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ORIGINAL RESEARCH

Trajectory Controllability of Hilfer Fractional Neutral Stochastic Differential Equations with Deviated Argument Using Rosenblatt Process and Poisson Jumps

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Abstract
The theoretical approach of Trajectory(T-)controllability of Hilfer fractional neutral stochastic differential equation with deviated arguments, Rosenblatt process and Poisson jumps has been elaborated. By knowing trajectories one can minimize the cost involved in the system. Our result extends the works of Chalishajar et al. (J Franklin Inst 347(7):1065–1075, 2010), Chalishajar et al. (Appl Math 3:1729–1738, 2012), Chalishajar et al. (Differential Equations and Dynamical Systems, Springer, Berlin, 2014) and the concept of Rieman-Liouville (R-L) and Caputo's derivatives. A Hilfer Fractional Stochastic differential equation proposed here remains untreated in the literature and its solvability is acquired by using fractional calculus, stochastic analysis, semigroup theory and Krasnoselskii's fixed point theorem. In the later part the T-controllability of the aforementioned system is calculated. An abstract phase space deinition has been used for the infinite delay T-control problem. Finally an illustration is given to validate our obtained results.

Keywords Deviated argument · Hilfer fractional derivative · Neutral Stochastic differential equation with infinite delay · Trajectory controllability

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Introduction

Fractional calculus is one of the beneficiary tools in demonstrating long-memory or nonlocal effects for validating physical occurrence more precisely. Fractional Differential Equations (FDEs) describe the potentiality of integer order models with fractional order. In real life phenomena, FDEs are applied and are modeled as HIV immune system, neural network and non-linear collision of earthquake, see [1–6]. Fractional derivatives includes Riemann-Liouville, Caputo, Hadamard, Coimbra, Atangana-Baleanu, Caputo-Hadamard derivatives where the finest fractional derivative relies on the experimental data more appropriate to the conceptual model. To overpower the already existing concepts of fractional derivative, Hilfer investigated a new generalized form referred as Hilfer fractional derivative (HFD) which combines number of definitions of fractional derivative operators, refer [7]. The type-parameter exhibits different kinds of stationary states and provides an excess degrees of freedom on initial data. Yang and Wang [8] established the existence results of a class of Hilfer fractional evolution equations with nonlocal conditions. Gu and Trujillo [9] obtained sufficient conditions ensuring existence of mild solution for evolution equation of HFD using Ascolli-Arzela theorem. Many authors have investigated solvability and controllability properties of Hilfer FDEs, refer the monographs [10–17] and the references therein.

The theory of neutral differential equations has considerable interest for its usefulness in various fields of science and engineering where it relies on past and present values of the function and also on the derivatives with delays. The neutral differential system in fractional order is extensive mainly in the infinite dimensional space, refer [18]. When an unknown quantity and its corresponding derivatives are present in different values in their parameters then differential equations with deviated arguments take place which is a generalized form of delay differential equation. Differential equations involving deviated arguments are used in various domains includes mathematical physics, mechanics, mathematical control models in integer and FDEs, see [19, 20].

Due to regular fluctuations in the deterministic models and due to noise which is random or appears to be so, researchers have moved on to the stochastic ones. Stochastic Differential Equations (SDEs) gain considerable popularity with greater interest because of their applications in many fields such as mechanical, electrical and control engineering. Moreover, fractional noise constitutes a vital tool in the description of credit risk sensitive instruments, see [21]. Mandelbrot and Van Ness [22] initiated that fractional Brownian motion (fBm) is a family of centered Gaussian processes with continuous sample paths indexed by Hurst parameter $\mathcal{H} \in (0, 1)$. The notable features includes similarity, stationary increments and long memory. The fBm being a generalization of classical Brownian motion has enormous applications in the fields of filtering theory, queuing networks and mathematical finance.

The concept of controllability [introduced by Kalman in 1960] leads to considerable results in the behavior of linear and nonlinear dynamical systems. There are many different definitions of controllability, including: complete controllability [23] [any two state vectors may possibly be connected by a trajectory], approximate controllability [24] [any state vector may be steered to the neighborhood of the desired state vector], exact controllability [25–27] [the control may steer the complete state $(\mathfrak{x}(\mathfrak{t}), \mathbf{u}(t))$ of a system to the desired state] and the null controllability [23] [any state vector may be steered to zero state]. Trajectory (T)-controllability is the new notion of the controllability initiated by Chalishajar [28]. In this context, we look for a control which steers the system along a prescribed trajectory rather than a control steering the given system from an initial state to desired final state. If we know the trajectories of the system then we can minimize the cost involved in steering the system and also we can safeguard the system. Based on this benefits, Chalishajar et al. [28] extended it to abstract nonlinear integro-differential systems in finite and infinite dimensional spaces. Chalishajar et al. [29] further studied numerical approach of T-controllability of nonlinear integro-differential system. Later on, Chalishajr et al. [30] further discussed numerical approach of T-controllability of second order nonlinear integro-differential system using sine and cosine operators with counter examples. Malik and George [31] then studied the Trajectory controllability of the nonlinear systems governed by fractional differential equations. In the view of the above mentioned works, still the T-controllability of Hilfer fractional neutral stochastic dynamical systems in Hilbert space with deviated arguments, Rosenblatt process and Poisson jumps remains untreated in the literature which serves as a motivation for this paper work. In this paper, we extend the works of Chalishajar et al. [28–30]. The main contribution and novelties of the article are listed below.

1. Initially, a Hilfer type fractional derivative neutral stochastic differential system is formulated and T-controllability which is new to the stochastic sense is investigated.
2. An abstract phase space deinition has been used for the infinite delay T-control problem.
3. By the already published works, there has not been any research that focuses on the theoretical approach of solvability and T-controllability of Hilfer fractional neutral SDEs with deviated argument using Rosenblatt process and Poisson jumps where the deviated argument generalizes the delay differential system.
4. The obtained concepts generalize the existing results of Caputo and R-L fractional derivative SDEs. Also, it extends the works of Chalishajar et al. [28–30].
5. The obtained results are validated using an example.

We may take into account the following Hilfer fractional neutral SDEs with deviated argument, Rosenblatt process and Poisson jump of the form:

$$\begin{aligned} \mathfrak{D}^{\alpha,\beta}_{0^+}\left[\mathfrak{x}(t)+\Xi(t,\mathfrak{x}_t)\right] &= \mathfrak{A}\mathfrak{x}(t)+\mathbb{B}(t)+\mathfrak{f}(t,\mathfrak{x}_t,\mathfrak{x}(\sigma(\mathfrak{x}(t),t)))+\gamma(t,\mathfrak{x}_t)\frac{d\mathcal{Z}_{\mathcal{H}}(t)}{dt} \\ &\quad + \int_{\mathscr{Z}} \mathfrak{h}(t,\mathfrak{x}_t,\vartheta)\tilde{\mathbb{N}}(dt,d\vartheta), \quad t\in(0,\mathrm{T}] := \mathcal{J}' \qquad (1.1) \\ \mathcal{I}^{(1-\delta)}_{0^+}\mathfrak{x}(t) &= \varphi\in\mathcal{B}, \quad t\in(-\infty,0], \quad \delta=\alpha+\beta-\alpha\beta; \end{aligned}$$

where $\mathfrak{D}^{\alpha,\beta}_{0^+}$ is the Hilfer fractional derivative of order α and type β with $0\le\alpha\le 1$, $\frac{1}{2}<\beta<1$. $\mathscr{X}$ be the real separable Hilbert space furnished with the inner product $<.,.>$ provided with the norm $\|.\|$ and the state variable $\mathfrak{x}(.)\in\mathscr{X}$. $\mathfrak{A}:\mathbb{D}(\mathfrak{A})\subset\mathscr{X}\to\mathscr{X}$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\mathcal{T}(t)$, $t\ge 0$ on $\mathscr{X}$. The time history values $\mathfrak{x}_t:(-\infty,0]\to\mathscr{X}$ being defined by $\mathfrak{x}_t(\theta)=\mathfrak{x}(t+\theta)$ that belongs to an abstract phase space $\mathcal{B}$ and $\theta\le 0$. The deviated argument σ is the mapping from $\mathscr{X}\times\mathcal{J}'\to\mathscr{R}^+$. Assuming $\mathcal{J}:=[0,\mathrm{T}]$, $\mathrm{T}>0$. The nonlnear functions $\mathfrak{f}:\mathcal{J}\times\mathcal{B}\times\mathscr{X}\to\mathscr{X}$, $\Xi:\mathcal{J}\times\mathcal{B}\to\mathscr{X}$ and $\mathfrak{h}:\mathcal{J}\times\mathcal{B}\times\mathscr{Z}\to\mathscr{X}$ are continuous and $\gamma:\mathcal{J}\to\mathcal{L}^0_2$ is the deterministic function where $\mathcal{L}^0_2(\mathscr{Y},\mathscr{X})$ represents the Q-Hilbert Schmidt operators from $\mathscr{Y}$ to $\mathscr{X}$. Let the closed subspace of $\mathbb{W}$ be $\mathcal{L}^2_{\mathfrak{I}}(\mathcal{J},\mathbb{W})$, with all measurable and $\mathfrak{I}-$adapted, $\mathbb{W}-$valued stochastic process satisfying $\mathbb{E}\int_0^t\|\mathbb{B}(t)\|^2_{\mathbb{W}}dt<\infty$ and $\tilde{\mathbb{N}}$ is a Poisson random measure. Let $\mathscr{W}$ be a non-empty bounded closed convex subset of $\mathbb{W}$. Represent $\mathscr{W}_{ad}=\{\mathbb{B}(.)\in\mathcal{L}^2_{\mathfrak{I}}(\mathcal{J},\mathbb{W}):\mathbb{B}(t)\in\mathscr{W}\ a.e,\ \ t\in\mathcal{J}\}$ as the set of admissible control where the control variables take their values. The initial values $\varphi = \{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathfrak{I}$−measurable, $\mathcal{B}$−valued stochastic process.

This article is summarized as follows: Section 2 demonstrates certain basic definitions and preliminary notions. Section 3 demonstrates the solvability of the proposed system by using fixed point theorem and fractional calculus. Section 4 is devoted to acquire T-controllability results for the considered Hilfer fractional system 1.1 by employing Gronwall's inequality. Moreover Section 5 represents an example to validate our obtained results.

Preliminaries and Notations

Hilfer Fractional Derivative and Space Representation

Definition 2.1 The left-hand side HFD of order $0 \leq \alpha \leq 1$ and type $0 < \beta < 1$ of a function $\mathfrak{g} : [a, +\infty) \to \mathscr{R}$ defined as

$$\mathfrak{D}_{a^+}^{\alpha,\beta}\mathfrak{g}(t) = \mathcal{I}_{a^+}^{\alpha(1-\beta)} \frac{d}{dt} \mathcal{I}_{a^+}^{(1-\alpha)(1-\beta)} \mathfrak{g}(t),$$

for the functions on the RHS exists.

Let $\mathcal{L}^2(\mathfrak{I}, \mathscr{X}) = \mathcal{L}^2(\Omega, \mathfrak{I}_t, \mathbb{P}, \mathscr{X})$ symbolizes the Hilbert space of all strongly $\mathfrak{I}_t$−measurable square integrable $\mathscr{X}$−valued random variable satisfying $\mathbb{E}\|\mathfrak{x}\|^2 < \infty$, $\mathbb{E}(.)$ denotes expectation of the random variable. $\mathcal{L}^2_{\mathfrak{I}}(\mathcal{J}, \mathscr{X})$ denotes the Hibert space of all stochastic processes $\mathfrak{I}_t$−adapted measurable function defined on $\mathcal{J}$ with values in $\mathscr{X}$ with the norm $\|\mathfrak{x}\|_{\mathcal{L}^2_{\mathfrak{I}}} = \left[\int_0^{\mathrm{T}} \mathbb{E}\|\mathfrak{x}(t)\|^2 dt\right]^{1/2} < \infty$. Let $\mathscr{C}((-\infty, \mathrm{T}], \mathcal{L}^2(\mathfrak{I}, \mathscr{X}))$ be the closed subspace of all continuous functions from $(-\infty, \mathrm{T}]$ into $\mathcal{L}^2(\mathfrak{I}, \mathscr{X}) \ni$ the restriction $\mathfrak{x} \in \mathscr{C}(\mathcal{J}', \mathcal{L}^2(\mathfrak{I}, \mathscr{X}))$. Let us define $\mathcal{C}_{1-\delta} = \{\mathfrak{x} \in \mathscr{C}(\mathcal{J}', \mathcal{L}^2(\mathfrak{I}, \mathscr{X})) : t^{(1-\delta)}\mathfrak{x}(t) \in \mathscr{C}(\mathcal{J}, \mathcal{L}^2(\mathfrak{I}, \mathscr{X}))\}$ equipped with the norm

$$\|\mathfrak{x}\|_{\mathcal{C}_{1-\delta}} = \left[\sup_{0 \leq t \leq \mathrm{T}} \mathbb{E}\left\|t^{(1-\delta)\mathfrak{x}(t)}\right\|^2\right]^{\frac{1}{2}}.$$

It is obvious that $\mathcal{C}_{1-\delta}$ is a Banach space.

In this manuscript, we suppose that the phase space is defined axiomatically [32]. Assume that $\mathcal{B}$ are developed for $\mathfrak{I}_0$-measurable functions from $(-\infty, 0]$ equipped with the norm $\|\cdot\|_{\mathcal{B}}$. Define the abstract phase space for an infinite time delay process by

$$\mathcal{B} = \Big\{\zeta : (-\infty, 0] \to \mathscr{X} \text{ for any } \tau > 0\ (\mathbf{E}\|\zeta\|^2)^{1/2} \text{ is bounded and measurable function } [\tau, 0] \text{ and } \int_{-\infty}^0 h(\iota) \sup_{\iota \leq \tau \leq 0} (\mathbf{E}\|\zeta(s)\|^2)^{1/2} d\iota < +\infty\Big\}$$

Clearly, $\mathcal{B}$ is a complete Banach space equipped with the norm $\|\zeta\|_{\mathcal{B}} = \int_{-\infty}^0 h(\iota) \sup_{\iota \leq \chi \leq 0} (\mathbf{E}\|\zeta\|^2)^{1/2} d\iota$

Lemma 2.1 *Presume* $\mathfrak{x} \in \mathscr{X}$, *then* $\forall\ \iota \in [0, \mathrm{T}]$, $\mathfrak{x}_\iota \in \mathcal{B}$ *and*

$$l(\mathbf{E}(\|\mathfrak{x}(\iota)\|^2))^{\frac{1}{2}} \le l_1 \sup_{0\le\chi\le\iota} (\mathbf{E}\|\mathfrak{x}(\chi)\|^2)^{\frac{1}{2}} + \|\mathfrak{x}_0\|_{\mathcal{B}},$$

where $l_1 = \int_{-\infty}^{0} h(\chi)\mathrm{d}\chi < \infty$.

The phase space defined above also satisfies the conditions of the following axioms of the phase space $\mathcal{B}$ developed by Hale and Kato [33] for $\mathfrak{I}_0$−measurable functions from $(-\infty, 0]$ equipped with the seminorm $\|.\|_{\mathcal{B}}.\mathcal{B}$:

(A1) If $\mathfrak{x} : (-\infty, a) \to \mathscr{X}, \;\; a \ge 0$ is continuous on $[0, a)$ and $\varphi \in \mathcal{B}$, then for every $t \in [0, a)$ the following holds.

- (i) $\mathfrak{x}_t$ is in $\mathcal{B}$,
- (ii) $\|\mathfrak{x}(t)\| \le \tilde{\mathfrak{h}}\|\mathfrak{x}_t\|_{\mathcal{B}}$,
- (iii) $\|\mathfrak{x}_t\|_{\mathcal{B}} \le \mathfrak{K}(t-\kappa)\sup\{\|\mathfrak{x}(s)\|, 0 \le s \le t\} + \mathfrak{M}(t-\kappa)\|\varphi\|_{\mathcal{B}}, \tilde{\mathfrak{h}} > 0$ being constant, $\mathfrak{K}, \mathfrak{M} : [0,\infty) \to [0,\infty)$. $\mathfrak{K}$ is continuous, $\mathfrak{M}$ is locally bounded. Also $\mathfrak{K}, \mathfrak{M}, \tilde{\mathfrak{h}}$ are independent of $\mathfrak{x}(.)$.

(A2) The space $\mathcal{B}$ is complete.

Rosenblatt Process

Consider a time interval $[0, \mathrm{T}]$ with arbitrary fixed horizon T and let $\left\{\mathcal{Z}_{\mathcal{H}(t)},\ t \in [0, \mathrm{T}]\right\}$ be one-dimensional Rosenblatt process with parameter $\mathcal{H} \in \left(\frac{1}{2}, 1\right)$. Also, the Rosenblatt process with parameter $\mathcal{H} > \frac{1}{2}$ admits the following representation [34]:

$$\mathcal{Z}_{\mathcal{H}}(t) = \mathrm{d}(\mathcal{H}) \int_0^t \int_0^t \left[\int_{y_1\vee y_2}^t \frac{\partial \mathrm{K}^{\mathcal{H}'}}{\partial \mathfrak{x}}(\mathfrak{x}, y_1)\frac{\partial \mathrm{K}^{\mathcal{H}'}}{\partial \mathfrak{x}}(\mathfrak{x}, y_2)\right] \mathrm{d}\mathcal{B}(y_1)\mathrm{d}\mathcal{B}(y_2),$$

where $\mathrm{K}^{\mathcal{H}}(t, s)$ is defined by

$$\mathrm{K}^{\mathcal{H}} = c_{\mathcal{H}} s^{\frac{1}{2}-\mathcal{H}} \int_s^t (\mathfrak{x} - s)^{\mathcal{H}-\frac{3}{2}} \mathfrak{x}^{\mathcal{H}-\frac{1}{2}} \mathrm{d}\mathfrak{x}, \text{ for } t > s.$$

for $s > t$, $\mathrm{d}_{\mathcal{H}} = \sqrt{\frac{\mathcal{H}(\mathcal{H}-1)}{\beta(2-2\mathcal{H},\mathcal{H}-\frac{1}{2})}}$, $\mathcal{H}' = \frac{\mathcal{H}+1}{2}$ and $c_{\mathcal{H}} = \frac{1}{\mathcal{H}+1}\sqrt{\frac{\mathcal{H}}{2(2\mathcal{H}-1)}}$. For basic preliminaries and fundamental results on Rosenblatt process, one can refer the articles [35, 36] and the references therein.

Poisson Jump

The Poisson random measure $\mathbb{N}$ represents $\sigma-$finite stationary $\mathfrak{I}_t$- adapted Poisson point process $\tilde{\mathfrak{p}}(.)$ taking values in a measurable space $(, \mathscr{X})$, and $\tilde{\mathbb{N}}$ denotes the compensated Poisson random measure $\tilde{\mathbb{N}}(dt, dy) = \mathbb{N}(dt, dy) - \pi(dy)dt$, where $\mathbb{N}((0, t] \times \triangle) : \sum_{s\in(0,t]} 1_{\triangle}(\tilde{\mathfrak{p}}(s))$ for $\triangle$ and π is the characteristic measure of $\mathbb{N}$.

Lemma 2.2 [37] *Let* $\mathfrak{x} : (-\infty, 0] \to \mathscr{X}$ *be an* $\mathfrak{I}_t$*–adapted measurable process such that the* $\mathfrak{I}_0$*–adapted process* $\mathfrak{x}(0) = \varphi \in \mathcal{B}$, *then*

$$\mathbb{E}\|\mathfrak{x}_t\|_{\mathcal{B}} \leq \mathfrak{K}_1 \sup_{0\leq s\leq t} \mathbb{E}\|\mathfrak{x}(s)\| + \mathfrak{K}_2 \mathbb{E}\|\varphi\|_{\mathcal{B}},$$

where $\mathfrak{K}_1 = \sup_{t\in\mathcal{J}}\{\mathfrak{K}(t)\}$ *and* $\mathfrak{K}_2 = \sup_{t\in\mathcal{J}}\{\mathfrak{M}(t)\}$.

For $\mu \in (0,1)$, the fractional power operator $\mathfrak{A}^{\mu}$ is defined as a closed linear operator on its domain $\mathbb{D}(\mathfrak{A}^{\mu})$ being dense in $\mathscr{X}$.

Theorem 2.1 [37]

(i) *For* $\mu \in (0,1]$, $\mathbb{D}(\mathfrak{A}^{\mu})$ *is a Banach space equipped with the norm* $\|\mathfrak{x}\|_{\mu} = \|\mathfrak{A}^{\mu}\mathfrak{x}\|$, $\mathfrak{x} \in \mathbb{D}(\mathfrak{A}^{\mu})$.

(ii) *If* $0 < \nu < \mu \leq 1$, *then* $\mathbb{D}(\mathfrak{A}^{\mu}) \hookrightarrow \mathbb{D}(\mathfrak{A}^{\nu})$ *and the embedding is compact, whenever* $\mathfrak{A}$ *is compact.*

(iii) *For all* $0 < \mu \leq 1, \exists\, \mathcal{C}_{\mu} > 0 \ni$

$$\|\mathfrak{A}^{\mu}\mathcal{T}(t)\| \leq \frac{\mathcal{C}_{\mu}}{t^{\mu}}, \quad 0 \leq t \leq \mathrm{T}.$$

Lemma 2.3 [37] *For any* $\mathfrak{x} \in \mathscr{X}$, $0 < y < 1$ *and* $\mu \in (0,1]$, *we have*

$$\mathfrak{A}\mathscr{T}_{\beta}(t)\mathfrak{x} = \mathfrak{A}^{1-y}\mathscr{T}_{\beta}(t)\mathfrak{A}^{y}\mathfrak{x}$$

and

$$\left\|\mathfrak{A}^{\mu}\mathscr{T}_{\beta}\mathfrak{x}\right\| \leq \frac{\beta\mathcal{C}_{\mu}}{t^{\beta\mu}} \frac{\Gamma(2-\mu)}{\Gamma(1+\beta(1-\mu))}, \quad t \in \mathcal{J}.$$

The family of operators $\{\mathfrak{S}_{\alpha,\beta}(t) = \mathcal{I}^{\alpha(1-\beta)} : t \geq 0\}$ and $\mathfrak{P}_{\beta}(t) : t \geq 0$ can be specified as

$$\begin{aligned} \mathfrak{S}_{\alpha,\beta}(t) &= \mathcal{I}_{0^+}^{\alpha(1-\beta)}\mathfrak{P}_{\beta}(t), \quad \mathfrak{P}_{\beta}(t) = t^{\beta-1}\mathscr{T}_{\beta}(t), \\ \mathscr{T}_{\beta}(t) &= \int_0^{\infty} \beta\Theta\varsigma_{\beta}(\Theta)\mathcal{T}(t^{\beta}\Theta)d\Theta, \end{aligned} \tag{2.1}$$

where $\varsigma_{\beta}(\Theta) = \sum_{n=1}^{\infty} \frac{(-\Theta)^{n-1}}{(n-1)!\Gamma(1-n\beta)}$, $0 < \beta < 1$, $\Theta \in (0,\infty)$ is a function of Wright type fulfilling,

$$\int_0^{\infty} \Theta^{\tau}\varsigma_{\beta}(\Theta)d\Theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\beta\tau)}, \quad \text{for } \Theta \geq 0.$$

Lemma 2.4 [37] *The considered operators* $\mathfrak{S}_{\alpha,\beta}$ *and* $\mathfrak{P}_{\beta}$ *posses the following properties*

(i) $\mathfrak{P}_{\beta}(t) : t > 0$ *being continuous in the uniform operator topology.*

(ii) $\mathfrak{S}_{\alpha,\beta}(t)$ *and* $\mathfrak{P}_{\beta}(t)$ *are linear bounded operators, for* $t > 0$ *being fixed*

$$\|\mathfrak{P}_\beta(t)\mathfrak{x}\| \le \frac{\mathscr{M}t^{\beta-1}}{\Gamma(\beta)}\|\mathfrak{x}\|, \quad \|\mathfrak{S}_{\alpha,\beta}(t)\mathfrak{x}\| \le \frac{\mathscr{M}t^{\delta-1}}{\Gamma(\delta)}\|\mathfrak{x}\|.$$

(iii) $\{\mathfrak{S}_{\alpha,\beta} : t > 0\}$ *and* $\{\mathfrak{P}_\beta(t) : t > 0\}$ *are strongly continuous.*

Let $\mathcal{V}$ be the collection of functions $\rho(.)$ defined on $\mathcal{J}$ with $\rho(0) = \mathfrak{x}_0$ and $\rho(\mathrm{T}) = \mathfrak{x}_\mathrm{T}, \forall t \in \mathcal{J}$. Moreover HFD $\mathfrak{D}_{0^+}^{\alpha,\beta}\mathfrak{x}(t)$ exists a.e on $\mathcal{J}'$. The set of all feasible trajectories for the control model 1.1 being denoted by $\mathcal{V}$.

Definition 2.2 The control system 1.1 is said to be trajectory controllable on $\mathcal{J}$, if for every $\rho \in \mathcal{V}$, such that the mild solution $\mathfrak{x}(.)$ of 1.1 satisfies $\rho(t) = \mathfrak{x}(t)$ almost everywhere.

Lemma 2.5 (Generalized Gronwall's inequality) *If* $\beta > 0$, $\tilde{a}(t)$ *is a non-negative function locally integrable on* $0 \le t \le \mathrm{T}$ *and* $\mathfrak{q}(t)$ *is a non-negative, non-decreasing continuous function on* $0 \le t \le \mathrm{T}, \mathfrak{q}(t) \le \mathfrak{c}$ *and suppose* $\tilde{\mathfrak{u}}(t) \le \tilde{a}(t) + \mathfrak{q}(t)\int_0^t (t-s)^{\beta-1}\tilde{\mathfrak{u}}(s)ds$, *on this interval. Then*

$$\tilde{\mathfrak{u}}(t) \le \tilde{a}(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(\mathfrak{q}(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{\beta-1}\tilde{a}(s)ds, \quad 0 \le t \le \mathrm{T}.$$

In particular, when $\tilde{a}(t) = 0$, *then* $\tilde{\mathfrak{u}}(t) = 0 \ \forall \ 0 \le t < \mathrm{T}$.

Theorem 2.2 (Bochner's theorem) *A measurable function* $\mathfrak{U} : \mathcal{J} \to \mathscr{X}$ *is Bochner integrable if* $\|\mathfrak{U}\|$ *is Lebesgue integrable.*

Theorem 2.3 (Krasnoselskii's fixed point theorem) *Let* $\mathscr{X}$ *be a Banach space and let* $\mathbb{Y}$ *be a non-empty bounded closed convex subset of* $\mathscr{X}$. *Suppose* Φ_1 *and* Φ_2 *be the mapping from* $\mathbb{Y}$ *onto* $\mathscr{X}$, $\ni$

(a) $\Phi_1\mathfrak{x} + \Phi_2 y \in \mathbb{Y}$ *whenever* $\mathfrak{x}, y \in \mathbb{Y}$,
(b) Φ_1 *is a contraction mapping*,
(c) Φ_2 *is compact and continuous.*

Then there exists $\mathfrak{A} \in \mathbb{Y} \ni \mathfrak{A} = \Phi_1\mathfrak{A} + \Phi_2\mathfrak{A}$.

Existence of Mild Solution

This section examines the existence of mild solution for 1.1 using Theorem 2.3.

Definition 3.1 An $\mathfrak{I}_t$–adapted stochastic processes $\{\mathfrak{x}(t) : t \in (-\infty, \mathrm{T}]\}$ is a mild solution of 1.1, if

(i) $\mathfrak{x}(t) = \varphi(t)$ on $(-\infty, 0]$ with $\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 < \infty$,
(ii) $\mathfrak{x}(t)$ is continuous on $[0, \mathrm{T}]$ a.s, and for each $s \in [0, \mathrm{T})$ the function $\mathfrak{P}_{\mathcal{B}}(t-s)\mathfrak{A}\Xi(s, \mathfrak{x}_s)$ is integrable with the following stochastic integral equation

$$\begin{aligned}\mathfrak{x}(t) = {} & \mathfrak{S}_{\alpha,\beta}(t)[\varphi(0) + \Xi(0,\varphi(0))] - \Xi(t,\mathfrak{x}_t) - \int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\mathfrak{x}_s)ds \\ & + \int_0^t \mathfrak{P}_\beta(t-s)\big[\mathbb{B}(s) + \mathfrak{f}(s,\mathfrak{x}_s,\mathfrak{x}(\sigma(\mathfrak{x}(s),s)))\big]ds \\ & + \int_0^t \mathfrak{P}_\beta(t-s)\gamma(s,\mathfrak{x}_s)d\mathcal{Z}_{\mathcal{H}}(s)ds \\ & + \int_0^t \int_{\mathscr{Z}} \mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\mathfrak{x}_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta).\end{aligned} \tag{3.1}$$

Remark 3.1 The HFD the Riemann-Liouville and Caputo derivative. For $\alpha = 0,\ 0 < \beta < 1$, the HFD coincides with the classical Riemann-Liouville fractional derivative as:

$$\mathfrak{D}_{a^+}^{0,\beta}\mathfrak{g}(t) = \frac{d}{dt}\mathcal{I}_{a^+}^{(1-\beta)}\mathfrak{g}(t) = ^L \mathfrak{D}_{a^+}^{\beta}\mathfrak{g}(t).$$

When $\alpha = 1, 0 < \beta < 1$, the HFD coincides with classical Caputo fractional derivative as:

$$\mathfrak{D}_{a^+}^{1,\beta}\mathfrak{g}(t) = \mathcal{I}_{a^+}^{(1-\beta)}\frac{d}{dt}\mathfrak{g}(t) = ^C \mathfrak{D}_{a^+}^{\beta}\mathfrak{g}(t).$$

As $\alpha = 1$, the fractional stochastic differential system simplifies into the classical Caputo fractional stochastic model see [38].

Assumptions

The following assumptions are considered to prove our main results:

(A1) The semigroup $\mathcal{T}(t)$ is compact for $t > 0$, and $\|\mathcal{T}(t)\| \le \mathtt{M},\ \ \mathtt{M} > 0$.

(A2) The nonlinear map $\mathfrak{f} : \mathcal{J} \times \mathcal{B} \times \mathscr{X} \to \mathscr{X}$ is Lipschitz continuous. For $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{B}$, $y_1, y_2 \in \mathfrak{x}$ and $\mathscr{C}_\mathfrak{f} > 0$.

$$\mathbb{E}\big\|\mathfrak{f}(t,\mathfrak{x}_1,y_1) - \mathfrak{f}(t,\mathfrak{x}_2,y_2)\big\|^2 \le \mathscr{C}_\mathfrak{f}\Big[\mathbb{E}\big\|\mathfrak{x}_1 - \mathfrak{x}_2\big\|_{\mathcal{B}}^2 + \big\|y_1 - y_2\big\|^2\Big].$$

Also,

$$\mathbb{E}\|\mathfrak{f}(.,0,\mathfrak{x}(0))\|^2 \le \tilde{\mathfrak{f}}_0.$$

(A3) Let $\sigma : \mathscr{X} \times \mathcal{J} \to \mathscr{R}^+$ be Lipschitz then for $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{X}$ and $\mathscr{C}_\sigma > 0$,

$$\mathbb{E}|\sigma(\mathfrak{x}_1,t) - \sigma(\mathfrak{x}_2,t)|_{\mathscr{R}^+}^2 \le \mathscr{C}_\sigma \mathbb{E}\big\|\mathfrak{x}_1 - \mathfrak{x}_2\big\|^2$$

and

$$\sigma(.,0) = 0.$$

(A4) The non-linear function $\Xi : \mathcal{J} \times \mathcal{B} \to \mathscr{X}$ is continuous and $\exists\, 0 < \mu < 1$ and $\mathscr{C}_\Xi > 0$ being constant, for every $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{B}$, $\mathfrak{A}^\nu\Xi(t,.)$ satisfies

$$\mathbb{E}\big\|\mathfrak{A}^\mu\Xi(t,\mathfrak{x}_1) - \mathfrak{A}^\mu\Xi(t,\mathfrak{x}_2)\big\|^2 \le \mathscr{C}_\Xi\mathbb{E}\big\|\mathfrak{x}_1 - \mathfrak{x}_2\big\|_{\mathcal{B}}^2$$
$$\mathbb{E}\|\mathfrak{A}^\mu\Xi(t,\mathfrak{x})\|^2 \le \mathfrak{l}_1\mathbb{E}\|\mathfrak{x}\|_{\mathcal{B}}^2 + \mathfrak{l}_2.$$

(A5) For each $t \in \mathcal{J}$ the function $\gamma(t,.) : \mathcal{B} \to \mathcal{L}_2^0$ is continuous and for each $\mathfrak{x} \in \mathcal{B}$, the map $\gamma(.,\mathfrak{x}) : \mathcal{J} \to \mathcal{L}_2^0$ is strongly measurable

$$\mathbb{E}\left\|\gamma(t,\mathfrak{x}_t)\right\|^2_{\mathcal{L}_2^0} \le \mathscr{C}_\gamma \|\mathfrak{x}\|^2.$$

(A6) The map $\mathfrak{h} : \mathcal{J} \times \mathcal{B} \times \mathscr{Z} \to \mathscr{X}$ satisfies $\forall\, t \in [0,\mathrm{T}], \mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{B}$, $\mathscr{C}_\mathfrak{h} > 0$ being constant

$$\int_{\mathscr{Z}} \left\|\mathfrak{h}(s,\mathfrak{x}_1,\eta) - \mathfrak{h}(s,\mathfrak{x}_2,\eta)\right\|^2 \lambda(d\eta)ds \vee \left(\int_{\mathscr{Z}} \left\|\mathfrak{h}(s,\mathfrak{x}_1,\eta) - \mathfrak{h}(s,\mathfrak{x}_2,\eta)\right\|^4 \lambda(d\eta)ds\right)^{1/2}$$
$$\le \mathscr{C}_\mathfrak{h}\mathbb{E}\left\|\mathfrak{x}_1 - \mathfrak{x}_2\right\|^2$$

Also,

$$\left(\int_{\mathscr{Z}} \left\|\mathfrak{h}(s,\mathfrak{x}_1,\eta) - \mathfrak{h}(s,\mathfrak{x}_2,\eta)\right\|^4 \lambda(d\eta)ds\right)^{1/2} \le \mathscr{C}_\mathfrak{h}\mathbb{E}\left\|\mathfrak{x}_1 - \mathfrak{x}_2\right\|^2.$$

Theorem 3.1 *If the hypotheses* (*A*1)-(*A*6) *holds*, *then for any* $\varphi \in \mathcal{B}$ *then* 1.1 *has a mild solution and*

$$\mathscr{C} = 3\mathrm{T}^{2-2\delta}\left[\mathfrak{K}_1^2\|\mathfrak{A}^{-\mu}\|^2\mathscr{C}_\Xi + \mathfrak{K}_1^2\mathscr{C}_\Xi\left[\frac{\beta\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu-1} + 2\mathscr{C}_\mathfrak{f}\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2 \times(2k\sigma + \mathfrak{K}_1^2)\frac{\mathrm{T}^{2\beta}}{2\beta-1}\right] < 1. \tag{3.2}$$

Proof Let $\mathscr{B}_\mathrm{T}$ be the set defined by

$$\mathscr{B}_\mathrm{T} = \{\mathfrak{x} : (-\infty,\mathrm{T}] \to \mathscr{X} \ni \mathfrak{x} \mid (-\infty,0] \in \mathcal{B},\ \mathfrak{x} \mid \mathcal{J} \in \mathcal{C}_{1-\delta}\}.$$

An operator Π is defined from $\mathscr{B}_\mathrm{T}$ to itself with the condition that $(\Pi\mathfrak{x})(t) = 0$, for $t \in (-\infty,0]$ and

$$(\Pi\mathfrak{x})(t) = \mathfrak{S}_{\alpha,\beta}(t)[\varphi(0) + \Xi(0,\varphi(0))] - \Xi(t,\mathfrak{x}_t) - \int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\mathfrak{x}_s)ds$$
$$+ \int_0^t \mathfrak{P}_\beta(t-s)\left[\mathbb{B}(s) + \mathfrak{f}(s,\mathfrak{x}_s,\mathfrak{x}(\sigma(\mathfrak{x}(s),s)))\right]d$$
$$+ \int_0^t \mathfrak{P}_\beta\gamma(s,\mathfrak{x}_s)d\mathcal{Z}_\mathcal{H}(s)ds$$
$$+ \int_0^t\int_{\mathscr{Z}} \mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\mathfrak{x}_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta), \qquad t \in \mathcal{J}$$

From Bochner's theorem, it follows that the functions $\mathfrak{f}, \gamma$ and $\mathfrak{h}$ being continuous are integrable on $\mathcal{J}$. Also,

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t \mathfrak{A}\mathfrak{P}_\beta(t-s)\Xi(s,\mathfrak{x}_s)ds\right\|^2\\
&\leq \mathbb{E}\left[\int_0^t \left\|(t-s)^{\beta-1}\mathfrak{A}^{1-\mu}\mathscr{T}_\beta(t-s)\mathfrak{A}^\mu\Xi(s,\mathfrak{x}_s)\right\|ds\right]^2\\
&\leq \left[\frac{\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\beta\mu)}\right]^2\frac{\beta\mathrm{T}^{\beta\mu}}{\mu}\int_0^t (t-s)^{\beta\mu-1}[\mathfrak{l}_1\mathbb{E}\|\mathfrak{x}_s\|^2_{\mathcal{B}}+\mathfrak{l}_2]ds
\end{aligned}$$

yields $\mathfrak{A}\mathfrak{P}_\beta(t-s)\Xi(s,\mathfrak{x}_s)$ is integrable on $\mathcal{J}'$. Thus the set Π is well-defined on $\mathscr{B}_\mathrm{T}$. For $\varphi \in \mathcal{B}$,

$$\eta(t)=\begin{cases}\varphi(t), & t\in(-\infty,0],\\ \mathfrak{S}_{\alpha,\beta}(t)\varphi(0), & t\in\mathcal{J}'.\end{cases} \tag{3.3}$$

Then $\eta \in \mathscr{B}_\mathrm{T}$. Let $\mathfrak{x}(t)=\eta(t)+y(t), -\infty<t\leq \mathrm{T}$. Clearly, $\mathfrak{x}(t)$ satisfies 3.1 if and only if $y(t)$ satisfies $y(0)=0$ and

$$\begin{aligned}
y(t)=&\,\mathfrak{S}_{\alpha,\beta}(t)\Xi(0,\varphi(0))-\Xi(t,\eta_t+y_t)\\
&-\int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\eta_s+y_s)ds+\int_0^t \mathfrak{P}_\beta(t-s)\\
&\times\left[\mathbb{B}(s)+\mathfrak{f}(s,\eta_s+y_s,(\eta+y)\sigma(\eta_s+y(s),s))\right]ds\\
&+\int_0^t \mathfrak{P}_\beta(t-s)\gamma(s,\eta_s+y_s)d\mathcal{Z}_\mathcal{H}(s)ds\\
&+\int_0^t\int_\mathscr{Z} \mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta).
\end{aligned}$$

Let $\mathfrak{U}_\mathrm{T}=\{y\in\mathscr{B}_\mathrm{T}, y(0)=0\in\mathcal{B}\}$. For every $y\in\mathfrak{U}_\mathrm{T}$,

$$\|y\|=\|y(0)\|_\mathcal{B}+\sup_{0\leq t\leq \mathrm{T}}\mathbb{E}\|y(t)\|=\sup_{0\leq t\leq \mathrm{T}}\mathbb{E}\|y(t)\|.$$

Thus, $(\mathfrak{U}_\mathrm{T},\|.\|)$ is a Banach space. Let us define the set $\mathtt{B}=\{y\in\mathfrak{U}_\mathrm{T},\mathbb{E}\|y\|^2\leq k\}$ for some $k\geq 0$.

It is obvious that $\mathtt{B}\subset\mathfrak{U}_\mathrm{T}$ is a bounded closed convex set. Moreover, for $y\in\mathtt{B}$, we have

$$\begin{aligned}
\|y_t+\eta_t\|^2_\mathcal{B}\leq&\, 2\left(\|y_t\|^2_\mathcal{B}+\|\eta_t\|^2_\mathcal{B}\right),\\
\leq&\, 2\mathfrak{K}_1^2\left(k+\left(\frac{\mathscr{M}\mathrm{T}^{\delta-1}}{\Gamma(\delta)}\right)^2\|\varphi\|^2_\mathcal{B}\right)+2\mathfrak{K}_2^2\|\varphi\|^2_\mathcal{B}:=\mathfrak{s}.
\end{aligned}$$

We may define an operator $\Lambda:\mathfrak{U}_\mathrm{T}\to\mathfrak{U}_\mathrm{T}$ by

$$(\Lambda y)(t)=\begin{cases}0, & t\in(-\infty,0],\\
\mathfrak{S}_{\alpha,\beta}(t)\Xi(0,\varphi(0))-\Xi(t,\eta_t+y_t)-\int_0^t\mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\eta_s+y_s)ds+\int_0^t\mathfrak{P}_\beta(t-s)\\
.\left[\mathbb{B}(s)+\mathfrak{f}(s,\eta_s+y_s,(\eta+y)\sigma(\eta_s+y(s),s))\right]ds+\int_0^t\mathfrak{P}_\beta(t-s)\gamma(s,\eta_s+y_s)d\mathcal{Z}_\mathcal{H}(s)ds\\
+\int_0^t\int_\mathscr{Z}\mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta), & t\in\mathcal{J}'.\end{cases}$$

Observing that Λ is well defined on $\mathtt{B}$, for $k>0$. Π has a fixed point if and only if Λ has a fixed point. Decomposing $\Lambda=\Lambda_1+\Lambda_2$, given by

$$
\begin{aligned}
(\Lambda_1 y)(t) = {} & \mathfrak{S}_{\alpha,\beta}(t)\Xi(0,\varphi(0)) - \Xi(t,\eta_t + y_t) \\
& - \int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\eta_s + y_s)ds + \int_0^t \mathfrak{P}_\beta(t-s) \\
& \times \mathfrak{f}(s,\eta_s + y_s,(\eta + y)\sigma(\eta_s + y(s),s))ds, \\
(\Lambda_2 y)(t) = {} & \int_0^t \mathfrak{P}_\beta(t-s)\mathbb{B}(s)ds + \int_0^t \mathfrak{P}_\beta(t-s)\gamma(s,\eta_s + y_s)d\mathcal{Z}_{\mathcal{H}}(s)ds \\
& + \int_0^t \int_{\mathscr{Z}} \mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\eta_s + y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta).
\end{aligned}
$$

Now we need to prove that Λ_1 and Λ_2 satisfy all the conditions of the Kransnoselskii's fixed point theorem.

Step 1: Claim that Λ_1 is a contraction mapping. Let $t \in \mathcal{J}$ and $y_1, y_2 \in \mathtt{B}$, consider

$$
\begin{aligned}
& \mathbb{E}\|(\Lambda_1 y_1) - (\Lambda_1 y_2)\|^2_{\mathcal{C}_{1-\delta}} \\
& = \sup_{t\in\mathcal{J}} t^{2-2\delta}\mathbb{E}\|(\Lambda_1 y_1)(t) - (\Lambda_1 y_2(t))\|^2 \\
& \le 3\sup_{t\in\mathcal{J}} t^{2-2\delta}\Bigg[\mathbb{E}\|\Xi(t,\eta_t,y_{1t}) - \Xi(t,\eta_t,y_{2t})\|^2 \\
& \quad + \mathbb{E}\Bigg\|\int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}^{1-\mu}\mathfrak{A}^{\mu}\Big[\Xi(s,\eta_s + y_{1s}) \\
& \quad - \Xi(s,\eta_s + y_{2s})\Big]ds\Bigg\|^2 + \mathbb{E}\Bigg\|\int_0^t \mathfrak{P}_\beta(t-s) \\
& \quad \Big[\mathfrak{f}(s,\eta_s + y_{1s},(\eta + y_1)(\sigma(\eta(s) + y_1(s),s))) \\
& \quad - \mathfrak{f}(s,\eta_s + y_{1s},(\eta + y_1)(\sigma(\eta(s) + y_2(s),s))) \\
& \quad - \mathfrak{f}(s,\eta_s + y_{2s},(\eta + y_2)(\sigma(\eta(s) + y_2(s),s))) \\
& \quad + \mathfrak{f}(s,\eta_s + y_{2s},(\eta + y_2)(\sigma(\eta(s) + y_2(s),s)))\Big]ds\Bigg\|^2\Bigg] \\
& \le 3\mathrm{T}^{2-2\delta}\Bigg[\|\mathfrak{A}^{-\mu}\|^2\Xi\mathfrak{K}_1^2\sup_{t\in\mathcal{J}}\mathbb{E}\|y_1(t) - y_2(t)\|^2 + \mathscr{C}_\Xi\left[\frac{\beta\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu - 1}\mathfrak{K}_1^2 \\
& \quad \times \sup_{s\in\mathcal{J}}\mathbb{E}\|y_1(s) - y_2(s)\|^2 + 2\mathscr{C}_{\mathfrak{f}} \\
& \quad \left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2(2k\mathscr{C}_\sigma + \mathfrak{K}_1^2)\frac{\mathrm{T}^{2\beta}}{2\beta - 1}\sup_{s\in\mathcal{J}}\mathbb{E}\|y_1(s) - y_2(s)\|^2\Bigg] \\
& \le \chi\sup_{t\in\mathcal{J}}\mathbb{E}\|y_1(t) - y_2(t)\|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\chi = {} & 3\mathrm{T}^{2-2\delta}\Bigg[\|\mathfrak{A}^{-\mu}\|^2\Xi\mathfrak{K}_1^2 + \mathscr{C}_\Xi\left[\frac{\beta\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu - 1}\mathfrak{K}_1^2 \\
& + 2\mathscr{C}_{\mathfrak{f}}\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2(2k\mathscr{C}_\sigma + \mathfrak{K}_1^2)\frac{\mathrm{T}^{2\beta}}{2\beta - 1}\Bigg].
\end{aligned}
$$

Thus, we have obtained $\chi < 1$. So the operator Λ_1 is a contraction mapping.

Step 2: To prove $\forall\ y, y^* \in \mathtt{B}$, $(\Lambda_1 y)(t) + (\Lambda_2 y^*)(t) \in \mathtt{B}$.

$$
\begin{aligned}
&\mathbb{E}\left\|(\Lambda_1(y)(t)+(\Lambda_2 y^*)(t)\right\|^2\\
&\quad\le 7\Bigg[\mathbb{E}\left\|\mathfrak{S}_{\alpha,\beta}(t)\Xi(0,\varphi(0))\right\|^2+\mathbb{E}\left\|\Xi(t,\eta_t\right.\\
&\qquad\left.+y_t)\right\|^2+\mathrm{T}\int_0^t\mathbb{E}\left\|\mathfrak{P}_\beta(t-s)\mathfrak{A}\Xi(s,\eta_s+y_s)\right\|^2ds\\
&\qquad+\mathrm{T}\int_0^t\mathbb{E}\left\|\mathfrak{P}_\beta(t-s)\right\|^2\mathbb{E}\Big\|\mathfrak{f}(s,\eta_s+y_s,(\eta+y)(\sigma(\eta_s+y(s),s)))\\
&\qquad-\mathfrak{f}(s,0,(\eta+y)(\sigma(\eta(0),0)))\\
&\qquad+\mathfrak{f}(s,0,(\eta+y)(\sigma(\eta(0),0)))\Big\|^2ds+\mathbb{E}\left\|\int_0^t\mathfrak{P}_\beta(t-s)\gamma(s,\eta_s+y^*_s)d\mathcal{Z}_{\mathcal{H}}(s)\right\|^2\\
&\qquad+\mathrm{T}\int_0^t\mathbb{E}\left\|\mathfrak{P}_\beta(t-s)\mathbb{B}(s)\right\|^2ds+\mathbb{E}\left\|\int_0^t\int_{\mathscr{Z}}\mathfrak{P}_\beta(t-s)\mathfrak{h}(s,\eta_s+y^*_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)\right\|^2\Bigg]\\
&\quad\le 7\Bigg[\left(\frac{\mathrm{T}^{\delta-1}\mathscr{M}}{\Gamma(\delta)}\right)^2\|\mathfrak{A}^{-\mu}\|^2(\mathfrak{l}_1\mathbb{E}\|\varphi\|^2+\mathfrak{l}_2)+(\mathfrak{l}_1\mathfrak{s}+\mathfrak{l}_2)\Big[\|\mathfrak{A}^{-\mu}\|^2\\
&\qquad+\left(\frac{\beta\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right)^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu-1}\Big]\\
&\qquad+\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2\frac{\mathrm{T}^{2\beta}}{2\beta-1}\Big[\|\mathbb{B}\|^2_{\mathcal{L}^2_{\mathfrak{F}}}+2(\mathscr{C}_{\mathfrak{f}}(\mathfrak{s}+k^2\mathscr{C}_\sigma)+\tilde{\mathfrak{f}}_0)+\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\mathscr{C}_\gamma\mathfrak{s}+\mathscr{C}_{\mathfrak{h}}\mathfrak{s}\mathrm{T}^{-1}\Big]\Bigg].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\left\|(\Lambda_1(y)(t)+(\Lambda_2 y^*)(t)\right\|^2_{\mathcal{C}_{1-\delta}}\\
&\quad\le 7\Bigg[\left(\frac{\mathscr{M}}{\Gamma(\delta)}\right)^2\|\mathfrak{A}^{-\mu}\|^2(\mathfrak{l}_1\mathbb{E}\|\varphi\|^2+\mathfrak{l}_2)+\mathrm{T}^{2-2\delta}(\mathfrak{l}_1\mathfrak{s}+\mathfrak{l}_2)\Big[\|\mathfrak{A}^{-\mu}\|^2\\
&\qquad+\left(\frac{\beta\mathcal{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right)^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu-1}\Big]\\
&\qquad+\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2\frac{\mathrm{T}^{2\beta+2-2\delta}}{2\beta-1}\Big[\|\mathbb{B}\|^2_{\mathcal{L}^2_{\mathfrak{F}}}+2(\mathscr{C}_{\mathfrak{f}}(\mathfrak{s}+k^2\mathscr{C}_\sigma)+\tilde{\mathfrak{f}}_0)+\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\mathscr{C}_\gamma\mathfrak{s}+\mathscr{C}_{\mathfrak{h}}\mathfrak{s}\mathrm{T}^{-1}\Big]\Bigg]\\
&\quad\le\mathfrak{l}_0\ \text{ (a constant).}
\end{aligned}
$$

This proves that $\forall\, y, y^*\in\mathtt{B}$, $(\Lambda_1 y)(t)+(\Lambda_2 y^*)(t)\in\mathtt{B}$.

Step 3: To prove $\{(\Lambda_2 y)(t) : t\in\mathcal{J}',\ y\in\mathtt{B}\}$ is bounded. For $k>0$ and $y\in\mathtt{B}$,

$$
\begin{aligned}
\mathbb{E}\left\|(\Lambda_2 y)(t)\right\|^2_{\mathcal{C}_{1-\delta}}&\le 3\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2\frac{\mathrm{T}^{2\beta+2-2\delta}}{2\beta-1}\\
&\qquad\Big[\|\mathbb{B}\|^2_{\mathcal{L}^2_{\mathfrak{F}}}+2(\mathscr{C}_{\mathfrak{f}}(\mathfrak{s}+k^2\mathscr{C}_\sigma)+\tilde{\mathfrak{f}}_0)+\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\mathscr{C}_\gamma\mathfrak{s}+\mathscr{C}_{\mathfrak{h}}\mathfrak{s}\mathrm{T}^{-1}\Big]\\
&\le\tilde{\mathfrak{l}}_0\ \text{(a constant).}
\end{aligned}
$$

This implies that $(\Lambda_2 y)(t)$ is bounded $\forall\, t\in\mathcal{J}'$ and $y\in\mathtt{B}$.

Step 4: To prove $\{(\Lambda_2 y) : y \in \mathtt{B}\}$ is equicontinuous. Choose $\epsilon > 0$ to be small, $y \in \mathtt{B}$ and $0 < t_1 < t_2$, then

$$
\begin{aligned}
&\mathbb{E}\|(\Lambda_2 y)(t_2) - (\Lambda_2 y)(t_1)\|^2_{\mathcal{C}_{1-\delta}} \\
&= \sup_{t\in\mathcal{J}} t^{2-2\delta}\mathbb{E}\|(\Lambda_2 y)(t_2) - (\Lambda_2 y)(t_1)\|^2 \\
&\leq \sup_{t\in\mathcal{J}} t^{2-2\delta}\mathbb{E}\Big\| \int_0^{t_2} \mathfrak{P}_\beta(t-s)\mathbb{B}(s)ds + \int_0^{t_2} \mathfrak{P}_\beta(t-s)\gamma(s,\eta_s+y_s)d\mathcal{Z}_{\mathcal{H}}(s) \\
&\quad + \int_0^{t_2} \mathfrak{P}_\beta(t-s) \\
&\quad \times \int_{\mathscr{Z}} \mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta) - \int_0^{t_1} \mathfrak{P}_\beta(t-s)\mathbb{B}(s)ds \\
&\quad - \int_0^{t_1} \mathfrak{P}_\beta(t-s)\gamma(s,\eta_s+y_s)d\mathcal{Z}_{\mathcal{H}}(s) \\
&\quad - \int_0^{t_1} \mathfrak{P}_\beta(t-s)\int_{\mathscr{Z}} \mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)\Big\|^2 \\
&\leq \mathrm{T}^{2-2\delta}\Big[\mathbb{E}\Big\|\int_0^{t_1-\epsilon}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\mathbb{B}(s)ds \\
&\quad + \int_0^{t_1-\epsilon}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)] \\
&\quad \times \gamma(s,\eta_{s+y_s})d\mathcal{Z}_{\mathcal{H}}(s) + \int_0^{t_1-\epsilon}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\int_{\mathscr{Z}} \mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta) \\
&\quad + \int_{t_1-\epsilon}^{t_1}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\mathbb{B}(s)ds \\
&\quad + \int_{t_1-\epsilon}^{t_1}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\gamma(s,\eta_{s+y_s})d\mathcal{Z}_{\mathcal{H}}(s) \\
&\quad + \int_{t_1-\epsilon}^{t_1}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\int_{\mathscr{Z}} \mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta) \\
&\quad + \int_{t_1}^{t_2}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)] \\
&\quad \times \mathbb{B}(s)ds + \int_{t_1}^{t_2}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)]\gamma(s,\eta_{s+y_s})d\mathcal{Z}_{\mathcal{H}}(s) \\
&\quad + \int_{t_1}^{t_2}[\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)] \\
&\quad \times \int_{\mathscr{Z}} \mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)\Big\|^2\Big] \\
&\leq 9\mathrm{T}^{2-2\delta}\Big[(t_1-\epsilon)\int_0^{t_1-\epsilon}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)\Big\|^2\mathbb{E}\|\mathbb{B}(s)\|^2ds + [\mathfrak{s}\mathscr{C}_{\mathfrak{h}} \\
&\quad + \mathfrak{C}_{\mathcal{H}}(t_1-\epsilon)^{2\mathcal{H}}\mathscr{C}_\gamma\mathfrak{s}] \\
&\quad \times \int_0^{t_1-\epsilon}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)\Big\|^2ds + \epsilon\int_{t_1-\epsilon}^{t_1}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)\Big\|^2ds \\
&\quad + [\mathfrak{s}\mathscr{C}_{\mathfrak{h}} + \mathfrak{C}_{\mathcal{H}}\epsilon^{2\mathcal{H}}\mathscr{C}_\gamma\mathfrak{s}]\int_{t_1-\epsilon}^{t_1}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)\Big\|^2ds \\
&\quad + (t_2-t_1)\int_{t_1}^{t_2}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) \\
&\quad - \mathfrak{P}_\beta(t_1-s)\Big\|^2\mathbb{E}\|\mathbb{B}(s)\|^2ds + [\mathfrak{s}\mathfrak{C}_{\mathcal{H}}(t_2-t_1)^{2\mathcal{H}}\mathscr{C}_\gamma\mathfrak{s}] \\
&\quad \int_{t_1}^{t_2}\mathbb{E}\Big\|\mathfrak{P}_\beta(t_2-s) - \mathfrak{P}_\beta(t_1-s)\Big\|^2ds\Big].
\end{aligned}
$$

By the continuity of the operator $\mathfrak{P}_\beta(t),\ \ t > 0$, in the uniform operator topology, the RHS tends to zero independent of $y \in \mathtt{B}$ as $t_2 \to t_1$ and with sufficiently small ϵ. This implies $\{(\Lambda_2 y) : y \in \mathtt{B}\}$ is equicontinuous.

Step 5: To depict $\Phi(t) = \{(\Lambda_2 y) : y \in \mathtt{B}\}$ is relatively compact.

For $t = 0$, $\Phi(0)$ is relatively compact in $\mathtt{B}$. Let $0 < t \le \mathtt{T}$ be fixed and $0 < \xi < t$. For arbitrary $\Upsilon > 0$, we may define

$$
\begin{aligned}
&(\Lambda^{\xi,\Upsilon} y)(t)\\
&= \beta \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1} \varpi_\beta(\theta) \mathtt{T}((t-s)^\beta \theta) \mathbb{B}(s) d\theta ds\\
&\quad + \beta \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1} \varpi_\beta(\theta) \mathtt{T}((t-s)^\beta \theta)\\
&\quad \times \gamma(s, \eta_s + y_s) d\mathcal{Z}_{\mathcal{H}}(s) d\theta + \beta \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1} \varpi_\beta(\theta) \mathtt{T}((t-s)^\beta \theta)\\
&\quad \int_{\mathscr{Z}} \mathfrak{h}(s, \eta_s + y_s, \vartheta) d\theta d\tilde{\mathbb{N}}(ds, d\vartheta)\\
&\le \beta \mathtt{T}(\xi^\beta \Upsilon) \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1} \varpi_\beta(\theta) \mathtt{T}((t-s)^\beta \theta - \xi^\beta \Upsilon) \mathbb{B}(s) d\theta ds\\
&\quad + \beta \mathtt{T}(\xi^\beta \Upsilon) \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1}\\
&\quad \times \varpi_\beta(\theta) \mathtt{T}((t-s)^\beta \theta - \xi^\beta \Upsilon) \gamma(s, \eta_s + y_s) d\mathcal{Z}_{\mathcal{H}}(s) d\theta + \beta \mathtt{T}(\xi^\beta \Upsilon)\\
&\quad \int_0^{t-\epsilon} \int_\Upsilon^\infty \theta(t-s)^{\beta-1} \varpi_\beta(\theta)\\
&\quad \times \mathtt{T}((t-s)^\beta \theta - \xi^\beta \Upsilon) \int_{\mathscr{Z}} \mathfrak{h}(s, \eta_s + y_s, \vartheta) d\theta d\tilde{\mathbb{N}}(ds, d\vartheta)
\end{aligned}
$$

Since, $\mathtt{T}(\xi^\beta \Upsilon)$ being a compact operator for $\xi^\beta \Upsilon > 0$, the set $\Phi^{\xi,\Upsilon}(t) = \{(\Lambda^{\xi,\Upsilon} y)(t) : y \in \mathtt{B}\}$ is relatively compact on $\mathscr{X}$, for every ξ, $0 < \xi < t$ and $\forall\ \Upsilon > 0$.

Furthermore,

$$
\begin{aligned}
&\mathbb{E}\left\|(\Lambda_2 y)(t)-(\Lambda_2^{\xi,\Upsilon}y)(t)\right\|^2_{\mathcal{C}_{1-\delta}}\\
&\leq 6\sup_{t\in\mathcal{J}} t^{2-2\delta}\Bigg(\beta^2\mathbb{E}\left\|\int_0^t\int_0^{\Upsilon}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\mathbb{B}(s)d\theta ds\right\|^2\\
&\quad+\beta^2\mathbb{E}\left\|\int_0^t\int_0^{\Upsilon}\theta(t-s)^{\beta-1}\times\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\gamma(s,\eta_s+y_s)d\theta d\mathcal{Z}_{\mathcal{H}}(s)\right\|^2\\
&\quad+\beta^2\mathbb{E}\left\|\int_0^t\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\mathbb{B}(s)d\theta ds\right.\\
&\quad\left.-\int_0^{t-\xi}\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\mathbb{B}(s)d\theta ds\right\|^2\\
&\quad+\beta^2\mathbb{E}\left\|\int_0^t\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\times\gamma(s,\eta_s+y_s)d\theta d\mathcal{Z}_{\mathcal{H}}(s)-\int_0^{t-\xi}\right.\\
&\quad\left.\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\gamma(s,\eta_s+y_s)d\theta d\mathcal{Z}_{\mathcal{H}}(s)\right\|^2\\
&\quad+\beta^2\mathbb{E}\left\|\int_0^t\int_0^{\Upsilon}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\int_{\mathscr{Z}}\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)d\theta\right\|^2\\
&\quad+\beta^2\mathbb{E}\left\|\int_0^t\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\int_{\mathscr{Z}}\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)d\theta\right.\\
&\quad\left.-\int_0^{t-\xi}\int_{\Upsilon}^{\infty}\theta(t-s)^{\beta-1}\varpi_\beta(\theta)\mathrm{T}((t-s)^\beta\theta)\int_{\mathscr{Z}}\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)d\theta\right\|^2\Bigg)\\
&\leq 6\mathrm{T}^{2-2\delta}\Bigg(\beta^2\mathscr{M}^2\int_0^t(t-s)^{\beta-1}ds\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2\int_0^t(t-s)^{\beta-1}\|\mathbb{B}(s)\|^2ds\\
&\quad+\beta^2\mathscr{M}^2\int_{t-\xi}^t(t-s)^{\beta-1}ds\times\left(\int_{\Upsilon}^{\infty}\theta\varpi_\beta(\theta)d\theta\right)^2\int_{t-\xi}^t(t-s)^{\beta-1}\|\mathbb{B}(s)\|^2ds+\beta^2\mathscr{M}^2\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\int_0^t(t-s)^{\beta-1}ds\\
&\quad\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2\\
&\quad\times\int_0^t(t-s)^{\beta-1}\mathbb{E}\left\|\gamma(s,\eta_s+y_s)\right\|^2ds+\beta^2\mathscr{M}^2\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\\
&\quad\int_{t-\xi}^t(t-s)^{\beta-1}ds\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2\times\int_0^t(t-s)^{\beta-1}\mathbb{E}\left\|\gamma(s,\eta_s+y_s)\right\|^2ds+\beta^2\mathscr{M}^2\\
&\quad\int_0^t(t-s)^{\beta-1}ds\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2\int_0^t(t-s)^{\beta-1}\\
&\quad\times\mathbb{E}\left\|\int_{\mathscr{Z}}\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)\right\|^2+\beta^2\mathscr{M}^2\int_{t-\xi}^t(t-s)^{\beta-1}ds\\
&\quad\left(\int_{\Upsilon}^{\infty}\theta\varpi_\beta(\theta)d\theta\right)^2\int_{t-\xi}^t(t-s)^{\beta-1}\times\mathbb{E}\left\|\int_{\mathscr{Z}}\mathfrak{h}(s,\eta_s+y_s,\vartheta)\tilde{\mathbb{N}}(ds,d\vartheta)\right\|^2\Bigg)\\
&\leq 6\mathrm{T}^{2-2\delta}\Bigg(\beta\mathscr{M}^2\|\mathbb{B}\|^2_{\mathcal{L}^2_{\mathfrak{Z}}}\Bigg[\mathrm{T}^\beta\int_0^t(t-s)^{\beta-1}\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2ds\\
&\quad+\xi^\beta\int_{t-\xi}^t(t-s)^{\beta-1}\left(\int_{\Upsilon}^{\infty}\theta\varpi_\beta(\theta)d\theta\right)^2ds\Bigg]\\
&\quad+\beta\mathscr{M}^2\mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\mathscr{C}_\gamma\mathfrak{s}\Bigg[\mathrm{T}^\beta\int_0^t(t-s)^{\beta-1}\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2ds\\
&\quad+\xi^\beta\int_{t-\xi}^t(t-s)^{\beta-1}\left(\int_{\Upsilon}^{\infty}\theta\varpi_\beta(\theta)d\theta\right)^2ds\Bigg]+\beta\mathscr{M}^2\mathfrak{C}_{\mathfrak{h}}\mathfrak{s}\Bigg[\mathrm{T}^\beta\int_0^t(t-s)^{\beta-1}\left(\int_0^{\Upsilon}\theta\varpi_\beta(\theta)d\theta\right)^2ds\\
&\quad+\xi^\beta\int_{t-\xi}^t(t-s)^{\beta-1}\left(\int_{\Upsilon}^{\infty}\theta\varpi_\beta(\theta)d\theta\right)^2ds\Bigg]\Bigg)\to 0\quad\text{as}\quad\xi\to 0^+\quad\text{as}\quad\Upsilon\to 0^+.
\end{aligned}
$$

Thus, there are relatively compact sets that are arbitrarily close to the set $\Phi(t) = \{(\Lambda_2 y) : y \in \mathbb{B}\}$. Hence $\Phi(t)$ is relatively compact in $\mathbb{B}$. Conceptually by Arzela-Ascoli theorem Λ_2 is completely continuous. Therefore there exist a fixed point. □

Trajectory Controllability

By applying Gronwall's inequality, the T-controllability of the system 1.1 are investigated.

Theorem 4.1 *If the assumptions* (A1)-(A6) *hold, the Hilfer fractional system* 1.1 *is trajectory controllable on* $\mathcal{J}'$, *provided* $(2\beta\mu - 1) > 0$ *and*

$$\mathfrak{Z}_1 = 5\mathscr{C}_\Xi\left(5\|\mathfrak{A}^{-\mu}\|^2\mathscr{C}_\Xi + 5\mathscr{C}_\Xi\left[\frac{\beta\mathfrak{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu - 1}\right) < 1.$$

Proof Let $\aleph(t)$ be the given trajectory on $\mathfrak{t}$. For $\alpha \in (0,1)$, choose the feedback control $\mathbb{B}(t)$ as,

$$\mathbb{B}(t) = \mathfrak{D}_{0^+}^{\alpha,\beta}[\aleph(t) + \Xi(t,\aleph_t)] - \mathfrak{A}\aleph(t) - \mathfrak{f}(t,\aleph_t,\aleph(\sigma(\aleph(t),t))) - \gamma(t,\aleph_t)\frac{d\mathcal{Z}_{\mathcal{H}}(t)}{dt} \\ - \int_{\mathscr{Z}} \mathfrak{h}(t,\aleph_t,\vartheta)\tilde{\mathbb{N}}(dt,d\vartheta).$$

Therefore 1.1 becomes

$$\mathfrak{D}_{0^+}^{\alpha,\beta}[\mathfrak{x}(t) + \Xi(t,\mathfrak{x}_t)] \\ = \mathfrak{A}\mathfrak{x}(t) + \Bigg[\mathfrak{D}_{0^+}^{\alpha,\beta}[\aleph(t) + \Xi(t,\aleph_t)] - \mathfrak{A}\aleph(t) - \mathfrak{f}(t,\aleph_t,\aleph(\sigma(\aleph(t),t))) - \gamma(t,\aleph_t)\frac{d\mathcal{Z}_{\mathcal{H}}(t)}{dt} \\ - \int_{\mathscr{Z}} \mathfrak{h}(t,\aleph_t,\vartheta)\tilde{\mathbb{N}}(dt,d\vartheta)\Bigg] + \mathfrak{f}(t,\mathfrak{x}_t,\mathfrak{x}(\sigma(\mathfrak{x}(t),t))) + \gamma(t,\mathfrak{x}_t)\frac{d\mathcal{Z}_{\mathcal{H}}t)}{dt} \\ + \int_{\mathscr{Z}} \mathfrak{h}(t,\mathfrak{x}_t,\vartheta)\tilde{\mathbb{N}}(dt,d\vartheta).$$

Put $\psi(t) = \mathfrak{x}(t) - \aleph(t)$, we obtain that

$$\mathfrak{D}_{0^+}^{\alpha,\beta}[\psi(t) + \Xi(t,\mathfrak{x}_t) - \Xi(t,\aleph_t)] = \mathfrak{A}\psi(t) + \mathfrak{f}(t,\mathfrak{x}_t,\mathfrak{x}(\sigma(\mathfrak{x}(t),t))) - \mathfrak{f}(t,\aleph_t,\aleph(\sigma(\aleph(t),t))) \\ + [\gamma(t,\mathfrak{x}_t) - \gamma(t,\aleph_t)]\frac{d\mathcal{Z}_{\mathcal{H}}(t)}{dt} \\ \int_{\mathscr{Z}} [\mathfrak{h}(t,\mathfrak{x}_t,\vartheta) - \mathfrak{h}(t,\aleph_t,\vartheta)]\tilde{\mathbb{N}}(dt,d\vartheta) \\ \mathcal{I}_{0^+}^{(1-\delta)}\psi(t) = \mathcal{I}_{0^+}^{(1-\delta)}\varphi - \mathcal{I}_{0^+}^{(1-\delta)}\varphi = 0, \quad t \in (-\infty, 0].$$

Thus the mild solution is

$$\psi(t) = \Xi(t, \aleph_t) - \Xi(t, \mathfrak{x}_t) - \int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}[\Xi(s, \mathfrak{x}_s) - \Xi(s, \aleph_s)]ds$$
$$+ \int_0^t \mathfrak{P}_\beta(t-s)\Big[\mathfrak{f}(s, \mathfrak{x}_s, \mathfrak{x}(\sigma(\mathfrak{x}(s), s)))$$
$$- \mathfrak{f}(s, \aleph_s, \aleph(\sigma(\aleph(s), s)))\Big]ds + \int_0^t \mathfrak{P}_\beta(t-s)\Big[\gamma(s, \mathfrak{x}_s) - \gamma(s, \aleph_s)\Big]d\mathcal{Z}_{\mathcal{H}}(s)$$
$$+ \int_0^t \mathfrak{P}_\beta(t-s)$$
$$\times \int_{\mathscr{Z}} [\mathfrak{h}(s, \mathfrak{x}_s, \vartheta) - \mathfrak{h}(s, \aleph_s, \vartheta)]\tilde{\mathbb{N}}(dt, d\vartheta), \quad t \in \mathcal{J}'.$$

Hence for $t \in (-\infty, 0]$, the initial data to be zero, we obtain $\eta(t) = 0, t \in (-\infty, 0]$ and $t \in \mathcal{J}'$ (i.e) $\eta(t) = 0$ for $t \in (-\infty, \mathrm{T}]$. Therefore $\mathfrak{x}_t = \eta_t + \mathfrak{x}_t = \mathfrak{x}_t$ and $\aleph_t = \eta_t + \aleph_t = \aleph_t$ on $\mathcal{J}'$. Now,

$$\mathbb{E}\|\psi(t)\|^2 \leq 5\Bigg[\mathbb{E}\big\|\Xi(t, \aleph_t) - \Xi(t, \mathfrak{x}_t)\big\|^2 + \mathbb{E}\Big\|\int_0^t \mathfrak{P}_\beta(t-s)\mathfrak{A}[\Xi(s, \mathfrak{x}_s) - \Xi(s, \aleph_s)]ds\Big\|^2$$
$$+ \mathbb{E}\Big\|\int_0^t \mathfrak{P}_\beta(t-s)$$
$$\times \Big[\mathfrak{f}(s, \mathfrak{x}_s, \mathfrak{x}(\sigma(\mathfrak{x}(s), s))) - \mathfrak{f}(s, \aleph_s, \aleph(\sigma(\aleph(s), s)))\Big]ds\Big\|^2$$
$$+ \mathbb{E}\Big\|\int_0^t \mathfrak{P}_\beta(t-s)\Big[\gamma(s, \mathfrak{x}_s) - \gamma(s, \aleph_s)\Big]d\mathcal{Z}_{\mathcal{H}}(s)\Big\|^2$$
$$+ \mathbb{E}\Big\|\int_0^t \mathfrak{P}_\beta(t-s)\int_{\mathscr{Z}}[\mathfrak{h}(s, \mathfrak{x}_s, \vartheta) - \mathfrak{h}(s, \aleph_s, \vartheta)]\tilde{\mathbb{N}}(dt, d\vartheta)\Big\|^2\Bigg]$$
$$\leq 5\Bigg(\|\mathfrak{A}^{-\mu}\|^2\mathscr{C}_\Xi\mathbb{E}\|\psi(t)\|^2 + \mathrm{T}\mathscr{C}_\Xi\left[\frac{\beta C_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2$$
$$\int_0^t (t-s)^{2\beta\mu-2}\mathbb{E}\|\psi(s)\|^2 ds$$
$$+ 2\mathrm{T}\mathscr{C}_\mathfrak{f}\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2 \int_0^t (t-s)^{2\beta-2}(\mathscr{C}_\sigma\mathfrak{S} + 2)\mathbb{E}\|\psi(s)\|^2 ds$$
$$+ \mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}-1}\mathscr{C}_\gamma\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2$$
$$\times \int_0^t (t-s)^{2\beta-2}\mathbb{E}\|\psi(2)\|^2 ds + \mathscr{C}_\mathfrak{h}\mathrm{T}\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2 \int_0^t (t-s)^{2\beta-2}\mathbb{E}\|\psi(s)\|^2 ds\Bigg)$$
$$\leq \mathfrak{Z}_1\mathbb{E}\|\psi(t)\|^2 + \mathfrak{Z}_2\int_0^t (t-s)^{2\beta-2}\mathbb{E}\|\psi(s)\|^2 ds,$$

where,

$$\mathfrak{Z}_1 = 5\mathscr{C}_\Xi 5\|\mathfrak{A}^{-\mu}\|^2\mathscr{C}_\Xi + 5\mathscr{C}_\Xi\left[\frac{\beta\mathfrak{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu-1},$$

with $\mathfrak{Z}_1 < 1$ and

$$\mathfrak{Z}_2 = 5\left(\frac{\mathscr{M}}{\Gamma(\beta)}\right)^2\left[2\mathrm{T}\mathscr{C}_{\mathfrak{f}}(\mathscr{C}_\sigma\mathfrak{s}+2) + \mathfrak{C}_{\mathcal{H}}\mathrm{T}^{2\mathcal{H}}\mathscr{C}_\gamma + \mathrm{T}\mathscr{C}_{\mathfrak{h}}\right].$$

Hence,

$$\mathbb{E}\|\psi(t)\|_{\mathcal{C}_{1-\delta}} = \sup_{t\in\mathcal{J}}\mathbb{E}\|\psi(t)\|^2 \le \frac{\mathrm{T}^{2-2\delta}\mathfrak{Z}_2}{1-\mathfrak{Z}_1}\int_0^t (t-s)^{2\beta-2}\mathbb{E}\|\psi\|^2 ds.$$

In view of generalized Gronwall's inequality, $\mathbb{E}\|\psi(t)\|_{\mathcal{C}_{1-\delta}} = 0$ (i.e), $\mathfrak{f}(t) = \aleph(t)$ a.e, $t \in \mathcal{J}'$. Thus 1.1 is T-controllable on $\mathcal{J}'$. □

Illustration

Consider the Hilfer fractional Stochastic differential equations of neutral type driven by Rosenblatt process and Poisson jumps of the form:

$$\begin{aligned}
\mathfrak{D}^{\frac{1}{3},\frac{4}{5}}\left[\varpi(t,\theta) + \frac{t^2+e^t|\varpi(t-\phi,\theta)|^2}{18}\right] &= \frac{\partial^2}{\partial\theta^2}\varpi(t,\theta) + \mathbb{B}(t,\theta)\\
&\quad + \left[\frac{t\sin\theta}{8\pi} + \frac{e^t|\varphi(t-\varphi,\theta)|}{2+|\varphi(t,\theta)|}\right]d\mathcal{Z}_{\mathcal{H}}(\tau)\\
&\quad + \frac{e^t\varpi(t-\phi,\theta)/\sqrt{2} + \varpi(t,\sin t|\varpi(t,\theta)|/\sqrt{2})}{9}\\
&\quad + \int_{\mathscr{Z}}\frac{e^t\cos|y\eta||\mathfrak{x}(t-\varphi,y)|}{9(1+|\mathfrak{x}(t-\varphi,y)|)}\tilde{\mathbb{N}}(dt,d\eta),\\
&\quad t\in(0,1]\ and\ \theta\in[0,\pi]\\
\varpi(t,\theta) &= 0, \quad on \quad [0,1]\times[0,\pi],\\
\mathcal{I}_{0^+}^{\frac{2}{15}}\varpi(t,\theta) &= \varphi(t,\theta), \quad \theta\in[0,\pi]\ and\ t\in(-\infty,0].
\end{aligned} \tag{5.1}$$

Assume that $[0,\pi]$ be a bounded domain in $\mathbb{R}^n$, $(n \ge 2)$ provided with the Lipschitz boundary. $\mathfrak{D}^{\frac{1}{3},\frac{4}{5}}$ is the HFD of order $\frac{1}{3}$ and type $\beta = \frac{4}{5}$. $\mathcal{I}_{0^+}^{\frac{2}{15}}$ is the R-L integral of order $\frac{2}{15}$. Let $\mathscr{X} = \mathcal{L}^2([0,\pi])$, $\varphi(t,\theta)\in\mathscr{C}$, the phase space and let $\mathfrak{A} : \mathcal{L}^2([0,\pi]) \to \mathcal{L}^2([0,\pi])$ be defined as $\mathfrak{D}(\mathfrak{A}) = \mathscr{X}_0^1([0,\pi]) \cap \mathscr{X}^2([0,\pi])$, $\mathfrak{A}\varpi = \triangle\ \varpi$, $\triangle\in\mathfrak{D}(\mathfrak{A})$.

Define$\mathfrak{D}(\mathfrak{A}) = \{y\in\mathscr{X} : y, \frac{dy}{d\mathfrak{x}}$ are absolutely continuous and $\frac{d^2\mathfrak{x}}{d\mathfrak{x}^2}\in\mathscr{X}, y(0) = y(\pi) = 0\}$. $\mathfrak{A}$ generates a compact semigroup $\{\mathcal{T}(t)\}_{t\ge0}$ being analytic and self-adjoint. Since $\mathfrak{A}$ has a discrete spectrum, there exist eigen values $-n^2$, $n\in\mathbb{R}$ with the orthogonal eigenvectors $\varpi_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny)$ then $\mathfrak{A}y = \sum_{n=1}^{\infty} -n^2\langle y,\varphi_n\rangle\varphi_n$. Also, $\mathcal{T}(t)y = \sum_{n=1}^{\infty} e^{-n^2t}\langle y,\varphi_n\rangle\varphi_n$, $y\in\mathscr{X}$

and $\forall\ t > 0$. Furthermore, $\|\mathcal{T}(t)\| \leq 1 = \mathtt{M}$ represents that $\{\mathcal{T}(t)\}_{t\geq 0}$ is uniformly bounded compact semigroup and hence, $\mathcal{R}(\Upsilon, \mathfrak{A}) = (\Upsilon - \mathfrak{A})^{-1}$ is a compact operator for every $\Upsilon \in \aleph(\mathfrak{A})$. We may present the phase space $\mathscr{C}$ with the norm,

$$\|\mathfrak{x}\|_{\mathscr{C}} = \int_{-\infty}^{0} \mathfrak{s}(s) \sup_{s\leq\varsigma\leq 0} \left(\mathbb{E}\|\varphi(\varsigma)\|^2\right)^{\frac{1}{2}} ds,$$

where $\mathfrak{s}(s) = e^{2s}$, for $s < 0$, $\mathfrak{l} = \int_{-\infty}^{0} \mathfrak{s}(s)ds = \frac{1}{2}$. The non-linear functions are defined as follows:

$$\begin{aligned}
\mathfrak{f}(t, \varpi(t,\theta), \varpi(\sigma(\varpi(t,\theta), t))) &= \frac{e^t \varpi(t-\phi,\theta)/\sqrt{2} + \varpi(t, \sin t|\varpi(t,\theta)|/\sqrt{2})}{9},\\
\sigma(\varpi(t,\theta), t) &= \sin t|\varpi(t,\theta)|/\sqrt{2}, \quad \gamma(t, \varpi(t,\theta))\\
&= \frac{t\sin\theta}{8\pi} + \frac{e^t|\varpi(t-\varphi,\theta)|}{2+|\varpi(t,\theta)|}\\
\mathfrak{h}(t, \varpi(t,\theta)) &= \frac{e^t \cos|y\eta||\mathfrak{x}(t-\varpi, y)|}{9(1+|\mathfrak{x}(t-\varpi,y)|)}
\end{aligned}$$

We may define the ball $\hat{r} > 0$, $B_{\hat{r}} = \{\varpi \in \mathfrak{x} : \mathbb{E}\|\varphi\|^2 \leq \hat{r}\}$. Now for $\varphi_1, \varphi_2 \in B_{\hat{r}}$,

$$\begin{aligned}
|\sigma(\varpi_1(t,\theta), t) - \sigma(\varpi_2(t,\theta), t)|^2 &= \left|\frac{\sin t|\varpi_1(t,\theta)|}{\sqrt{2}} - \frac{\sin t|\varpi_2(t,\theta)|}{\sqrt{2}}\right|^2\\
&\leq \frac{1}{2}|\varpi_1(t,\theta) - \varpi_2(t,\theta)|^2.
\end{aligned}$$

Moreover, $\sigma(.,0) = 0$, σ satisfies the assumption (A3) with $\mathscr{C}_\sigma = \frac{1}{2}$.

$$\begin{aligned}
&\mathbb{E}\left\|\mathfrak{f}(t, \varpi_1(t,\theta), \varpi_1(\sigma(\varpi_1(t,\theta), t))) - \mathfrak{f}(t, \varpi_2(t,\theta), \varpi_1(\sigma(\varpi_2(t,\theta), t)))\right\|^2\\
&\leq \frac{1}{81}\int_{-\infty}^{0} \mathbb{E}\left\|\frac{e^s \varpi_1(s-\varphi,\theta)}{\sqrt{2}} + \varpi_1(s, \sin s|\varpi_1(s,\theta)|/\sqrt{2})\right.\\
&\quad - \frac{e^s \varpi_2(s-\varphi,\theta)}{\sqrt{2}} - \varpi_2(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2})\\
&\quad \left.+ \varpi_1(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2}) - \varpi_1(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2})\right\|^2 ds\\
&\leq \frac{1}{27}\int_{\infty}^{0} \left[\frac{e^{2s}}{2} \sup_{s-\varphi\leq\varsigma\leq 0} \mathbb{E}\left\|\varpi_1(\varsigma,\theta) - \varpi_2(\varsigma,\theta)\right\|^2 + \mathscr{C}_\sigma \mathbb{E}\left\|\varpi_1(s,\theta) - \varpi_2(s,\theta)\right\|^2\right.\\
&\quad \left.+ \mathbb{E}\left\|\varpi_1(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2}) - \varpi_2(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2})\right\|^2\right] ds\\
&\leq \frac{1}{27}\left[\mathbb{E}\left\|\varpi_1(t,\theta) - \varpi_2(t,\theta)\right\|_{\mathscr{C}}^2\right.\\
&\quad \left.+ \mathbb{E}\left\|\varpi_1(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2}) - \varpi_2(s, \sin s|\varpi_2(s,\theta)|/\sqrt{2})\right\|^2\right]
\end{aligned}$$

Hence, the map $\mathfrak{f} : (0,1] \times \mathscr{C} \times \mathscr{X} \to \mathscr{X}$ satisfies the assumption (A2) with $\mathscr{C}_{\mathfrak{f}} = \frac{1}{27}$. Also,

$$
\begin{aligned}
&\mathbb{E}\left\|\gamma(t,\varpi_1(t,\theta))-\gamma(t,\varpi_2(t,\theta))\right\|^2\\
&\quad\le\int_{-\infty}^{0}e^{2s}\mathbb{E}\left\|\frac{|\varpi_1(s-\varphi,\theta)|}{2+|\varpi_1(s,\theta)|}-\frac{|\varpi_2(s-\varphi,\theta)|}{2+|\varpi_2(s,\theta)|}\right\|^2ds\\
&\quad\le 2\int_{\infty}^{0}e^{2s}\mathbb{E}\left\|\frac{|\varpi_1(s-\varphi,\theta)|-|\varpi_2(s-\varphi,\theta)|}{(1+|\varpi_1(s,\theta)|)(1+|\varpi_2(s,\theta)|)}\right\|^2ds\\
&\quad\le 2\int_{-\infty}^{0}e^{2s}\sup_{s-\varphi\le\varsigma\le 0}\mathbb{E}\left\||\varpi_1(\varsigma,\theta)|-|\varpi_2(\varsigma,\theta)|\right\|^2,\ \text{ since }\frac{1}{2+|\varpi|}<\frac{1}{1+|\varpi|}\\
&\quad\le 2\mathbb{E}\left\|\varpi_{t,\theta}-\varpi_2(t,\theta)\right\|^2_{\mathscr{C}},
\end{aligned}
$$

Thus $\gamma:(0,1]\times\mathscr{C}\to\mathcal{L}^2(\mathfrak{K},\mathscr{X})$ satisfies the assumptions (A5) with $\mathscr{C}_\gamma=2$.

Take $\mu=0.9$ and

$$
\mathbb{E}\left\|\mathfrak{A}^\mu\Xi(t,\varpi_1)-\mathfrak{A}^\mu\Xi(t,\varpi_2)\right\|^2\le\frac{1}{324}\|\mathfrak{A}^\mu\|^2\mathbb{E}\left\|\varpi_1(t,\theta)-\varpi_2(t,\theta)\right\|^2_{\mathscr{C}}
$$

with $\mathscr{C}_\Xi=\frac{\|\mathfrak{A}^{0.9}\|^2}{324}$. Also, $(2\beta-1)=0.6>0$ and $(2\beta\mu-1)=0.44>0$. $\Gamma(\beta)=\Gamma(0.8)=1.1642$, $\Gamma(1+\mu)=\Gamma(1.9)=0.9618$, $\Gamma(1+\beta\mu)=\Gamma(1.72)=0.9126$. We may consider $\mathtt{M}=\mathcal{T}=1$, $\delta=1-\frac{2}{15}=0.86$. By substituting the assumed values in 3.1, we can obtain $\mathscr{C}<1$, if $(0.00372)\|\mathfrak{A}^{0.9}\|^2\mathscr{C}_{0.1}^2+.08\hat{r}<0.9$. Hence, this satisfies all the assumptions of Theorem 3.1 thereby there occurs a mild solution for the system 5.1. Also,

$$
\begin{aligned}
\mathfrak{Z}_1&=\mathscr{C}_\Xi\left(5\|\mathfrak{A}^{-\mu}\|^2\mathscr{C}_\Xi+5\mathscr{C}_\Xi\left[\frac{\beta\mathfrak{C}_{1-\mu}\Gamma(1+\mu)}{\Gamma(1+\mu\beta)}\right]^2\frac{\mathrm{T}^{2\beta\mu}}{2\beta\mu-1}\right)\\
&\le 0.015+0.015\|\mathfrak{A}^{0.9}\|^2\left(\frac{0.592\times 2.2727\times\mathscr{C}_{0.1}^2}{0.8328}\right).
\end{aligned}
$$

$\mathfrak{Z}_1<1$ thereby fulfilling the hypotheses of Theorem 4.1. Thus 5.1 is T-controllable on (0, 1]. □

Conclusion

The theoretical approach of T-controllability of Hilfer fractional neutral stochastic differential equation with deviated arguments, Rosenblatt process and Poisson jumps is studied. This result extended the works of Chalishajar et al. [28–30]. R-L and Caputo's derivatives are extended by Hilfer fractional derivatives using fractional calculus. Numerical estimation of the system is the future work to be studied. T-contollability of Hilfer fractional parabolic, elliptic and the hyperbolic system will be the challenging work.

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Declarations

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