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Approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by Rosenblatt process via resolvent operators

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ABSTRACT

In this work, we investigate the existence of a mild solution and the approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert space with the Hurst parameter $\mathsf{H} \in (1/2, 1)$. We achieve the result using the semigroup theory of bounded linear operators, Grimmer's resolvent operator theory, and stochastic analysis. Using Krasnoselskii's and Schauder's fixed point theorems, we demonstrate the existence of mild solutions and the approximate controllability of the system. Finally, an example shows the potential for significant results.

RESUMEN

En este trabajo investigamos la existencia de una solución mild y la controlabilidad aproximada de ecuaciones integrodiferenciales estocásticas no-instantáneas impulsivas dirigidas por el proceso de Rosenblatt en espacios de Hilbert con el parámetro de Hurst $\mathsf{H} \in (1/2, 1)$. Logramos este resultado usando la teoría de semigrupos de operadores lineales acotados, la teoría del operador resolvente de Grimmer y análisis estocástico. Usando los teoremas de punto fijo de Krasnoselskii y Schauder, demostramos la existencia de soluciones mild y la controlabilidad aproximada del sistema. Finalmente, un ejemplo muestra el potencial para resultados significativos.

Keywords and Phrases: Approximate controllability, fixed point theorem, Rosenblatt process, stochastic integrodifferential equations, resolvent operator, non-instantaneous impulses.

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1 Introduction

Stochastic differential equations have become an active field of study because of their various applications in fields such as electrical engineering, mechanics, medical biology, economic systems, etc. For more information, see [2, 11, 18, 29]. The mathematical description of the phenomenon under investigation must account for randomness since many real-world events, such as stock prices, heat conduction in memory materials, and rising population, are unpredictable or noisy. It has been demonstrated that stochastic differential systems are especially powerful methods for describing and understanding this kind of event. Stochastic differential systems theory has been applied to model various phenomena in this life. Numerous authors have also investigated the existence, uniqueness, stability, controllability, approximate controllability, and other qualitative and quantitative properties of SDEs and stochastic integrodifferential equations (SIEs) using stochastic analysis, the fixed point approach, and the concept of resolvent operators in the case of SIEs. See for example, [6, 9, 15, 17]. In the last decades the theory of impulsive partial equations or inclusions seems to be a natural description of many real processes that are exposed to some disturbances, the duration of which is insignificant in comparison to the duration of the process. In addition to impulsive effects, stochastic effects also exist in real systems. Thus, impulsive stochastic differential equations describing these dynamical systems subject to both impulsive and stochastic changes have attracted significant attention. In particular, the papers [3, 26, 40] have studied the existence of smooth solutions for certain impulsive neutral stochastic functional integrodifferential equations with infinite delay in Hilbert spaces.

Let us consider $(\zeta_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with correlation function holds $\mathfrak{R}(n) = \mathbb{E}(\zeta_0\zeta_n) = n^{\frac{2\mathsf{H}-2}{k}}\mathscr{L}(n)$, with $\mathsf{H} \in \left(\frac{1}{2}, 1\right)$ and $\mathscr{L} \to \infty$. Let $\mathscr{G}$ denote the Hermite function of rank H. Also, if $\mathscr{G}$ admits the following,

$$\mathscr{G}(\rho) = \sum_{j\geq 0} c_j \mathsf{H}_j(\rho), \quad c_j = \frac{1}{j!}\mathbb{E}(\mathscr{G}(\zeta_0 \mathsf{H}(\zeta_0))),$$

then $\mathsf{H} = \min\{j | c_j \neq 0\} \geq 1$. $\mathsf{H}_j(\rho) = (-1)^j e^{\frac{\rho^2}{2}} \frac{\partial_j}{\partial \rho_j} e^{-\frac{\rho^2}{2}}$, where $\mathsf{H}_j(\rho)$ is the Hermite polynomial of degree j. Then by the Non-central Limit Theorem, $\frac{1}{n^{\mathsf{H}}}\sum_{[nt]}^{j=1}\mathscr{G}(\zeta_j)$ converges as $n \to \infty$ in the sense of finite-dimensional distributions to the process

$$\mathsf{R}_{\mathsf{K}}^{\mathsf{H}}(\rho) = c(\mathsf{H},\mathsf{K}) \int_{\mathbb{R}} \int_0^1 \left(\prod_{j=1}^{\mathsf{K}} (\xi - \vartheta)_+^{-\left(\frac{1}{2}+\frac{1-\mathsf{H}}{\mathsf{K}}\right)} \right) dW(\vartheta_1) \cdots dW(\vartheta_{\mathsf{K}}), \tag{1.1}$$

The (1.1) is a Wiener integral of order K with respect to the standard Brownian motion $(W(\vartheta))_{\vartheta\in\mathbb{R}}$ and $c(\mathsf{H},\mathsf{K})$ is normalizing constant depends on H and K. The process $(\mathsf{R}_{\mathsf{K}}^{\mathsf{H}}(\rho))_{\rho\geq 0}$ is known as the Hermite process.

- If $\mathsf{K} = 1$, the process (1.1) is the fractional Brownian motion with Hurst index $\mathsf{H} \in \left(\frac{1}{2}, 1\right)$.
- If $\mathsf{K} = 2$, the process given by (1.1) is called the Rosenblatt process, and it is not a Gaussian process, see [36, 37].

Fractional Brownian motion is a Gaussian stochastic process, which depends on a parameter $\mathsf{H} \in (0,1)$ called the Hurst index established by Kolmogorov [24]. For further reference on fractional Brownian motion, we refer the reader to [28]. There is another process like Rosenblatt's process with a non-Gaussian character, which contributes to the other properties for $\mathsf{H} > 1/2$, the long memory property. Self-similar processes with long-range dependence are seen in a variety of fields, including econometrics, internet traffic, hydrology, turbulence, and finance. The Rosenblatt process is a self-similar process with stationary increments that occurs as the limit of long-range-dependent stationary series. Still, it is not a Gaussian process, however, in real situations when the Gaussianity is not plausible for the model, one can use the Rosenblatt process. Comparatively, Rosenblatt process gains its interest due to its convolution of the dependence structures and the property of non-Gaussianity. Therefore, it seems stimulating to establish the SDEs with Rosenblatt process. Observations of stock price processes suggest that they are not self-similar. In particular, in [5, 22], the authors established the existence and uniqueness of mild solutions for stochastic differential equations driven by the Rosenblatt process with finite delay. Recently, in [7, 8, 34, 35, 38], the authors analyzed the stability and controllability of the stochastic functional differential equation driven by the Rosenblatt process. Also, many real-life phenomena and processes are characterized by abrupt changes in their state variable. These changes can be classified into two types: (i) In the first type, the changes take place over a relatively short period compared to the overall duration of the whole process, known as instantaneous impulses. (ii) In the second type, these changes start impulsively at certain times and remain active for certain intervals, known as non-instantaneous impulses. A well-known application of non-instantaneous impulses is the introduction of insulin into the bloodstream, which is an abrupt change. The resulting absorption is gradual because it remains active for a finite time interval. Models of this situation are created using differential and integrodifferential equations of non-instantaneous pulses detailed in [21, 23].

Approximate controllability refers to moving a system from an arbitrary initial state to a state arbitrarily close to a final state using only certain admissible controls. Recently, many authors have established results on the approximate controllability of first, second, and fractional-order differential equations with impulses; [1, 14, 32], and the references cited there. In references [12, 16, 39], the authors studied the approximate controllability of fractional stochastic Hilfer integrodifferential equations.

Motivated by this consideration, in this paper, we investigate the existence of mild solutions and approximate controllability of non-instantaneous impulsive stochastic integrodifferential equations driven by the Rosenblatt process having the following form:

$$\begin{cases} d\vartheta(\rho) = [\mathrm{A}\vartheta(\rho) + \int_0^\rho \mathbf{\Gamma}(\rho - s)\vartheta(s)ds + \mathsf{Bu}(\rho) + \mathsf{F}(\rho, \vartheta(\rho))]d\rho + \mathsf{G}(\rho, \vartheta(\rho))d\mathsf{R}^{\mathsf{H}}(\rho), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rho \in \cup_{i=0}^{m}(s_i, \rho_{i+1}), \\ \vartheta(\rho) = \mathsf{p}_i(\rho, \vartheta(\rho_i^-)), \quad \rho \in \cup_{i=1}^{m}(\rho_i, s_i], \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{1.2}$$

where $0 = \rho_0 = s_0 < \rho_1 < \cdots < s_m < \rho_{m+1} = \mathsf{b}$, $\mathcal{J} = [0, \mathsf{b}]$, $\vartheta(\cdot)$ takes values in the separable Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\mathrm{A} : \mathcal{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ and $\mathbf{\Gamma}(\rho) : \mathcal{D}(\mathbf{\Gamma}(\rho)) \subset \mathbb{H} \to \mathbb{H}$ are closed linear unbounded operators with $\mathcal{D}(\mathbf{\Gamma}(\rho)) \supset \mathcal{D}(\mathrm{A})$. $\{\mathsf{R}_{\mathsf{H}}(\rho)\}_{\rho \geq 0}$ is Q-Rosenblatt process with Hurst index $\mathsf{H} \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, , \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho \geq 0}; \mathbb{P})$ with values in a Hilbert space $\mathbb{K}$. The functions $\mathsf{p}_i(\rho, \vartheta(\rho_i^-))$ represent non-instantaneous impulses in the intervals $(\rho_i, s_i], i = 1, 2, \ldots, m$, and the functions $\mathsf{F} : [0, \mathsf{b}] \times \mathbb{H} \to \mathbb{H}$, $\mathsf{G} : [0, \mathsf{b}] \times \mathbb{H} \to L_0^2(\mathbb{K}, \mathbb{H})$ are appropriate functions wich will be specified later. The control function $\mathsf{u}(\cdot)$ is given in $L^2_{\mathcal{F}_\rho}([0, \mathsf{b}], \mathfrak{U})$ of admissible control functions, where $L^2_{\mathcal{F}_\rho}([0, \mathsf{b}], \mathfrak{U})$ is the Hilbert space of all $\mathcal{F}_\rho$-adopted, square integrable processes; $\mathfrak{U}$ is a Hilbert space; B is a bounded linear operator from $\mathfrak{U}$ into $\mathbb{H}$.

More specifically, our work focuses on developing a set of new, good criteria for the existence of mild solutions and approximate controllability of non-instantaneous impulsive stochastic integro-differential equations driven by the Rosenblatt process having the following abstract form (1.2).

The main contributions of our work, in particular, are summarized in the three aspects listed below:

- A new class of non-instantaneous impulsive partial stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert spaces is formulated.
- Initially, we establish the existence and uniqueness of mild solutions of the system above using stochastic analysis theory and the fixed point technique combined with resolvent operator theory.
- In comparison to [6, 17, 23], we enhance the approach and ease the conditions.
- Non-instantaneous impulsive partial stochastic integrodifferential equations driven by the Rosenblatt process in Hilbert spaces have received little attention in the literature. In order to bridge this gap, we have looked into the approximate controllability of (1.2).

This paper is organized as follows. In Section 2, we give some preliminaries, basic definitions, and results, which will be used in the sequel. In Section 3, the existence and approximate controllability outcomes of the considered system (1.2) are discussed. Section 4 illustrates the derived theoretical results through an example. Section 5 presents the conclusion and future direction of works in the last part of this work.

2 Preliminaries

Throughout this paper, $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{H}$ represent the real separable Hilbert spaces and $(\Omega, \mathcal{F}, \{\mathcal{F}_\rho\}_{\rho\geq 0}; \mathbb{P})$ be a complete probability space with natural filtration $(\mathcal{F}_\rho)_{\rho\geq 0}$, where $\mathcal{F}_\rho$, the Random variables generate σ-algebra $\Big\{\beta^{\mathsf{H}}(s), W(s), s \in [0,\rho]\Big\}$ and $\mathbb{P}$-null sets. We denote by $L^2_{\mathcal{F}_\rho}([0,\mathsf{b}],\mathbb{H})$ the space of all square integrable and $\mathcal{F}_\rho$-adapted process from $[0,\mathsf{b}]$ to $\mathbb{H}$ and $\mathcal{L}(\mathbb{X},\mathbb{H})$, $\mathcal{L}(\mathbb{Y},\mathbb{H})$ are respectively, the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{H}$ and $\mathbb{Y}$ to $\mathbb{H}$. For convenience, the same notation $\|\cdot\|$ is used to denote the norms in $\mathbb{X}$, $\mathbb{H}$, $\mathbb{Y}$, $\mathcal{L}(\mathbb{X},\mathbb{H})$ and $\mathcal{L}(\mathbb{Y},\mathbb{H})$ and the inner product of $\mathbb{X}$, $\mathbb{H}$, $\mathbb{Y}$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $\mathbb{C}([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ be the space of all continuous $\mathcal{F}_\rho$-adapted measurable processes from $[0,\mathsf{b}]$ to $L^2(\Omega,\mathbb{H})$ that satisfy $\sup\limits_{\rho\in[0,\mathsf{b}]} \mathbb{E}\|\vartheta(\rho)\|^2 < \infty$. Then, it is easy to see that $\mathbb{C}\Big([0,\mathsf{b}], L^2(\Omega,\mathbb{H})\Big)$ is a Banach space equipped with the following norm:

$$\|\vartheta\|_{\mathbb{C}} = \left(\sup_{\rho\in[0,\mathsf{b}]} \mathbb{E}\|\vartheta(\rho)\|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Let

$$\mathbf{V_q} = \Big\{\vartheta \in \mathbb{C}([0,\mathsf{b}], L^2(\Omega,\mathbb{H})) \,:\, \|\vartheta\|^2_{\mathbb{C}} \leq \mathsf{q}\Big\}. \tag{2.2}$$

2.1 Rosenblatt process

Consider a time interval $[0,\mathsf{b}]$ with arbitrary fixed horizon b and $\{\mathsf{R}^{\mathsf{H}}(\rho), \rho\in[0,\mathsf{b}]\}$ the one dimensional Rosenblatt process with parameter $\mathsf{H}\in(\frac{1}{2},1)$, R^{H} has the following integral representation [37]

$$\mathsf{R}_{\mathsf{H}}(\rho) = q(\mathsf{H})\int_0^\rho\int_0^\rho \left[\int_{\vartheta_1\vee\vartheta_2}^\rho \frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_1)\frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_2)du\right] dW_1(\vartheta_1)dW_1(\vartheta_2), \tag{2.3}$$

where $K^{\mathsf{H}}(\rho,s)$ is given by

$$K^{\mathsf{H}}(\rho,s) = c_{\mathsf{H}} s^{\frac{1}{2}-\mathsf{H}}\int_s^\rho (u-s)^{\mathsf{H}-3/2}u^{\mathsf{H}-1/2}du \quad \text{for} \quad \rho > s,$$

with

$$c_{\mathsf{H}} = \sqrt{\frac{\mathsf{H}(2\mathsf{H}-1)}{\beta(2-2\mathsf{H},\mathsf{H}-\frac{1}{2})}},$$

$\beta(\cdot,\cdot)$ denotes the Beta function, $K^{\mathsf{H}}(\rho,s)=0$ when $\rho\le s$, $\{W_1(\rho),\rho\in[0,\mathsf{b}]\}$ is a Brownian motion, $\mathsf{H}'=\frac{\mathsf{H}+1}{2}$ and $q(\mathsf{H})=\frac{1}{\mathsf{H}+1}\sqrt{\frac{\mathsf{H}}{2(2\mathsf{H}-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{\mathsf{R}_{\mathsf{H}}(\rho),\rho\in[0,\mathsf{b}]\}$ satisfies

$$\mathbb{E}(\mathsf{R}_{\mathsf{H}}(\rho)\mathsf{R}_{\mathsf{H}}(s))=\frac{1}{2}\left(s^{2\mathsf{H}}+\rho^{2\mathsf{H}}-|s-\rho|^{2\mathsf{H}}\right)$$

and this structure of $\{\mathsf{R}_{\mathsf{H}}(\rho)\}_{\rho\in[0,\,\mathsf{b}]}$ allows us to represent it as a Wiener integral.

Let $\mathsf{R}_Q^{\mathsf{H}}(\rho)$ be a $\mathbb{K}$-valued Rosenblatt process with covariance Q as

$$\mathsf{R}_Q^{\mathsf{H}}(\rho)=\mathsf{R}_Q(\rho)=\sum_{n=1}^{\infty}\sqrt{\delta_n}\xi_n(\rho)e_n,\quad \rho\ge 0.$$

Next, we introduce the space $L_0^2(\mathbb{K},\mathbb{H})$ of all Q-Hilbert-Schmidt operators $\Psi:\mathbb{K}\to\mathbb{H}$. Recall that $\Psi\in L(\mathbb{K},\mathbb{H})$ is called a Q-Hilbert-Schmidt operator if

$$\|\Psi\|=\sum_{n=1}^{\infty}\|\sqrt{\delta_n}\Psi e_n\|^2<\infty,$$

and that the space L_0^2 equipped with the inner product $<\phi,\psi>_{L_0^2}=\sum_{n=1}^{\infty}<\phi e_n,\psi e_n>$, is a Hilbert space.

Let $\rho:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ such that

$$\sum_{n=1}^{\infty}\|K_{\mathsf{H}}^*(\rho Q^{1/2}e_n)\|_{L^2([0,\mathsf{b}];\mathbb{H})}<\infty. \tag{2.4}$$

Definition 2.1 (Tudor [37])**.** *Let $\kappa(l):[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ satisfy (2.4). In that case, the stochastic integral of κ with respect to the Rosenblatt process $\mathsf{R}_Q^{\mathsf{H}}(\rho)$ is defined for $\rho\ge 0$ as follows*

$$\int_0^{\rho}\kappa(l)d\mathsf{R}_Q^{\mathsf{H}}(l):=\sum_{n=1}^{\infty}\int_0^{\rho}\kappa(s)Q^{1/2}e_n d\mathsf{R}_n(l)=\sum_{n=1}^{\infty}\int_0^{\rho}\int_0^{\tau}(K_{\mathsf{H}}^*(\kappa Q^{1/2}e_n))(\vartheta_1,\vartheta_2)dW_1(\vartheta_1)dW_1(\vartheta_2).$$

Lemma 2.2 ([34])**.** *For any $\kappa:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{H})$ such that $\sum_{n=1}^{\infty}\|\kappa Q^{1/2}e_n\|_{L^{1/\mathsf{H}}([0,\mathsf{b}];\mathbb{V})}<\infty$ holds, and for any $\alpha,\beta\in[0,\mathsf{b}]$ with $\beta>\alpha$, we have*

$$\mathbb{E}\left\|\int_{\alpha}^{\beta}\kappa(\rho)d\mathsf{R}_Q(\rho)\right\|^2\le c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\sum_{n=1}^{\infty}\int_{\alpha}^{\beta}\|\kappa(\rho)Q^{1/2}e_n\|^2d\rho.$$

If, in addition,

$$\sum_{n=1}^{\infty}\|\kappa(\rho)Q^{1/2}e_n\| \text{ is uniformly convergent for } \rho\in[0,\mathsf{b}],$$

then, it holds that

$$\mathbb{E}\left\|\int_{\alpha}^{\beta}\kappa(\rho)d\mathsf{R}_Q(\rho)\right\|^2 \leq c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\int_{\alpha}^{\beta}\|\kappa(\rho)\|^2_{L^2(Q^{1/2}\mathbb{K},\mathbb{V})}d\rho.$$

For further references, we refer to [19, 37].

2.2 Integrodifferential equations in Banach spaces

We recall some knowledge of partial integrodifferential equations and the related resolvent operators. Let $\mathcal{D}$ be the Banach space $\mathcal{D}(\mathrm{A})$ equipped with the graph norm defined by

$$\|\vartheta\|_{\mathcal{D}} := \|\mathrm{A}\vartheta\| + \|\vartheta\| \quad \text{for } \vartheta \in \mathcal{D}.$$

We denote by $\mathcal{C}(\mathbb{R}^+,\mathcal{D})$, the space of all functions from $\mathbb{R}^+$ into $\mathcal{D}$ which are continuous. Let us consider the following system for further purposes:

$$\begin{cases} \vartheta'(\rho) &= \mathrm{A}\vartheta(\rho) + \displaystyle\int_0^{\rho}\mathbf{\Gamma}(\rho-s)\vartheta(s)ds \quad \text{for } \rho \geq 0 \\ \vartheta(0) &= \vartheta_0 \in \mathcal{D}. \end{cases} \tag{2.5}$$

Definition 2.3 ([20])**.** *A resolvent operator for equation* (2.5) *is a bounded linear operator valued function* $\Psi(\rho) \in \mathcal{L}(\mathbb{H})$ *for* $\rho \geq 0$*, having the following properties :*

(*i*) $\Psi(0) = I$ *(the identity map of* $\mathbb{H}$*) and* $\|\Psi(\rho)\| \leq Ne^{\beta\rho}$ *for some constants* $N > 0$ *and* $\beta \in \mathbb{R}$.

(*ii*) *For each* $\vartheta \in \mathbb{H}$, $\Psi(\rho)\vartheta$ *is strongly continuous for* $\rho \geq 0$.

(*iii*) *For* $\vartheta \in \mathbb{H}, \Psi(\cdot)\vartheta \in \mathbb{C}^1(\mathbb{R}^+;\mathbb{H}) \cap \mathbb{C}(\mathbb{R}^+;\mathcal{D})$ *and*

$$\Psi'(\rho)\vartheta = \mathrm{A}\Psi(\rho)\vartheta + \int_0^{\rho}\mathbf{\Gamma}(\rho-s)\Psi(s)\vartheta ds = \Psi(\rho)\mathrm{A}\vartheta + \int_0^{\rho}\Psi(\rho-s)\mathbf{\Gamma}(s)\vartheta ds, \quad \text{for } \rho \in [0,\mathsf{b}].$$

Next, we assume A and $(\mathbf{\Gamma}(\rho))_{\rho\geq 0}$ satisfy the following conditions:

($\mathbf{R_1}$) The operator A is the infinitesimal generator of a strongly continuous semigroup $(\mathsf{T}(\rho))_{\rho\geq 0}$ on $\mathbb{H}$.

($\mathbf{R_2}$) For all $\rho \geq 0$, the operator $\mathbf{\Gamma}(\rho)$ is closed and linear from $\mathcal{D}(\mathrm{A})$ to $\mathbb{Y}$ and $\mathbf{\Gamma}(\rho) \in \mathcal{L}(\mathcal{B},\mathbb{H})$. For any $\vartheta \in \mathbb{H}$, the map $\rho \mapsto \mathbf{\Gamma}(\rho)\vartheta$ is bounded, differentiable and the derivative $\rho \mapsto \mathbf{\Gamma}'(\rho)\vartheta$ is bounded and uniformly continuous for $\rho \geq 0$.

Theorem 2.4 ([20])**.** *Assume that* ($\mathbf{R_1}$)-($\mathbf{R_2}$) *hold. Then, there exists a unique resolvent operator of the Cauchy problem* (2.5).

We have the following useful results.

Theorem 2.5 ([13]). *Let the assumptions* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *be satisfied. Let the* C_0*-semigroup* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *generated by* A *be compact for* $\rho > 0$. *Then the corresponding resolvent operator* $(\Psi(\rho))_{\rho\geq 0}$ *of equation* (1.2) *is also compact for* $\rho > 0$.

Lemma 2.6 ([13]). *Let the assumptions* $(\mathbf{R_1})$ *and* $(\mathbf{R_2})$ *be satisfied. Then, there exists a constant* $L = L(\mathsf{b})$ *such that*

$$\|\Psi(\rho+\varepsilon) - \Psi(\varepsilon)\Psi(\rho)\|_{\mathcal{L}(\mathbb{H})} \leq L(\varepsilon), \quad \text{for } 0 < \varepsilon \leq \rho \leq \mathsf{b}.$$

Based on these, we have the following Theorem establishing the equivalence between operator-norm continuity of the semigroup generated by A and the resolvent operator $(\Psi(\rho))_{\rho\geq 0}$ corresponding to the linear equation (2.5).

Theorem 2.7 ([25]). *Let* A *be the infinitesimal generator of a* C_0*-semigroup* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *and let* $(\mathbf{\Gamma}(\rho))_{\rho\geq 0}$ *satisfy* $(\mathbf{R_2})$. *Then the resolvent operator* $(\Psi(\rho))_{\rho\geq 0}$ *for Eq.* (2.5) *is operator-norm continuous (or continuous in the uniform operator topology) for* $\rho > 0$ *if and only if* $(\mathsf{T}(\rho))_{\rho\geq 0}$ *is operator-norm continuous for* $\rho > 0$.

Now, we introduce the space $C_{\mathsf{b}} = \mathcal{PC}([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ formed by all $\mathbb{H}-$valued stochastic processes

$$\{\vartheta(\rho), \rho\in[0,\mathsf{b}] \text{ such that } \vartheta|_{I_i}\in C(I_i,\mathbb{H}) \text{ for all } w\in\Omega, i=0,1,\dots,m, \text{ and there exist}$$
$$\vartheta(\rho_i^-) \text{ and } \vartheta(\rho_i^+), i=1,2,\dots,m \text{ with } \vartheta(\rho_i^-)=\vartheta(\rho_i) \text{ and } \sup_{\rho\in[0,\mathsf{b}]}\mathbb{E}\|\vartheta(\rho)\|^2<\infty\}$$

endowed with the norm

$$\|\vartheta\|_{\mathcal{PC}} = \left(\sup_{\rho\in[0,\mathsf{b}]}\mathbb{E}\|\vartheta(\rho)\|^2\right)^{1/2}, \tag{2.6}$$

where $I_i = (\rho_i, \rho_{i+1}]$, $i = 0, 1, \dots, m$.

Now, we define the mild solution of Eq. (1.2) expressed by the resolvent operator $\Psi(\rho)$ as follows.

Definition 2.8. *A* $\mathbb{H}$*-valued stochastic process* $\vartheta \in C([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ *is called a mild solution of the stochastic problem* (1.2)*, if*

(1) $\vartheta(\rho)$ *is* $\mathcal{F}_\rho-$*adapted and measurable for each* $\rho \geq 0$.

(2) $\vartheta(\rho)$ *has càdlàg paths on* $\rho \in [0,\mathsf{b}]$ *a.s. and for each* $\rho \in [0,\mathsf{b}], \vartheta(\rho)$ *satisfies* $\vartheta(\rho) = \mathsf{p}_i(\rho, \vartheta(\rho_i^-))$ *for all* $\rho \in (\rho_i, s_i]$, $i = 1, 2, \dots, m$ *and* $\vartheta(\rho)$ *is the solution of the following integral equations*

$$\begin{aligned}\vartheta(\rho) &= \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho-s)\ \mathsf{F}(s,\vartheta(s))\ \mathrm{ds} + \int_0^\rho \Psi(\rho-s)\ \mathsf{B}\mathsf{u}(s)\ \mathrm{ds} \\ &\quad + \int_0^\rho \Psi(\rho-s)G(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\,\mathrm{ds}, \quad \textit{for } \rho\in[0,\rho_1], \\ \vartheta(\rho) &= \Psi(\rho-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho-s)\ \mathsf{F}(s,\vartheta(s))\ \mathrm{ds} + \int_{s_i}^\rho \Psi(\rho-s)\ \mathsf{B}\mathsf{u}(s)\ \mathrm{ds} \\ &\quad + \int_{s_i}^\rho \Psi(\rho-s)G(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\,\mathrm{ds}, \quad \textit{for } \rho\in[s_i,\rho_{i+1}], \quad i=1,2,\ldots,m.\end{aligned} \tag{2.7}$$

Let us denote the state value of the system (1.2) at the time ρ by $\vartheta_\rho = \vartheta(\rho;\vartheta_0,\mathsf{u})$ with respect to initial value ϑ_0 and the control function u. The set of all final states is known as reachable set of the system (1.2) and defined as $\mathfrak{M}(\mathsf{b},\vartheta_0,\mathsf{u}) = \Big\{\vartheta_{\mathsf{b}} = \vartheta(\mathsf{b};\vartheta_0,\mathsf{u}) : \mathsf{u}\in L^2([0,\mathsf{b}],\mathfrak{U})\Big\}$.

Definition 2.9. *Eq.* (1.2) *is said to be approximately controllable on the interval* $[0,\mathsf{b}]$*, if*

$$\overline{\mathfrak{M}(\mathsf{b},\vartheta_0,\mathsf{u})} = L^2(\Omega,\mathbb{H}),$$

that is, for arbitrary $\varepsilon>0$*, it is possible to steer the state from the point* ϑ_0 *to within a distance* ε *from all points in the state space* $L^2(\Omega,\mathbb{H})$ *at time* b.

To discuss the approximate controllability of system (1.2) we introduce the following operators.

(1) The controllability Grammian $\mathbf{\Pi}_0^{\mathsf{b}}$ is defined by:

$$\mathbf{\Pi}_{s_i}^{\rho_{i+1}} = \int_{s_i}^{\rho_{i+1}} \Psi(\rho_{i+1}-s)\mathsf{B}\,\mathsf{B}^*\Psi^*(\rho_{i+1}-s)ds,$$

where B^* and $\Psi^*(\rho)$ denote the adjoint of the operators B and $\Psi(\rho)$.

(2) $W(\gamma,\mathbf{\Pi}_{s_i}^{\rho_{i+1}}) = (\gamma\,\mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}$.

In the sequel we assume that the operator $W(\gamma,\mathbf{\Pi}_{s_i}^{\rho_{i+1}})$ satisfies

($\mathbf{H}_0$) $\gamma W(\gamma,\mathbf{\Pi}_{s_i}^{\rho_{i+1}}) \to 0$ as $\gamma\to 0^+$ in the strong operator topology.

The above condition ($\mathbf{H}_0$) is equivalent to the approximate controllability of the linear system.

$$\begin{cases} \dfrac{d\vartheta(\rho)}{d\rho} = \mathrm{A}\vartheta(\rho) + \displaystyle\int_0^\rho \mathbf{\Gamma}(\rho-s)\vartheta(s)ds + \mathsf{B}\mathsf{u}(\rho), \qquad \rho\in[0,\mathsf{b}], \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{2.8}$$

In fact, we have that

Theorem 2.10 ([4, 10]). *The following statements are equivalent:*

(*i*) *The control system* (2.8) *is approximately controllable on* $[0, \mathsf{b}]$.

(*ii*) $\mathsf{B}^*\Psi^*(\rho)\vartheta = 0$ *for all* $\rho \in [0, \mathsf{b}]$ *imply* $\vartheta = 0$.

(*iii*) *The condition* $(\mathbf{H}_0)$ *holds.*

Lemma 2.11 ([27]). *For any* $\vartheta_{\rho_{i+1}} \in L^2(\Omega, \mathcal{F}_{\rho_{i+1}}, \mathbb{H})$, *there exist* $\Phi_i \in L^2(\Omega;\ L^2([s_i, \rho_{i+1}]; L_2^0(\mathbb{Y}, \mathbb{H})))$, *such that* $\vartheta_{\rho_{i+1}} = \mathbb{E}\vartheta_{\rho_{i+1}} + \int_{s_i}^{\rho_{i+1}} \Phi_i(s) d\mathsf{R}^{\mathsf{H}}(s)$.

Our results are based on the following Krasnoselskii's and Schauder's fixed point theorem.

Theorem 2.12 (Krasnoselskii's theorem [32]). *Let $\mathcal{B}$ be a closed, bounded and convex subset of a Banach space $\mathbb{H}$, and let Φ_1, Φ_2 be maps of $\mathcal{B}$ into $\mathbb{H}$ such that $\Phi_1\vartheta_1 + \Phi_2\vartheta_2 \in \mathcal{B}$, for all $\vartheta_1, \vartheta_2 \in \mathcal{B}$. If Φ_1 is a contraction and Φ_2 is continuous and compact, then the equation $\vartheta = \Phi_1\vartheta + \Phi_2\vartheta$ has a solution on $\mathcal{B}$.*

Theorem 2.13 (Schauder's theorem [33]). *If $\mathcal{B}$ is a closed, bounded and convex subset of a Banach space $\mathbb{H}$ and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is completely continuous, then $\mathcal{F}$ has a fixed point in $\mathcal{B}$.*

3 Approximate controllability results

This section proves the approximate controllability of the stochastic control system (1.2). Let $M = \sup_{\rho\in[0,\mathsf{b}]} \|\Psi(\rho)\|$. In order to establish the results, we impose the following hypotheses.

(C$_1$) $\mathsf{T}(\rho)$ is compact for $\rho > 0$.

(C$_2$) The maps $\mathsf{p}_i : \mathsf{b}_i \times \mathbb{H} \to \mathbb{H}$, $\mathsf{b}_i = [\rho_i, s_i]$, $i = 1, 2, \ldots, m$ are continuous functions and satisfy

(a) There exist constants $D_{\mathsf{p}_i} > 0$, $i = 1, 2, \ldots, m$, such that

$$\mathbb{E}\|\mathsf{p}_i(\rho, \vartheta)\|^2 \le D_{\mathsf{p}_i}(1 + \mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_i \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exist constants $R_{\mathsf{p}_i} > 0$, $i = 1, 2, \ldots, m$, such that

$$\mathbb{E}\|\mathsf{p}_i(\rho, \vartheta_1) - \mathsf{p}_i(\rho, \vartheta_2)\|^2 \le R_{\mathsf{p}_i}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_i \text{ and } \vartheta_1, \vartheta_2 \in \mathbb{H}.$$

(C$_3$) The map $\mathsf{F} : \mathsf{b}_0 \times \mathbb{H} \to \mathbb{H}$, $\mathsf{b}_0 = \bigcup_{i=0}^{m} [s_i, \rho_{i+1}]$ is a continuous function and satisfies

(a) There exists a constant $\mathfrak{M}_{\mathsf{F}} > 0$ such that

$$\mathbb{E}\|\mathsf{F}(\rho,\vartheta)\|^2 \leq \mathfrak{M}_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exists a constant $R_{\mathsf{F}} > 0$ such that

$$\mathbb{E}\|\mathsf{F}(\rho,\vartheta_1)-\mathsf{F}(\rho,\vartheta_2)\|^2 \leq R_{\mathsf{F}}\mathbb{E}\|\vartheta_1-\vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta_1,\vartheta_2 \in \mathbb{H}.$$

(C$_4$) The map $\mathsf{G} : \mathsf{b}_0 \times \mathbb{H} \to L_2^0$, is a continuous function and satisfies

(a) There exists a constant $\mathfrak{M}_{\mathsf{G}} > 0$ such that

$$\mathbb{E}\|\mathsf{G}(\rho,\vartheta)\|^2 \leq \mathfrak{M}_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2), \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta \in \mathbb{H}.$$

(b) There exists a constant $R_{\mathsf{G}} > 0$ such that

$$\mathbb{E}\|\mathsf{G}(\rho,\vartheta_1)-\mathsf{G}(\rho,\vartheta_2)\|^2 \leq R_{\mathsf{G}}\mathbb{E}\|\vartheta_1-\vartheta_2\|^2, \quad \forall \rho \in \mathsf{b}_0 \text{ and } \vartheta_1,\vartheta_2 \in \mathbb{H}.$$

(C$_5$) The following inequalities hold

(a) $\max\limits_{0\leq i\leq m} N_i < 1$,

(b) $\max\limits_{1\leq i\leq m} D_{\mathsf{p}_i} < 1$,

(c) $\max\limits_{1\leq i\leq m} \{M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_0}\rho_1^2, R_{\mathsf{p}_i}, 2\left(M^2 R_{\mathsf{p}_i} + M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho_{i+1}^2\right)\} < 1.$

(C$_6$) The linear control system (2.8) is approximately controllable on $[0,\mathsf{b}]$.

For any $\gamma > 0$, we define the operator $\mathcal{S}^{(\gamma)} : C([0,\mathsf{b}], L^2(\Omega,\mathbb{H})) \to C([0,\mathsf{b}], L^2(\Omega,\mathbb{H}))$ by

$$\begin{aligned}(\mathcal{S}^{(\gamma)}\vartheta)(\rho) &= \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds} + \int_0^\rho \Psi(\rho-s)\,\mathsf{B}\mathsf{u}^{(\gamma)}(s,\vartheta)\,\mathrm{ds}\\ &+ \int_0^\rho \Psi(\rho-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad \forall \rho \in [0,\rho_1]\end{aligned}$$

and

$$\begin{aligned}(\mathcal{S}^{(\gamma)}\vartheta)(\rho) &= \Psi(\rho-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds} + \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{B}\mathsf{u}^{(\gamma)}(s,\vartheta)\,\mathrm{ds}\\ &+ \int_{s_i}^\rho \Psi(\rho-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad \forall \rho \in [s_i,\rho_{i+1}],\ i=1,2,\ldots,m,\end{aligned}$$

where,

$$\begin{aligned}\mathsf{u}^{(\gamma)}(s,\vartheta) &= \mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\left\{\mathbb{E}\,\vartheta_{\rho_{i+1}}+\int_{s_i}^{\rho_{i+1}}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s)-\Psi(\mathsf{b}-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))\right\}\\ &\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{F}(s,\vartheta(s))\,\mathrm{ds}\\ &\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\end{aligned}$$

and $J_0(0,\cdot)=\vartheta_0$, $\vartheta(\rho_{m+1})=\vartheta_{\rho_{m+1}}=\vartheta_{\mathsf{b}}$.

Lemma 3.1. *There exist positive constants R_{u_i} and R_{v_i} $i=0,1,\ldots,m$, such that for all $\vartheta_1,\vartheta_2\in C_{\mathsf{b}}$, we have*

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(\rho,\vartheta_1)-\mathsf{u}^{(\gamma)}(\rho,\vartheta_2)\|^2\leq R_{\mathsf{u}_i}\|\vartheta_1-\vartheta_2\|_{\mathcal{PC}}, \tag{3.1}$$

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(\rho,\vartheta)\|^2\leq R_{v_i}, \tag{3.2}$$

where

$$R_{\mathsf{u}_i}=3\,\frac{\|\mathsf{B}\|^2M^4}{\gamma^2}\left\{R_{\mathsf{p}_i}+(\rho_{i+1}-s_i)^2R_{\mathsf{F}}+2R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\right\}, \tag{3.3}$$

$$\begin{aligned}R_{v_i}&=\frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\left[\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+D_{\mathsf{p}_i}(1+\mathfrak{M})+(\rho_{i+1}-s_i)^2\,D_{\mathsf{F}}(1+\mathfrak{M})\right.\\ &\quad\left.+\,c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}D_{\mathsf{G}}(1+\mathfrak{M})\right],\quad\|\vartheta\|^2_{\mathcal{PC}}\leq\mathfrak{M}.\end{aligned} \tag{3.4}$$

Proof. Let $\vartheta_1,\vartheta_2\in C_{\mathsf{b}}$

$$\begin{aligned}&\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)-\mathsf{u}^{(\gamma)}(s,\vartheta_1)\|^2\\ &\quad\leq\mathbb{E}\Big\|\mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\big\{\Psi(\mathsf{b}-s_i)\big[\mathsf{p}_i(s_i,\vartheta_1(\rho_i^-))-\mathsf{p}_i(s_i,\vartheta_2(\rho_i^-))\big]\big\}\\ &\qquad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1}-s)\,\big[\mathsf{F}(s,\vartheta_1(s))-\mathsf{F}(s,\vartheta_2(s))\big]\,\mathrm{ds}\\ &\qquad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1}-s)\,\big[\mathsf{G}(s,\vartheta_1(s))-\mathsf{G}(s,\vartheta_2(s))\big]d\mathsf{R}^{\mathsf{H}}(s)\Big\|^2\\ &\quad\leq\frac{3\|\mathsf{B}\|^2M^2}{\gamma^2}\Big[M^2\mathbb{E}\big\|\mathsf{p}_i(s_i,\vartheta_1(\rho_i^-))-\mathsf{p}_i(s_i,\vartheta_2(\rho_i^-))\big\|^2\\ &\qquad+M^2(\rho_{i+1}-s_i)\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\mathsf{F}(s,\vartheta_1(s))-\mathsf{F}(s,\vartheta_2(s))\|^2\,\mathrm{ds}\\ &\qquad+M^2c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\mathsf{G}(s,\vartheta_1(s))-\mathsf{G}(s,\vartheta_2(s))\|^2\,\mathrm{ds}\Big]\\ &\quad\leq\;3\,\frac{\|\mathsf{B}\|^2\,M^4}{\gamma^2}R_{\mathsf{p}_i}\mathbb{E}\|\vartheta_2-\vartheta_1\|^2+\;3(\rho_{i+1}-s_i)\,R_{\mathsf{F}}\frac{\|\mathsf{B}\|^2\,M^4}{\gamma^2}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\vartheta_2(s)-\vartheta_1(s)\|^2\,\mathrm{ds}\\ &\qquad+6\;R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\frac{\|\mathsf{B}\|^2\,M^4}{\gamma^2}\int_{s_i}^{\rho_{i+1}}\mathbb{E}\|\vartheta_2(s)-\vartheta_1(s)\|^2\,\mathrm{ds}\end{aligned}$$

$$\leq 3\,\frac{\|\mathsf{B}\|^2\,M^4}{\gamma^2}\left\{R_{\mathsf{p}_i}+\ (\rho_{i+1}-s_i)^2\,R_{\mathsf{F}}+2\ R_{\mathsf{G}}c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\right\}\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}. \quad (3.5)$$

Hence,

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)-\mathsf{u}^{(\gamma)}(s,\vartheta_1)\|^2\leq R_{\mathsf{u}_i}\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}.$$

The proof of inequality (3.2) is

$$\begin{aligned}
\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta_2)\|^2 &\leq \mathbb{E}\bigg\|\mathsf{B}^*\Psi^*(\rho_{i+1}-s)(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\big\{\vartheta_{\rho_{i+1}}-\Psi(\mathsf{b}-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))\big\}\\
&\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1}-s)\ \mathsf{F}(s,\vartheta(s))\ \mathrm{ds}\\
&\quad-\mathsf{B}^*\Psi^*(\rho_{i+1}-s)\int_{s_i}^{\rho_{i+1}}(\gamma\mathrm{Id}+\mathbf{\Pi}_{s}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1}-s)\,\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\bigg\|^2\\
&\leq\frac{4\|\mathsf{B}\|^2M^2}{\gamma^2}\bigg[M^2D_{\mathsf{p}_i}(1+\mathbb{E}\|\vartheta\|^2)+M^2(\rho_{i+1}-s_i)\int_{s_i}^{\rho_{i+1}}\ D_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2)\ \mathrm{ds}\\
&\quad+M^2c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}-1}\int_{s_i}^{\rho_{i+1}}\ D_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2)\ \mathrm{ds}\bigg]\\
&\leq\frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\left[D_{\mathsf{p}_i}(1+\mathbb{E}\|\vartheta\|^2)+(\rho_{i+1}-s_i)^2\ D_{\mathsf{F}}(1+\mathbb{E}\|\vartheta\|^2)\right.\\
&\quad\left.c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\ D_{\mathsf{G}}(1+\mathbb{E}\|\vartheta\|^2)\ \right].\\
&\leq\frac{4\|\mathsf{B}\|^2M^4}{\gamma^2}\left[\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+D_{\mathsf{p}_i}(1+\mathfrak{M})+(\rho_{i+1}-s_i)^2\ D_{\mathsf{F}}(1+\mathfrak{M})\right.\\
&\quad\left.c_{\mathsf{H}}(\rho_{i+1}-s_i)^{2\mathsf{H}}\ D_{\mathsf{G}}(1+\mathfrak{M})\ \right].
\end{aligned}$$

Hence,

$$\mathbb{E}\|\mathsf{u}^{(\gamma)}(s,\vartheta)\|^2\leq R_{v_i}.$$

□

Let the constant $\mathfrak{M}$ satisfy the inequality

$$\mathfrak{M}\geq\max_{1\leq i\leq m}\left[\frac{Q_0}{1-N_0},\ \ \frac{D_{\mathsf{p}_i}}{1-D_{\mathsf{p}_i}},\ \ \frac{Q_i}{1-N_i}\right], \quad (3.6)$$

where

$$Q_i=\frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left[4M^2D_{\mathsf{p}_i}+4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right],$$

$D_{J_0}=0,$

$$N_i=\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left\{4M^2D_{\mathsf{p}_i}+4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right\}$$

$$N_0=\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)\left\{4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right\}$$

$$Q_0=\frac{16\rho_1^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)\left[4M^2\mathbb{E}\|\vartheta_0\|^2+4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}+4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right].$$

Theorem 3.2. *Assume that hypotheses* $(\mathbf{C}_1)$–$(\mathbf{C}_5)$ *hold. Then the system* (1.2) *has at least one mild solution on* $[0, \mathsf{b}]$.

Proof. First, we define two operators Φ_1 and Φ_2 on

$$S_{\mathfrak{M}} = \{\vartheta \in C_{\mathsf{b}} : \|\vartheta\|^2_{\mathcal{PC}} \leq \mathfrak{M}\} \subseteq C_{\mathsf{b}}.$$

as follows

$$(\Phi_1\vartheta)(\rho) = \begin{cases} \Psi(\rho)\vartheta_0 + \int_0^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds, & \rho \in [0, \rho_1], \\ \mathsf{p}_i(\rho, \vartheta(\rho_i^-)), & \rho \in (\rho_i, s_i], \\ \Psi(\rho - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-)) + \int_{s_i}^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds & \rho \in (s_i, \rho_{i+1}], \end{cases}$$

and

$$(\Phi_2\vartheta)(\rho) = \begin{cases} \int_0^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds + \int_0^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in [0, \rho_1], \\ 0 & \rho \in (\rho_i, s_i], \\ \int_{s_i}^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds + \int_{s_i}^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in (s_i, \rho_{i+1}]. \end{cases}$$

The set $S_{\mathfrak{M}}$ is a bounded closed and convex set in C_{b}. Next, we prove that the operators Φ_1 and Φ_2 satisfy all the conditions of Krasnoselskii's theorem. For the sake of convenience, we split the proof into several steps.

Step 1. We prove that $\Phi_1\vartheta_1 + \Phi_2\vartheta_2 \in S_{\mathfrak{M}}$ for any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$.

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in [0, \rho_1]$, we have

$$\begin{aligned} \mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 &\leq 4\mathbb{E}\|\Psi(\rho)\vartheta_0\|^2 + 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{B}\mathsf{u}^\gamma(s, \vartheta)ds\right\|^2 \\ &+ 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{F}(s, \vartheta(s))ds\right\|^2 + 4\mathbb{E}\left\|\int_0^\rho \Psi(\rho - s)\mathsf{G}(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\right\|^2 \\ &\leq 4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\|\mathsf{B}\|^2\rho\int_0^\rho \mathbb{E}\left\|\mathsf{u}^\gamma(s, \vartheta)\right\|^2 ds \\ &+ 4M^2\int_0^\rho \mathbb{E}\left\|\mathsf{F}(s, \vartheta(s))\right\|^2 ds + 4M^2 c_{\mathsf{H}}\rho^{2\mathsf{H}-1}\int_0^\rho \mathbb{E}\left\|\mathsf{G}(s, \vartheta(s))\right\|^2 ds \\ &\leq 4M^2\mathbb{E}\|\vartheta_0\|^2 + \frac{4\|\mathsf{B}\|^4 M^4\rho_1^2}{\gamma^2}\Big[4\mathbb{E}\|\vartheta_{\rho_1}\|^2 + 4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\rho_1^2\, D_{\mathsf{F}}(1 + \mathfrak{M}) \\ &+ 4M^2 c_{\mathsf{H}}\rho_1^{2\mathsf{H}}\, D_{\mathsf{G}}(1 + \mathfrak{M})\Big] + 4M^2\rho_1^2\, D_{\mathsf{F}}(1 + \mathfrak{M}) + 4M^2 c_{\mathsf{H}}\rho_1^{2\mathsf{H}}\, D_{\mathsf{G}}(1 + \mathfrak{M}) \end{aligned}$$

$$\leq \frac{16\rho_1^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2 + \left(1 + \frac{4\|\mathsf{B}\|^4M^4\rho_1^2}{\gamma^2}\right)$$
$$\times\left\{4M^2\mathbb{E}\|\vartheta_0\|^2 + 4M^2\rho_1^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 4c_{\mathsf{H}}\rho_1^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\right\}.$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 \leq Q_0 + N_0\mathfrak{M} \leq \mathfrak{M}. \tag{3.7}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$, and $\rho \in (\rho_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 = \mathbb{E}\|\mathsf{p}_i(\rho, \vartheta_1(\rho_i^-))\|^2 \leq D_{\mathsf{p}_i}(1 + \mathbb{E}\|\vartheta_1\|^2).$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 \leq D_{\mathsf{p}_i}(1+\mathfrak{M}) \leq \mathfrak{M}. \tag{3.8}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$, and $\rho \in (s_i, \rho_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 &\leq 4M^2\mathbb{E}\|\mathsf{p}_i(s_i, \vartheta(\rho_i^-))\|^2 + 4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho - s)\mathsf{B}\mathsf{u}^{\gamma}(s,\vartheta)ds\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho - s)\|\mathsf{F}(s,\vartheta(s))ds\right\|^2 + 4\mathbb{E}\left\|\int_{s_i}^{\rho}\Psi(\rho - s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s)\right\|^2 \\
&\leq \frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2 + \left(1 + \frac{3\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right) \\
&\quad \times\left\{4M^2D_{\mathsf{p}_i}(1+\mathfrak{M}) + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M}) + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\right\} \\
&\leq \frac{16\rho_{i+1}^4\|\mathsf{B}\|^4M^4}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2 \\
&\quad + \left(1 + \frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left[4M^2D_{\mathsf{p}_i} + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}} + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right] \\
&\quad + \left(1 + \frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\left\{4M^2D_{\mathsf{p}_i} + 4M^2\rho_{i+1}^2\mathfrak{M}_{\mathsf{F}} + 4c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}}M^2\mathfrak{M}_{\mathsf{G}}\right\}\mathfrak{M}.
\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) + (\Phi_2\vartheta_2)(\rho)\|^2 \leq Q_i + N_i\mathfrak{M} \leq \mathfrak{M}. \tag{3.9}$$

Equations (3.7)–(3.9) implies that

$$\|\Phi_1\vartheta_1 + \Phi_2\vartheta_2\|_{\mathcal{PC}}^2 \leq \mathfrak{M}.$$

Hence, $\Phi_1\vartheta_1 + \Phi_2\vartheta_2 \in S_{\mathfrak{M}}$.

Step 2. Φ_2 is continuous on $S_{\mathfrak{M}}$. Let $\{\vartheta_n\}_{n=1}^{\infty}$ be a sequence such that $\vartheta_n \to \vartheta$ in $S_{\mathfrak{M}}$.

For any $\rho \in (s_i,\, \rho_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_2\vartheta_n)(\rho) - (\Phi_2\vartheta)(\rho)\|^2 &\leq 2M^2\rho_{i+1}\int_{s_i}^{\rho}\mathbb{E}\|\mathsf{F}(s,\vartheta_n(s)) - \mathsf{F}(s,\vartheta(s))\|^2 ds \\ &\quad + 4c_{\mathsf{H}}M^2\rho_{i+1}^{2\mathsf{H}-1}\int_{s_i}^{\rho}\mathbb{E}\|\mathsf{G}(s,\vartheta_n(s) - \mathsf{G}(s,\vartheta(s))\|^2_{L_2^0} ds \\ &\leq \left[2M^2R_F\rho_{i+1}^2 + 4c_{\mathsf{H}}M^2\rho_{i+1}^{2\mathsf{H}}\right]\|\vartheta_n - \vartheta\|^2_{\mathcal{PC}}.\end{aligned}$$

Hence, $\mathbb{E}\|(\Phi_2\vartheta_n)(\rho) - (\Phi_2\vartheta)(\rho)\|^2 \to 0$ as $n \to \infty$, thus, Φ_2 is continuous on $S_{\mathfrak{M}}$.

Step 3. We show $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is equicontinuous.

For any $\tau_1, \tau_2 \in (s_i, \rho_{i+1}]$, $i = 0, 1, \ldots, m$, $\tau_1 < \tau_2$ and $\vartheta \in S_{\mathfrak{M}}$, we obtain

$$\begin{aligned}\mathbb{E}\|(\Phi_2\vartheta)(\tau_2) + (\Phi_2\vartheta)(\tau_1)\|^2 &\leq 4\mathfrak{M}_{\mathsf{F}}\tau_1\int_{s_i}^{\tau_1}\|\Psi(\tau_2 - s) - \Psi(\tau_1 - s)\|^2(1+\mathfrak{M})ds \\ &\quad + 4M^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})(\tau_2-\tau_1)^2 \\ &\quad + 8c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}-1}\mathfrak{M}_{\mathsf{G}}\int_{s_i}^{\tau_1}\|\Psi(\tau_2 - s) - \Psi(\tau_1 - s)\|^2(1+\mathfrak{M})ds \\ &\quad + 8M^2c_{\mathsf{H}}\rho_{i+1}^{2\mathsf{H}-1}\mathfrak{M}_{\mathsf{G}}(\tau_2-\tau_1). \end{aligned} \tag{3.10}$$

We conclude that $\mathbb{E}\|(\Phi_2\vartheta)(\tau_2) - (\Phi_2\vartheta)(\tau_1)\|^2 \to 0$ as $\tau_2 \to \tau_1$, since the operator $\Psi(\rho)$ is compact, which implies the continuity of the operator $\Psi(\rho)$. Hence $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is equicontinuous. Also, clearly $\{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is bounded.

Step 4. We show that $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is relatively compact in $\mathbb{H}$.

Clearly, $\mathcal{Z}(0) = \{0\}$ is compact. Let ε be a real number and $\rho \in (s_i, \rho_{i+1}]$, $i = 0, 1, \ldots, m$ be fixed with $0 < \varepsilon < \rho$. For any $\vartheta \in S_{\mathfrak{M}}$, we define

$$(\Phi_2^{\varepsilon}\vartheta)(\rho) = \begin{cases}\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\|F(s,\vartheta(s))ds + \int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in [0,\rho_1], \\ 0 & \rho \in (\rho_i, s_i], \\ \displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds + \int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho \in (s_i,\rho_{i+1}],\end{cases}$$

and

$$(\Phi_2^{*\varepsilon}\vartheta)(\rho)=\begin{cases}\Psi(\varepsilon)\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds\\ \qquad\qquad+\Psi(\varepsilon)\displaystyle\int_0^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho\in[0,\rho_1],\\ 0 & \rho\in(\rho_i,s_i],\\ \Psi(\varepsilon)\displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{F}(s,\vartheta(s))ds\\ \qquad\qquad+\Psi(\varepsilon)\displaystyle\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-\varepsilon-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s) & \rho\in(s_i,\rho_{i+1}].\end{cases}$$

By Lemma 2.6 and using the compactness of $\big(\Psi(\varepsilon)\big)_{\varepsilon>0}$, we deduce that the set $\mathcal{Z}^{\varepsilon}(\rho)=\{(\Phi_2^{\varepsilon}\vartheta)(\rho):\vartheta\in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ for every ε, $0<\varepsilon<\rho$. Moreover, by Lemma 2.6 and Hölder's inequality, for every $\rho\in(0,\rho_1]$, we obtain:

$$\begin{aligned}
&\mathbb{E}\|(\Phi_2^{\varepsilon}\vartheta)(\rho)-\big(\Phi_2^{*\varepsilon}\vartheta\big)(\rho)\|^2\\
&\le 2\mathbb{E}\left\|\Psi(\varepsilon)\int_0^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{F}(s,\vartheta(s))ds-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
&\quad+2\mathbb{E}\left\|\Psi(\varepsilon)\int_0^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\
&=2\mathbb{E}\left\|\int_0^{\rho-\varepsilon}\big[\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big]\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
&\quad+2\mathbb{E}\left\|\int_0^{\rho-\varepsilon}\big[\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big]\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\
&\le 2\mathbb{E}\int_0^{\rho-\varepsilon}\big\|\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big\|^2\big\|\mathsf{F}(s,\vartheta(s))\big\|^2ds\\
&\quad+2\mathbb{E}\int_0^{\rho-\varepsilon}\big\|\Psi(\varepsilon)\Psi(\rho-s-\varepsilon)-\Psi(\rho-s)\big\|^2\big\|\mathsf{G}(s,\vartheta(s))\big\|^2d\mathsf{R}_Q^{\mathsf{H}}(s)\\
&\le 2L(\varepsilon)^2\rho\int_0^{\rho-\varepsilon}\mathbb{E}\big\|\mathsf{F}(s,z(s))\big\|^2ds+2L(\varepsilon)^2\int_0^{\rho-\varepsilon}\mathbb{E}\big\|\mathsf{G}(s,\vartheta(s))\big\|^2d\mathsf{R}_Q^{\mathsf{H}}(s)\\
&\le 3L(\varepsilon)^2\int_0^{\rho-\varepsilon}\mathfrak{M}_{\mathsf{F}}(1+\|\vartheta(s)\|^2)ds+3L(\varepsilon)^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_0^{\rho-\varepsilon}\mathfrak{M}_{\mathsf{G}}(1+\|\vartheta(s)\|^2)ds\\
&\le 3L(\varepsilon)^2\mathsf{b}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})+3L(\varepsilon)^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\underset{\varepsilon\to0}{\longrightarrow}0.
\end{aligned}$$

So the set $\mathcal{Z}^{\varepsilon}(\rho)=\{(\Phi_2^{\varepsilon}\vartheta)(\rho):\vartheta\in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ by using the total boundedness.

Using this idea again, we obtain

$$\begin{aligned}
\mathbb{E}\|(\Phi_2\vartheta)(\rho)-(\Phi_2^{\varepsilon}\vartheta)(\rho)\|^2\le 3\mathbb{E}&\left\|\int_0^{\rho}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2\\
+3\mathbb{E}&\left\|\int_0^{\rho}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)-\int_0^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2
\end{aligned}$$

$$\begin{aligned}&\leq 3\mathbb{E}\left\|\int_{\rho-\varepsilon}^{\rho}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2+3\mathbb{E}\left\|\int_{\rho-\varepsilon}^{\rho}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\&\leq 3M^2\varepsilon\left[\int_{\rho-\varepsilon}^{\rho}\mathbb{E}\|\mathsf{F}(s,\vartheta(s))\|^2ds+2c_{\mathsf{H}}\varepsilon^{2\mathsf{H}-1}\int_{\rho-\varepsilon}^{\rho}\mathbb{E}\|\mathsf{G}(s,\vartheta(s))\|^2ds\right]\\&\leq 2M^2\varepsilon^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})+2M^2c_{\mathsf{H}}\varepsilon^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

Similarly, for any $\rho \in (e_i, \rho_{i+1}]$ with $i = 1, \ldots, m$. Let $e_i < \rho \leq s \leq \rho_{i+1}$ be fixed and let ε be a real number satisfying $0 < \varepsilon < \rho$. If we use Lemma 2.6 and compactness of $\big(\Psi(\varepsilon)\big)_{\varepsilon>0}$, we deduce that the set $\mathcal{Z}^{\varepsilon}(\rho)$ is precompact in $\mathbb{H}$ for every ε, $0 < \varepsilon < \rho$. Moreover, by Lemma 2.6 and Hölder's inequality, for every $\vartheta \in S_{\mathfrak{M}}$ we have:

$$\begin{aligned}&\mathbb{E}\|(\Phi_2^{\varepsilon}\vartheta)(\rho)-\big(\Phi_2^{*\varepsilon}\vartheta\big)(\rho)\|^2\\&\leq 2\mathbb{E}\left\|\Psi(\varepsilon)\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{F}(s,\vartheta(s))ds-\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2\\&\quad+2\mathbb{E}\left\|\Psi(\varepsilon)\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-s-\varepsilon)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)-\int_{s_i}^{\rho-\varepsilon}\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\&\leq 3\big(L(\varepsilon)\big)^2\left[\mathsf{b}\int_{s_i}^{\rho-\varepsilon}\mathbb{E}\|\|\mathsf{F}(s,\vartheta(s))\|^2ds+c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_{s_i}^{\rho-\varepsilon}\mathbb{E}\|\mathsf{G}(s,\vartheta(s))\|^2ds\right]\\&\leq 3\big(L(\varepsilon)\big)^2\left[\mathsf{b}^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})+c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\right]\underset{\varepsilon\to 0}{\longrightarrow}0.\end{aligned}$$

So the set $\mathcal{Z}^{\varepsilon}(\rho) = \{(\Phi_2^{\varepsilon}\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$ by using the total boundedness.

Using this idea again, we obtain

$$\begin{aligned}\mathbb{E}\|(\Phi_2\vartheta)(\rho)-(\Phi_2^{\varepsilon}\vartheta)(\rho)\|^2&\leq 2\mathbb{E}\left\|\int_{\rho-\varepsilon}^{\rho}\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds\right\|^2+2\mathbb{E}\left\|\int_{\rho-\varepsilon}^{\rho}\Psi(\rho-s)\mathsf{G}(s)d\mathsf{R}_Q^{\mathsf{H}}(s)\right\|^2\\&\leq 2M^2\varepsilon^2\mathfrak{M}_{\mathsf{F}}(1+\mathfrak{M})+2M^2c_{\mathsf{H}}\varepsilon^{2\mathsf{H}}\mathfrak{M}_{\mathsf{G}}(1+\mathfrak{M})\to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

Therefore, as $\varepsilon \to 0$, there are precompact sets arbitrarily close to the set $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$. Thus, the set $\mathcal{Z}(\rho) = \{(\Phi_2\vartheta)(\rho) : \vartheta \in S_{\mathfrak{M}}\}$ is precompact in $\mathbb{H}$. Finally, by the Arzelà-Ascoli theorem, we can conclude that the operator Φ_2 is continuous and compact.

Step 5. Φ_1 is a contraction.

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in [0, \rho_1]$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)-(\Phi_1\vartheta_2)(\rho)\|^2\leq M^2R_{\mathsf{u}_0}\rho_1^2\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}. \tag{3.11}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in (\rho_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\mathbb{E}\|(\Phi_1\vartheta_1)(\rho)-(\Phi_1\vartheta_2)(\rho)\|^2=\mathbb{E}\|\mathsf{p}_i(\rho,\vartheta_1(\rho_i^-))-\mathsf{p}_i(\rho,\vartheta_2(\rho_i^-))\|^2\leq R_{\mathsf{p}_i}\|\vartheta_2-\vartheta_1\|^2_{\mathcal{PC}}. \tag{3.12}$$

For any $\vartheta_1, \vartheta_2 \in S_{\mathfrak{M}}$ and $\rho \in (s_i, \rho_{i+1}]$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_1\vartheta_1)(\rho) - (\Phi_1\vartheta_2)(\rho)\|^2 &\le 2M^2 R_{\mathsf{p}_i}\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}} + 2M^2\|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho_{i+1}^2\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}} \\ &= 2M^2\left(R_{\mathsf{p}_i} + \|\mathsf{B}\|^2 R_{\mathsf{u}_i}\rho_{i+1}^2\right)\|\vartheta_2 - \vartheta_1\|^2_{\mathcal{PC}}. \end{aligned} \tag{3.13}$$

Equations (3.11)–(3.13) and hypothesis $(\mathbf{C}_5)$ imply that Φ_1 is a contraction. The operators Φ_1, Φ_2 satisfy all the conditions of Theorem 2.12, then there exists a fixed point ϑ on $S_{\mathfrak{M}}$. Therefore, the system (1.2) has at least on mild solution on $[0, \mathsf{b}]$. □

Theorem 3.3. *Assume that hypotheses* $(\mathbf{C}_1)$–$(\mathbf{C}_6)$ *hold and the functions* F, G *are uniformly bounded on their respective domains. Then the system* (1.2) *is approximately controllable on* $[0, \mathsf{b}]$.

Proof. Let ϑ^γ be a fixed point of $\Phi_1 + \Phi_2$. By using the stochastic Fubini theorem, we obtain

$$\begin{aligned}\vartheta^{(\gamma)}(\rho_{i+1}) &= \vartheta_{\rho_{i+1}} - \gamma(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\left\{\mathbb{E}\,\vartheta_{\rho_{i+1}} - \Psi(\rho_{i+1} - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-))\right\} \\ &\quad - \gamma\int_{s_i}^{\rho_{i+1}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s) + \gamma\int_0^{\mathsf{b}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s, \vartheta(s))\,\mathrm{ds} \\ &\quad + \gamma\int_{s_i}^{\rho_{i+1}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1} - s)\,G(s, \vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), \quad i = 0, 1, 2, \dots, m. \end{aligned} \tag{3.14}$$

Moreover, hypotheses F and G are uniformly bounded. Then there are subsequences, still denoted by $\mathsf{F}(s, \vartheta^\gamma)$ and $\mathsf{G}(s, \vartheta^\gamma)$, which converge weakly to say $\mathsf{F}(s)$ and $\mathsf{G}(s)$ respectively in $\mathbb{H}$ and L_2^0. From the above equation, we obtain

$$\begin{aligned}\mathbb{E}\|\vartheta^{(\gamma)}(\rho_{i+1}) - \vartheta_{\rho_{i+1}}\|^2 &\le 7\mathbb{E}\left\|\gamma(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\mathbb{E}\vartheta_{\rho_{i+1}}\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Phi_i(s)d\mathsf{R}^{\mathsf{H}}(s)\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1} - s_i)\mathsf{p}_i(s_i, \vartheta(\rho_i^-))\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma\int_0^b(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s)\,\mathrm{ds}\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{G}(s)d\mathsf{R}^{\mathsf{H}}(s)\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma\int_0^b(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_i+1})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{F}(s, \vartheta^\gamma(s))\,\mathrm{ds}\right\|^2 \\ &\quad + 7\mathbb{E}\left\|\gamma\int_{s_i}^{\rho_{i+1}}(\gamma \mathrm{Id} + \mathbf{\Pi}_{s_i}^{\rho_{i+1}})^{-1}\Psi(\rho_{i+1} - s)\,\mathsf{G}(s, \vartheta^\gamma(s))d\mathsf{R}^{\mathsf{H}}(s)\right\|^2. \end{aligned}$$

It follows from $(\mathbf{H}_0)$, for all $0 \le s \le \mathsf{b}$ the operator $\gamma(\gamma \mathrm{Id} + \mathbf{\Pi}_s^{\rho_{i+1}})^{-1} \to 0$ as $\gamma \to 0^+$, and $\|(\gamma \mathrm{Id} + \mathbf{\Pi}_s^{\rho_{i+1}})^{-1}\|^2 \le 1$ and by using the Arzelà-Ascoli theorem, one can prove that the operator

$\bar{l}(\cdot) \to \int_{s_i}^{\rho_{i+1}} \Psi(\cdot - s)\bar{l}(s)ds$ is compact, we obtain

$$\mathbb{E}\|\vartheta^{(\gamma)}(\rho_{i+1}) - \vartheta_{\rho_{i+1}}\|^2 \to 0 \text{ as } \gamma \to 0^+.$$

This gives the approximate controllability of system (1.2) on $[0, \mathsf{b}]$. □

Now, we are going to prove the approximate controllability of the stochastic system (1.2) by using another method, namely Schauder's fixed point theorem with some other hypotheses, which are different from hypotheses of the Theorems 3.2 and 3.3. In order to establish the approximate controllability results, we impose the following hypotheses.

($\mathbf{C}_7$) $\mathsf{T}(\rho)$ is compact for $\rho > 0$.

($\mathbf{C}_8$) The function $\mathsf{F} : \mathcal{J} \times \mathbb{H} \to \mathbb{H}$ satisfy the following conditions

(a) for each $\rho \in \mathcal{J}$ the function $\mathsf{F}(\rho, \cdot) : \mathbb{H} \to \mathbb{H}$ is continuous for each $\vartheta \in \mathbb{H}$ the function $\mathsf{F}(\cdot, \vartheta) : \mathcal{J} \to \mathbb{H}$ is strongly measurable,

(b) for each positive number $\mathfrak{M}$, there exists $\mu_{\mathfrak{M}} \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\sup_{\mathbb{E}\|\vartheta\|^2 \le \mathfrak{M}} \mathbb{E}\|\mathsf{F}(\rho, \vartheta)\|^2 \le \mu_{\mathfrak{M}}(\rho)$$

and there exists a $\Lambda_1 > 0$ such that

$$\lim_{\mathfrak{M} \longrightarrow \infty} \frac{\int_0^{\rho} \mu_{\mathfrak{M}}(s)d\rho}{\mathfrak{M}} = \Lambda_1 < \infty.$$

($\mathbf{C}_9$) The function $\mathsf{G} : \mathcal{J} \times L_2^0$ satisfies the following conditions

(a) for each $\rho \in \mathcal{J}$ the function $\mathsf{G}(\rho, \cdot) : \mathbb{H} \to L_2^0$ is continuous for each $\vartheta \in \mathbb{H}$ the function $\mathsf{G}(\cdot, \vartheta) : \mathcal{J} \to L_2^0$ is strongly measurable,

(b) for each positive number $\mathfrak{M}$, there exists $v_{\mathfrak{M}} \in L^1(\mathcal{J}, \mathbb{R}^+)$ such that

$$\sup_{\mathbb{E}\|\vartheta\|^2 \le \mathfrak{M}} \mathbb{E}\|\mathsf{G}(\rho, \vartheta)\|^2_{L_2^0} \le v_{\mathfrak{M}}(\rho)$$

and there exists a $\Lambda_2 > 0$ such that

$$\lim_{\mathfrak{M} \longrightarrow \infty} \frac{\int_0^{\rho} v_{\mathfrak{M}}(s)d\rho}{\mathfrak{M}} = \Lambda_2 < \infty.$$

Theorem 3.4. *Assume that hypotheses* $(\mathbf{C}_2)$ *and* $(\mathbf{C}_7)$–$(\mathbf{C}_9)$ *hold. Then, the system* (1.2) *has at least one mild solution on* $[0,\mathsf{b}]$, *provided that*

$$\max_{1\le i\le m}\left[D_{\mathsf{p}_i}\left(\frac{4\|\mathsf{B}\|^2M^4\mathsf{b}^2}{\gamma^2}\right)\left(4M^2D_{\mathsf{p}_i}+4M^2\mathsf{b}\Lambda_1+8M^2c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\Lambda_2\right)\right]<1. \tag{3.15}$$

Proof. Consider a set

$$S'_{\mathfrak{M}}=\{\vartheta\in C_{\mathsf{b}}:\|\vartheta\|^2_{\mathcal{PC}}\le\mathfrak{M}\}\subseteq C_{\mathsf{b}},$$

where $\mathfrak{M}$ is constant. The set $S'_{\mathfrak{M}}$ is a bounded closed and convex set in C_{b}.

Now, we define an operator $\mathcal{F}$ on C_{b} by

$$(\mathcal{F}\vartheta)(\rho)=\begin{cases}\Psi(\rho)\vartheta_0+\displaystyle\int_0^\rho\Psi(\rho-s)\mathsf{B}\mathsf{u}^\gamma(s,\vartheta)ds & \\ +\displaystyle\int_0^\rho\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds+\int_0^\rho\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), & \rho\in[0,\rho_1],\\ \mathsf{p}_i(\rho,\vartheta(\rho_i^-)), & \rho\in(\rho_i,s_i],\\ \Psi(\rho-s_i)\mathsf{p}_i(s_i,\vartheta(\rho_i^-))+\displaystyle\int_{s_i}^\rho\Psi(\rho-s)\mathsf{B}\mathsf{u}^\gamma(s,\vartheta)ds & \\ +\displaystyle\int_{s_i}^\rho\Psi(\rho-s)\mathsf{F}(s,\vartheta(s))ds+\int_{s_i}^\rho\Psi(\rho-s)\mathsf{G}(s,\vartheta(s))d\mathsf{R}^{\mathsf{H}}(s), & \rho\in(s_i,\rho_{i+1}].\end{cases}$$

Next, we prove that the operator $\mathcal{F}$ satisfies all Schauder's fixed point theorem conditions.

Now, we prove that there exists $\mathfrak{M}>0$ such that $\mathcal{F}(S'_{\mathfrak{M}})\subseteq S'_{\mathfrak{M}}$. If we assume that this assertion is false, then for any $\mathfrak{M}>0$, we can choose $\vartheta^{\mathfrak{M}}\in S'_{\mathfrak{M}}$ and $\rho\in[0,\mathsf{b}]$ such that $\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2>\mathfrak{M}$.

For any $\rho\in[0,\rho_1]$, we have

$$\begin{aligned}\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2&\le\frac{16\|\mathsf{B}\|^2M^4\rho_1^2}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_1^4}{\gamma^2}\right)\Bigg\{4M^2\mathbb{E}\|\vartheta_0\|^2\\&+4M^2\rho_1\int_0^\rho\mu_{\mathfrak{M}}(s)ds+8M^2c_{\mathsf{H}}\rho_1^{2\mathsf{H}-1}\int_0^\rho\upsilon_{\mathfrak{M}}(s)ds\Bigg\}.\end{aligned}$$

For any $\rho\in(\rho_i,s_i]$, $i=1,2,\dots,m$, we have

$$\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2=\mathbb{E}\|\mathsf{p}_i(\rho,\vartheta^{\mathfrak{M}}(\rho_i^-))\|^2\le D_{\mathsf{p}_i}(1+\mathfrak{M}).$$

Similarly, for $\rho\in(s_i,\rho_{i+1}]$, $i=1,2,\dots,m$, we have

$$\begin{aligned}\mathfrak{M}<\mathbb{E}\|\mathcal{F}(\vartheta^{\mathfrak{M}})(\rho)\|^2&\le\frac{16\|\mathsf{B}\|^2M^4\rho_1^2}{\gamma^2}\mathbb{E}\|\vartheta_{\rho_1}\|^2+\left(1+\frac{4\|\mathsf{B}\|^4M^4\rho_{i+1}^2}{\gamma^2}\right)\Bigg\{4M^2D_{\mathsf{p}_i}(1+\mathfrak{M})\\&+4M^2\rho_{i+1}\int_0^\rho\mu_{\mathfrak{M}}(s)ds+8M^2\rho_{i+1}^{2\mathsf{H}-1}\int_0^\rho\upsilon_{\mathfrak{M}}(s)ds\Bigg\}\end{aligned}$$

From the above equations we have for $\rho \in [0, \mathsf{b}]$

$$\mathfrak{M} < \mathbb{E}\|(\mathcal{F}\vartheta^{\mathfrak{M}})(\rho)\|^2 \leq Q + D_{\mathsf{p}_i}\mathfrak{M} + \left(1 + \frac{4\|\mathsf{B}\|^4 M^4 \mathsf{b}^2}{\gamma^2}\right)\Bigg\{4M^2 D_{\mathsf{p}_i}\mathfrak{M} + 4M^2\mathsf{b}\int_0^\rho \mu_{\mathfrak{M}}(s)ds + 8M^2 c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\int_0^\rho \upsilon_{\mathfrak{M}}(s)ds\Bigg\},$$

where

$$Q = \max_{1\leq i\leq m}\left[\frac{16\|\mathsf{B}\|^2 M^2}{\gamma^2}[\rho_1^2\mathbb{E}\|\vartheta_{\rho_1}\|^2 + \rho_{i+1}^2\mathbb{E}\|\vartheta_{\rho_{i+1}}\|^2] + 4c_0M^2\mathbb{E}\|\vartheta_0\|^2 + D_{\mathsf{p}_i} + 4c_iM^2D_{\mathsf{p}_i}\right].$$

Dividing both sides of above by $\mathfrak{M}$ and taking $\mathfrak{M} \to \infty$, we obtain

$$1 < D_{\mathsf{p}_i} + \left(1 + \frac{4\|\mathsf{B}\|^4 M^4 \mathsf{b}^2}{\gamma^2}\right)\left(4M^2 D_{\mathsf{p}_i} + 4M^2\mathsf{b}\Lambda_1 + 8M^2 c_{\mathsf{H}}\mathsf{b}^{2\mathsf{H}-1}\Lambda_2\right).$$

This contradicts (3.15). Hence, there exists $\mathfrak{M} > 0$ such that $\mathcal{F}(S'_{\mathfrak{M}}) \subseteq S'_{\mathfrak{M}}$.

Adopting the method used in the Theorem 3.1 of the paper [31], one can easily show that $\mathcal{F}$ is a continuous operator. Hence, operator $\mathcal{F}$ satisfies all the conditions of the Theorem 2.13, then there exists a fixed point ϑ on $S'_{\mathfrak{M}}$. Therefore, the system (1.2) has at least one mild solution on $[0, \mathsf{b}]$. □

Theorem 3.5. *Assume that hypotheses* $(\mathbf{C}_2)$, $(\mathbf{C}_7)$–$(\mathbf{C}_9)$ *hold and the functions* F, G *are uniformly bounded on their respective domains. Then the stochastic system* (1.2) *is approximately controllable on* $[0, \mathsf{b}]$.

Proof. Using the same arguments as in the Theorem 3.3, one can prove the approximate controllability of stochastic system (1.2). □

Remark 3.6. *We can see that the hypotheses of the Theorem* (1.2) *and Theorem* (3.5) *are sufficient conditions but not necessary to prove the approximate controllability of the stochastic system* (1.2).

4 Application

For an illustration of the obtained theory, we consider the following stochastic integrodifferential system

$$\begin{cases} d\,z(\rho,\xi) &= \left[\dfrac{\partial^2 z(\rho,\xi)}{\partial \xi^2} + \displaystyle\int_0^\rho \gamma(\rho-s)\dfrac{\partial^2 z(s,\xi)}{\partial \xi^2}ds + f_1(\rho, z(\rho,\xi)) + \mathsf{u}(\rho,\xi) \right. \\ & \left. \quad + g_1(\rho, z(\rho,\xi))\dfrac{d\mathsf{R}^{\mathsf{H}}(\rho)}{d\rho}\right] d\rho, \quad 0 \le \xi \le \pi, \quad \rho \in (2i, 2i+1], \quad i = 0,1,\dots,m, \\ z(\rho,0) &= z(\rho,\pi) = 0, \quad \rho \ge 0, \\ z(\rho,\xi) &= \sin it.z((2i-1)^-,\xi), \quad \rho \in (2i-1, 2i], \quad i = 1,2,\dots,m, \\ z(0,\xi) &= z_0(\xi), \quad \xi \in [0,\pi]. \end{cases} \tag{4.1}$$

where $0 = s_0 = \rho_0 < \rho_1 < s_1 < \cdots < s_m < \rho_{m+1} = \mathsf{b} < \infty$ with $\rho_1 = 1$, $s_i = 2i$, $\rho_i = 2i-1$, R^{H} is a Rosenblatt process. The functions f_1, g_1 and γ will be described below.

Let $\mathbb{H} = \mathbb{Y} = \mathbb{U} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $\mathrm{A} : \mathcal{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ by $\mathrm{A}\vartheta = \vartheta''$ with domain

$$\mathcal{D}(\mathrm{A}) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

The spectrum of A consists of the eigenvalues $-n^2$ for $n \in \mathbb{N}^\star$, with associated eigenvectors $e_n := \sqrt{\frac{2}{\pi}}\sin(n\vartheta)$, $(n = 1,2,3,\dots)$. Furthermore, the set $\{e_n : n \in \mathbb{N}^\star\}$ is an orthogonal basis in $\mathbb{H}$. Then

$$\mathrm{A}\vartheta = \sum_{n=1}^{\infty} -n^2\langle \vartheta, e_n\rangle e_n, \quad \vartheta \in \mathbb{H}.$$

It is well known that A is the infinitesimal generator of a strongly continuous semigroup $\{\mathsf{T}(\rho)\}_{\rho\ge 0}$ on $\mathbb{H}$, which is compact and is given by

$$\mathsf{T}(\rho)\vartheta = \sum_{n=1}^{\infty} e^{-n^2\rho}\langle \vartheta, e_n\rangle e_n, \quad \vartheta \in \mathbb{H}.$$

In order to define the operator $Q : \mathbb{Y} \to \mathbb{Y}$, we choose a sequence $\{v_n\}_{n\ge 1} \subset \mathbb{R}^+$, set $Qe_n = v_n e_n$ and assume that $Tr(Q) = \sum_{n=1}^{\infty}\sqrt{v_n} < \infty$. Define the process $\mathsf{R}_Q^{\mathsf{H}}(s)$ by

$$\mathsf{R}_Q^{\mathsf{H}}(s) = \sum_{n=1}^{\infty}\sqrt{v_n}\beta_n^{\mathsf{H}}(\rho)e_n$$

where $\mathsf{H} \in (\frac{1}{2}, 1)$ and $\{\beta_n^{\mathsf{H}}\}_{n\in\mathbb{N}}$ is a sequence of mutually independent two-sided one-dimensional fBm and an infinite dimensional space.

Let $\mathbf{\Gamma} : \mathcal{D}(\mathrm{A}) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by

$$\mathbf{\Gamma}(\rho)(\tilde{z}) = \gamma(\rho)\mathrm{A}\tilde{z} \quad \text{for} \quad \rho \geq 0 \quad \text{and} \quad \tilde{z} \in D(\mathrm{A}).$$

In order to rewrite system (4.1) in an abstract form in $\mathbb{H}$, we introduce the following notations

$$\begin{cases} \vartheta(\rho) &= z(\rho,\xi) \quad \text{for} \quad \rho \geq 0 \quad \text{and} \quad \xi \in [0,\pi], \\ \vartheta(0) &= z(0,\xi) \quad \text{for} \quad \xi \in [0,\pi], \end{cases}$$

and the bounded linear operator $\mathsf{B} : L^2([0,\pi]) \to L^2([0,\pi])$ as

$$\mathsf{Bu}(\rho)(\xi) = \mathsf{u}(\rho,\xi), \quad \rho \in [0,\mathsf{b}], \quad \xi \in [0,\pi].$$

Next, we define the functions $\mathsf{F} : \mathsf{b}_0 \times \mathbb{H} \to \mathbb{H}$ and $\mathsf{G} : \mathsf{b}_0 \times \mathbb{H} \to L_2(\mathbb{X},\mathbb{H})$ as

$$\mathsf{F}(\rho,\vartheta(\rho))(\xi) = f_1(\rho,\vartheta(\rho))(\xi), \quad \vartheta \in \mathbb{H}, \quad \xi \in [0,\pi], \tag{4.2}$$

$$\mathsf{G}(\rho,\vartheta)(\xi) = g_1(\rho,\vartheta(\rho))(\xi), \quad \vartheta \in \mathbb{H}, \quad \xi \in [0,\pi]. \tag{4.3}$$

The functions $\mathsf{p}_i : \mathsf{b}_i \times \mathbb{H} \to \mathbb{H}$ are given by $\mathsf{p}_i(\rho,\vartheta(t_i^-))(\xi) = \sin it.z((2i-1)^-,\xi)$. From the above choices of the functions and operator $\mathbf{\Gamma}(\rho)$ with $\mathsf{B} = Id$, the system (4.1) takes the following abstract form

$$\begin{cases} d\vartheta(\rho) = \mathrm{A}\vartheta(\rho) + \displaystyle\int_0^\rho \mathbf{\Gamma}(\rho-s)\vartheta(s)ds + \mathsf{F}(\rho,\vartheta(\rho)) + \mathsf{Bu}(\rho) + \mathsf{G}(s,\vartheta(s))\frac{d\mathsf{R}^{\mathsf{H}}(\rho)}{d\rho}, \quad \rho \in \cup_{i=0}^m (s_i,\rho_{i+1}), \\ \vartheta(\rho) = \mathsf{p}_i(\rho,\vartheta(\rho_i^-)), \quad \rho \in \cup_{i=1}^m (\rho_i,s_i], \\ \vartheta(0) = \vartheta_0. \end{cases} \tag{4.4}$$

Moreover, $\mathbf{\Gamma}(\rho)$ satisfies $(\mathbf{R_2})$ and hence, by Theorem 2.4, Eq. (2.5) has a resolvent operator $(\Psi(\rho))_{\rho\geq 0}$ on $\mathbb{H}$. In particular, if we take $\mathsf{F}(\rho,\vartheta(\rho))(\xi) = \dfrac{\sin\rho}{1+\sin\rho}\vartheta(\rho)(\xi)$, and $\mathsf{G}(\rho,\vartheta)(\xi) = \dfrac{\vartheta(\rho)(\xi)}{e(1+e^\rho)}$, we see that, F and G satisfy assumptions $(\mathbf{C_1})$ and $(\mathbf{C_2})$. Therefore all conditions of Theorem 3.2 are satisfied. Since the semigroup $\mathsf{T}(\rho)$ is compact for $\rho > 0$, it is clear from Theorem 2.5 that the resolvent operator $\Psi(\rho)$ is compact for all $\rho > 0$. Therefore, the associated linear system of (4.1) can not be exactly controllable but may be approximately controllable.

It remains now to verify that $(\mathbf{H}_0)$ is fulfilled. To this end, we have the following result:

Lemma 4.1 ([30])**.** *Let $\gamma(\rho) \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ with primitive $\mathbb{O}(\rho) \in L^1_{loc}(\mathbb{R}^+)$ such that $\mathbb{O}(\rho)$ is non-positive, non-decreasing and $\mathbb{O}(0) = -1$. If operator* A *is self-adjoint and positive semi-definite, then the resolvent operator $\Psi(\rho)$ associated to* (2.5) *is self-adjoint as well.*

By Lemma 4.1 above, the resolvent operator $\Psi(\rho)$ of (2.5) is self-adjoint. Thus

$$\Psi^*(\rho)y = \Psi(\rho)y, \quad y \in \mathbb{H}.$$

If $\Psi^*(\rho)y = 0$, for all $\rho \in \mathcal{J}$, thus

$$\Psi^*(\rho)y = \Psi(\rho)y = 0, \quad \rho \in \mathcal{J}.$$

It follows from the fact $\Psi(0) = \mathrm{Id}$ that $y = 0$, so by virtue of Theorem 2.10, $(\mathbf{H}_0)$ holds. Therefore, in view of Theorem 3.2 and Theorem 3.3, the stochastic integrodifferential system (4.4) is approximately controllable on $\mathcal{J}$.

Remark 4.2. *In this above example, if we choose* $\mathsf{F}(\rho, \vartheta) = \dfrac{1}{\rho^{1/3}} \sin \vartheta$*, we observe that* $F(\rho, \vartheta)$ *does not satisfy the Lipschitz condition* $(\mathbf{C}_3) - b$ *near* 0*, but it satisfies the hypotheses* $(\mathbf{C}_8)$ *(see [36]). With this setting, Theorem 3.2 can not be applied to the system* (4.4)*, but we can apply the Theorem 3.3 to the* (4.4)*.*

5 Conclusion

In this research, we investigated the approximate controllability for a class of non-instantaneous impulsive integrodifferential equations driven by the Rosenblatt process. The proposed results have been carried out using Grimmer resolvent operator, stochastic analysis theory, and fixed point techniques (Krasnoselskii's and Schauder's fixed point theorem). Finally, an example is provided to illustrate the applicability of our results. We believe our study can open new research routes in stochastic integrodifferential systems with state-dependent delay and fractional cases. This article will initiate future work in the above categories.

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