides by δ and taking the limit as $K\to\infty$ we get

$$M\frac{(\Xi(t)-\Xi(a))^{1-q(1-p)}}{\rho^p\Gamma(p+1)} \geq 1,$$

which contradicts Eq.(5.2) This shows that $K_2(D_{\Xi,\delta}) \subset D_{\Xi,\delta}$.
Step 4: Now we have to show that $K_2(D_{\Xi,\delta})$ is equicontinuous and bounded. From step 3, it is clear that $K_2(D_{\Xi,\delta})$ is bounded.
$\left\|(\Xi(t_2)-\Xi(a))^{1-\vartheta}(K_2)\mathcal{A}(t_2)-(\Xi(t_1)-\Xi(a))^{1-\vartheta}(K_1)\mathcal{A}(t_1)\right\|$

$$
\begin{aligned}
&= \left\| \frac{(\Xi(t_2)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right.\\
&\left. -\frac{(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right\|\\
&\leq \left\| \frac{(\Xi(t_2)-\Xi(a))^{p-1}-(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1}\Xi^{\triangle}(s) \right.\\
&\left. \mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right\|\\
&+ \left\| \frac{(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right\|\\
&+ \left\| \frac{(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1}\Xi^{\triangle}(s)\mathbb{Q}\left(s,\mathcal{A}(s),\mathcal{A}\left(\frac{s}{\eta}\right)\right)\triangle s \right\|\\
&\leq \frac{(\Xi(t_2)-\Xi(a))^{p-1}-(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} M \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t)-\Xi(s))^{p-1}\Xi'(s)ds\\
&+ \frac{(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} M \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1}\Xi'(s)ds\\
&+ \frac{(\Xi(t_1)-\Xi(a))^{1-\vartheta}}{\rho^p\Gamma(p)} M \int_{a^+}^{t} e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(s))} (\Xi(t_2)-\Xi(s))^{p-1}\Xi'(s)ds
\end{aligned}
$$

$$
\left\|(\Xi(t_2)-\Xi(a))^{1-\vartheta}(K_2)\mathcal{A}(t_2)-(\Xi(t_1)-\Xi(a))^{1-\vartheta}(K_1)\mathcal{A}(t_1)\right\| \to 0 \quad as \quad t_2 \to t_1.
$$

It follows that $\|K_2\mathcal{A}-K_1\mathcal{A}\|_C \to 0$ as $t_2 \to t_1$. Hence, $K_2(D_{\Xi,\delta})$ is equicontinuous.
As a consequence of Steps 2-4, together with the Arzela-Ascoli theorem, one has that K_2 is compact. We conclude that $K = K_1 + K_2$ is continuous and takes bounded sets into bounded sets. Also, one can verify the validity of $\lambda(K_2(D_{\Xi,\delta})) = 0$ since $(K_2(D_{\Xi,\delta}))$ is relatively compact. It follows from the inclusion $(K_1(D_{\Xi,\delta})) \subset (D_{\Xi,\delta})$ and the equality $\lambda(K_2(D_{\Xi,\delta})) = 0$ that

$$
\lambda(K(D_{\Xi,\delta})) \leq \lambda(K_1(D_{\Xi,\delta})) + \lambda(K_2(D_{\Xi,\delta})) \leq \lambda(D_{\Xi,\delta}),
$$

for every bounded set $(D_{\Xi,\delta})$ of $C_{1-\vartheta,\Xi}(J,\mathbb{R})$ with $\lambda(D_{\Xi,\delta}) > 0$. Since $K(D_{\Xi,\delta}) \subset D_{\Xi,\delta}$ for convex, closed and bounded set $(D_{\Xi,\delta})$ of $C_{1-\vartheta,\Xi}(J,\mathbb{R})$, all conditions of the Sadovskii fixed point theorem are satisfied and we conclude that the operator K has a fixed point $x \in (D_{\Xi,\delta})$ that is a solution of (2.8) with $\mathcal{A}(t) = \mathcal{A}_1$. Therefore the proposed problem (2.8) is controllable on J.

6 An example

Let us consider the Ambartsumian equation with generalized Hilfer PFD on timescale as

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{3},\frac{1}{7},\frac{2}{3},\Xi}\mathcal{A}(t) &= \frac{5t}{7}+\frac{1}{80}|sin(t)|+\mathcal{A}\left(\frac{sin(t)}{3}\right), \quad t\in J=[0,1], \\ I_{0^+}^{1-\vartheta,\frac{2}{3},\Xi}\mathcal{A}(0) &= 5\mathcal{A}\left(\frac{1}{3}\right)+\sqrt{3}\mathcal{A}\left(\frac{3}{5}\right). \end{cases} \quad (6.$$

Now comparing Equation (6.1) with our proposed problem (2.8), we get
$p=\frac{1}{3}, \quad q=\frac{1}{7}, \quad \rho=\frac{2}{3}, \quad \vartheta=\frac{3}{7}, \quad a=0, \quad b=1, \quad \mu_1=5, \quad \mu_2=\sqrt{3}$ as $m=2, \quad \tau_1=\frac{2}{5},\tau_2$
$\frac{3}{7}\in[0,1]$.
Also,$\mathbb{Q}:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. is a function defined by

$$\mathbb{Q}\left(t,\mathcal{A}(t),\mathcal{A}\left(\frac{t}{\eta}\right)\right)=\frac{t+3}{7}+\frac{1}{80}|\mathcal{A}(t)|+\mathcal{A}\left(\frac{2t}{3}\right), \quad t\in[0,1].$$

Clearly, $\mathbb{Q}$ is continuous function and

$$\left|\mathbb{Q}\left(t,\mathcal{A}_1(t),\mathcal{A}_1\left(\frac{t}{\eta}\right)\right)-\mathbb{Q}\left(t,\mathcal{A}_2(t),\mathcal{A}_2\left(\frac{t}{\eta}\right)\right)\right|\le\frac{1}{80}|\mathcal{A}_1-\mathcal{A}_2|.$$

Hence the hypotheses $(H_1),(H_2)$ hold with $k=\frac{1}{80}$.
Now choose $\Xi(t)=t^2+1$, then it implies that $\Xi(t)$ is positive increasing and continuous in $[0,1]$.
Next substituting the values that we mentioned above in $|\wedge|$

$$|\wedge|=\left|\frac{1}{\left(\frac{2}{3}\right)^{\frac{3}{7}-1}\Gamma\left(\frac{3}{7}\right)-\left(5e^{\left(\frac{-1}{18}\right)}\left(\frac{1}{9}\right)^{\left(\frac{2}{3}-1\right)}+\sqrt{3}e^{\left(\frac{-9}{50}\right)}\left(\frac{9}{25}\right)^{\left(\frac{3}{7}-1\right)}\right)}\right|\approx 0.1,$$

and

$$\phi=\frac{\beta\left(\frac{3}{7},\frac{1}{3}\right)}{\left(\frac{2}{3}\right)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}\left\{|\wedge|\left(5\left(\frac{1}{9}\right)+\sqrt{3}\left(\frac{9}{25}\right)\right)+1\right\}\approx 2.04.$$

This implies that $k\phi<1$, which is (H_3).

Further more $\lambda\approx 0.46$ and $k<1$, which means that the assumption (H_4) is also satisfied. So defined problem has at least one solution and hence is unique on J.

$$M\frac{e^{\frac{\rho-1}{\rho}(\Xi(t)-\Xi(a))}(\Xi(t)-\Xi(a))^{1-q(1-p)}}{\rho^p\Gamma(p+1)}=M(-5.102171888) \\ <1.$$

Hence our proposed problem is controllable on J.

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References

[1] I. Ahmed, P. Kumam, F. Jarad, P. Borisut, W. Jirakitpuwapat, On Hilfer generalized proportional fractional derivative, *Adv.Differ.Equ.,* **1**(2020),1-18.

[2] S. Ahmad, A. Ullah, A. Akgul, Manuel De la Sen, A study of fractional order Ambartsumian equation involving exponential decay kernal, *AIMS Math.,* **6(9)**(2021), 9981-9997.

[3] G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, *Math.Comput.Modelling* **52**(2010), 556-566.

[4] S. Harikrishnan, R. Ibrahim, K. Kanagarajan, An arbitrary order differential equations on time scales, *Univ. J. Math. Appl.,* **1**(2018), 262-266.

[5] F. Jarad, M. A. Alquadah, T. Abdeljawad, On more general forms of proportional fractional operators, *Open Math.,* **18**(2020),167-176.

[6] F. Jarad, T. Abedljawad, D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J.Nonlinear Sci.Appl.,* **10**(2017),2607-2619.

[7] F. Jarad, T. Abedljawad, S. Rashid, Z. Hammouch, More properties of the proportional fractional integrals and derivatives of a function with respect to another function, *Adv.Differ.Equ.,* **2020**(2020),1-16.

[8] U. N. Katugampola, New approach to a generalized fractional integral, *Appl.Math.Comput.,* **218**(2011), 860-865.

[9] U. N. Katugampola, A new approach to generalized fractional derivatives, *Appl.Math.Comput.,* **6**(2014), 1-15.

[10] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J.Comput.Appl.Math.,* **264**(2014),65-70.

[11] D. Kumar, J. Singh, D. Baleanu, S. Rathore, Analysis of a fractional model of the Ambartsumian equation, *Eur. Phys. J. Plus.,***133**(2018),259.

[12] I. Mallah, S. Alha, I. Ahmed, A. Akgul, F. Jarad, On ψ-Hilfer generalized proportional fractional operators, *AIMS.Math.,* **7**(2021),82-103

[13] J. Patade, S. Bhalekar, On analytical solution of Ambartsumian equation, *Natl. Acad. Sci. Lett.,* **40**(2017),291-293.

[14] S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, Y. M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Math.,* **7**(2019),1-16.

[15] J. V. C. Sousa, E. C. Oliveira, On the ψ-Hilfer fractional derivative, *Commun.Nonlinear Sci.,***60**(2018),72-91.

[16] J. Vanterler da C. Sousa, D S Oliveira, Gastão S. F. Frederico, Delfim F M Torres, Existence, uniqueness and controllability for g-Hilfer differential equations on time scales, *hal-03726777* 2022.

[17] D. Vivek, K. Kanagarajan, S. Sivasundaram, On the behavior of solutions of fractional differential equations on time scale via Hilfer fractional derivatives, *Fractional Calculus and Applied Analysis*, **21(4)**(2018), 1120-1138.

[18] D. Vivek, K. Kanagarajan, E. Elasyed, Some existence and stability results for Hilfer fractional implicit differential equations with nonlocal conditions, *Mediterr.J.Math.,* **15**(2018),15.