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Research article

Well posedness of second-order impulsive fractional neutral stochastic differential equations

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Abstract: In this manuscript, we investigate a class of second-order impulsive fractional neutral stochastic differential equations (IFNSDEs) driven by Poisson jumps in Banach space. Firstly, sufficient conditions of the existence and the uniqueness of the mild solution for this type of equations are driven by means of the successive approximation and the Bihari's inequality. Next we get the stability in mean square of the mild solution via continuous dependence on initial value.

Keywords: stochastic fractional differential equations; Bihari's inequality; successive approximation; Poisson jumps
Mathematics Subject Classification: 34K30, 60H60

1. Introduction

In recent years, fractional differential equations (FDEs) are an effective mathematical tool to model and analyze many real life problems; it has been used by researchers and scientists to get better results than the integer order differential equations. Classical theory and applications of FDEs are presented in the monographs [8, 11, 13]. Stochastic differential equations (SDEs) are the proper apparatus to model systems with external noise and suffered by uncertain or random facts, for more details on SDEs readers can can refer to [1, 2, 5, 6, 10, 14]. Very recently, many researchers were devoted to study impulsive fractional integro differential evolution equations Xie [18] investigated as follows,

$$\begin{aligned} \mathfrak{D}_t^{\alpha}[y'(t) - \mathfrak{g}(t, y_t)] &= \mathfrak{A}y(t) + \mathfrak{f}(t, y_t, \mathfrak{B}y(t)),\ t \in J,\ t \neq t_k, \\ \Delta y(t_i) &= I_i(y_{t_i}),\ \Delta y'(t_i) = J_i(x_{t_i}),\ i = 1, 2, ..., m, \\ y_0 &= \varphi \in \mathcal{B},\ y_0' = y_1 \in \mathfrak{H}, \end{aligned}$$

where $\mathfrak{B}y(t) = \int_0^t \mathfrak{k}(t, s)y(s)ds$, $k \in \mathfrak{C}(\mathfrak{D}, \mathfrak{R}^+)$, $\mathfrak{D} = \{(t, s) : 0 \leq s \leq t \leq b\}$.

On the other hand, Poisson jumps processes are used in modeling for several real life situations. Moreover, many practical applications are used in the field of market crashes, earthquakes, epidemics, etc,. In dynamical systems, a jump term is included to make the model a realistic one. Many literature have been study SDEs driven by Poisson jumps has [3,4,7,9,12,15–17]. However, there is no literature in IFNSDEs, we use of the successive method and Bihari's inequality. This paper is concerned with IFNSDEs driven by Poisson jump,

$$\begin{aligned} \mathfrak{D}_t^{\alpha}\left[y'(t) - \mathfrak{g}(t, y_t)\right] &= \mathfrak{A}y(t) + \mathfrak{f}(t, y_t) + \sigma(t, y_t)dw(t) + \int_{\mathcal{U}} \mathfrak{h}(t, y_t, u)\widetilde{\mathfrak{N}}(ds, du), J = t \in [0, a], \\ \Delta y(t_k) &= I_k(y_{t_k}),\ \Delta y'(t_k) = J_k(y_{t_k}),\quad k = 1, 2, ..., m, \\ y_0 &= \varphi \in \mathcal{B},\quad y_0' = y_1 \in \mathfrak{H}. \end{aligned} \tag{1.1}$$

Here, $\mathfrak{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order $0 < \alpha < 1$; $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathfrak{X} \to \mathfrak{X}$ denotes sectorial operator. The nonlinear maps $\mathfrak{f}, \mathfrak{g} : [0, a] \times \mathcal{B} \to \mathfrak{X}$, $\sigma : [0, a] \times \mathcal{B} \to \mathcal{L}(\mathfrak{Y}, \mathfrak{X})$ and $\mathfrak{h} : [0, a] \times \mathcal{B} \times \mathcal{U} \to \mathfrak{X}$ are appropriate mappings. Let $\mathcal{B}$ is an abstract phase space. Let $y_t : (-\infty, 0] \to \mathfrak{X}$, $y_t(s) = y(t + s)$, $s \leq 0 \in \mathcal{B}$. In $\widetilde{\mathfrak{N}}(ds, du) = \mathfrak{N}(ds, du) - v(du)ds$. the Poisson measure $\tilde{\mathcal{N}}(dt, du)$ denotes the Poisson counting measure associated with a characteristic measure λ. Moreover, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ for $0 \leq t_0 < t_1 < \cdots < t_n < t_{n+1} = a$, and $y(t_k^+)$ and $y(t_k^-)$ denote the right and the left limits of $y(t)$ at $t = t_k$, respectively.

2. Preliminaries

In this section, we dealt with basic definitions for FC and some of the lemmas that are useful for further derivation, (see [13, 19]).

We introduce the space $\mathscr{PC}$ formed by all $\mathfrak{X}$-valued stochastic processes $\{y(t) : t \in [0, a]\}$ such that y is continuous at $t \neq t_k$, $y(t_k^-) = y(t_k)$ and $y(t_k^+)$ exist for all $k = 1, 2, ..., m$. When $\mathscr{PC}$ is endowed with the norm $\left\|y\right\|_{\mathscr{PC}} = \left(\sup_{s\in[0,a]} \mathbb{E}\|y(s)\|^2\right)^{1/2}$, $(\mathscr{PC}, \|\cdot\|_{\mathscr{PC}})$ is a Banach space. Next, we present an axiomatic definition of the phase space $\mathcal{B}$ are established for $\mathfrak{I}_0$-measurable functions from $(-\infty, 0] \to \mathfrak{X}$, with a semi norm $\|\cdot\|_{\mathcal{B}}$ which satisfies:

(H1) If $y : (-\infty, a] \to \mathfrak{X}$, $a > 0$ is s.t $y_0 \in \mathcal{B}$ and $y|_{[0,a]} \in \mathscr{PC}$, then, for every $t \in [0, a]$, if the following conditions hold:

(1) $y_t \in \mathcal{B}$
(2) $\left|y(t)\right| < K \left\|y_t\right\|_{\mathcal{B}}$,
(3) $\left\|y_t\right\|_{\mathcal{B}} \leq M(t) \sup_{0\leq s\leq t} \left|y(s)\right| + N(t) \left\|y_0\right\|_{\mathcal{B}}$

where $K > 0$, $M, N : [0, +\infty) \to [1, +\infty)$ are mappings. M is continuous and N is locally bounded.

(H2) The space $\mathcal{B}$ is complete.

Lemma 2.1. *Let* $\mathrm{y} : (-\infty, a] \to \mathfrak{X}$ *be an* $\mathfrak{I}_{\mathrm{t}}$*-adapted measurable process s.t* $\mathrm{y}_0 = \varphi \in \mathcal{L}_2(\Omega, \mathcal{B})$,

$$\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}\right\|_{\mathcal{B}} \leq \mathrm{N}_a\mathbb{E}\,\|\varphi\|_{\mathcal{B}} + \mathrm{M}_a\mathbb{E}\left(\sup_{0\leq \mathrm{s}\leq a}\left\|\mathrm{y}(\mathrm{s})\right\|\right), \tag{2.1}$$

where $\mathrm{N}_a = \sup_{\mathrm{t}\in[0,a]}\{\mathrm{N}(\mathrm{t})\}$ *and* $\mathrm{M}_a = \sup_{\mathrm{t}\in[0,a]}\{\mathrm{M}(\mathrm{t})\}$.

Denoted by $\mathcal{M}^2((-\infty, a], \mathfrak{X})$, the space of $\mathfrak{X}$-valued cadlag processes $\mathrm{y} = \{\mathrm{y}(\mathrm{t})\}_{-\infty<\mathrm{t}<a}$ s.t

(i) $\mathrm{y}_0 = \varphi \in \mathcal{B}$, $\mathrm{y}(\mathrm{t})$ is $\mathfrak{I}_{\mathrm{t}}$-adapted on $[0, a]$
(ii) If $\mathcal{M}^2((-\infty, a], \mathfrak{X})$ with the norm

$$\left\|\mathrm{y}\right\|^2_{\mathcal{M}^2} = \mathbb{E}\,\|\varphi\|^2_{\mathcal{B}} + \mathbb{E}\left(\sup_{\mathrm{t}\in[0,a]}\left\|\mathrm{y}(\mathrm{t})\right\|^2\right) < \infty. \tag{2.2}$$

Definition 2.2. *An* $\mathfrak{X}$*-valued stochastic process* $\mathrm{y}(\mathrm{t}), (\mathrm{t} \in \mathrm{J})$ *is called a mild solution of (1.1), if*

(i) $\mathrm{y}(\mathrm{t})$ *is measurable and* $\mathfrak{I}_{\mathrm{t}}$*-adapted for* $\mathrm{t} \in [0, a]$,
(ii) $\mathrm{y}_0 = \varphi \in \mathcal{B}$.
(iii) For $\mathrm{t} \in [0, a]$, *a.s*

$$\mathrm{y}(\mathrm{t}) = \begin{cases} \mathrm{S}_\alpha(\mathrm{t})\varphi(0) + \int_0^{\mathrm{t}} \mathrm{S}_\alpha\left[\mathrm{y}_1 - \mathfrak{g}(0,\varphi)\right]d\mathrm{s} + \int_0^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{g}(\mathrm{s},\mathrm{y}_{\mathrm{s}})d\mathrm{s} \\ + \sum_{\mathrm{t}_{\mathrm{k}}<\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{t}_{\mathrm{k}})I_{\mathrm{k}}(\mathrm{y}_{\mathrm{t}_{\mathrm{k}}}) + \sum_{\mathrm{t}_{\mathrm{k}}<\mathrm{t}} \int_{\mathrm{t}_{\mathrm{k}}}^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\left[J_{\mathrm{k}}(\mathrm{y}_{\mathrm{t}_{\mathrm{k}}}) - \mathfrak{g}(\mathrm{t}_{\mathrm{k}}, \mathrm{y}_{\mathrm{t}_{\mathrm{k}}} + I_{\mathrm{k}}(\mathrm{y}_{\mathrm{t}_{\mathrm{k}}}))\right]d\mathrm{s} \\ + \int_0^{\mathrm{t}} \mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{f}(\mathrm{s},\mathrm{y}_{\mathrm{s}})d\mathrm{s} + \int_0^{\mathrm{t}} \mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\sigma(\mathrm{s},\mathrm{y}_{\mathrm{s}})dw(\mathrm{s}) \\ + \int_0^{\mathrm{t}}\int_{\mathcal{U}} \mathrm{T}_\alpha(\mathrm{t}-s)\mathfrak{h}(s,\mathrm{y}_{\mathrm{s}},\mathrm{u})\widetilde{\mathfrak{N}}(d\mathrm{s}, d\mathrm{u}), \end{cases} \tag{2.3}$$

where $\mathrm{S}_\alpha(\mathrm{t}), \mathrm{T}_\alpha(\mathrm{t}) : \mathfrak{R}_+ \to \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ $(\zeta = 1 + \alpha)$ are given by

$$\mathrm{S}_\alpha(\mathrm{t}) = \mathrm{E}_{\alpha,1}(\mathfrak{A}\mathrm{t}^\alpha) = \frac{1}{2\pi i}\int_{\mathbb{B}_r} \frac{e^{\lambda t}\lambda^{\alpha-1}}{\lambda^\alpha - \mathfrak{A}}d\lambda, \tag{2.4}$$

$$\mathrm{T}_\alpha(\mathrm{t}) = \mathrm{t}^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(\mathrm{A}\mathrm{t}^\alpha) = \frac{1}{2\pi i}\int_{\mathbb{B}_r} \frac{e^{\lambda\mathrm{t}}}{\lambda^\alpha - \mathrm{A}}d\lambda, \tag{2.5}$$

and $\mathbb{B}_r$ denotes the Bromwich path [18].

3. Main results

In order to prove our main results, we enforce the following hypotheses,

(H3) $\mathfrak{A}$ is the infinitesimal generator of an α-order cosine families $\mathrm{S}_\alpha(\mathrm{t})$ and $\mathrm{T}_\alpha(\mathrm{t})$ on $\mathfrak{X}$ and $\exists$ $\mathrm{L} > 0$, $\mathrm{L}_a \geq 1$

$$\|\mathrm{S}_\alpha(\mathrm{t})\| \leq \mathrm{L}$$
$$\|\mathrm{T}_\alpha(\mathrm{t})\| \leq \mathrm{t}^{\alpha-1}\mathrm{L}_a$$

(H4) $\mathfrak{g}, \mathfrak{f} : [0, a] \times \mathcal{B} \to \mathfrak{X}$, $\sigma : [0, a] \times \mathcal{B} \to \mathcal{L}_2^0$ and $\mathfrak{h} : [0, a] \times \mathcal{B} \to \mathfrak{X}$ satisfy

$$(1)\ \left\|\mathfrak{g}(t, y) - \mathfrak{g}(t, z)\right\|^2 \vee \left\|\mathfrak{f}(t, y) - \mathfrak{f}(t, z)\right\|^2 \vee \left\|\sigma(t, y) - \sigma(t, z)\right\|^2 \leq k\left(\left\|y - z\right\|_{\mathcal{B}}^2\right),\ t \in [0, a], y, z \in \mathcal{B},$$

$$(2) \int_{\mathcal{U}} \left\|\mathfrak{h}(t, y, u) - \mathfrak{h}(t, z, u)\right\|^2 v(du)ds \vee \left(\int_{\mathcal{U}} \left\|\mathfrak{h}(t, y, u) - \mathfrak{h}(t, z, u)\right\|^4 v(du)ds\right)^{1/2} \leq k\left(\left\|y - z\right\|_{\mathcal{B}}^2\right),\ t \in [0, a], y, z \in \mathcal{B},$$

$$(3) \left(\int_{\mathcal{U}} \left\|\mathfrak{h}(t, y, u)\right\|^4 v(du)ds\right)^{1/2} \leq k\left|y\right|^2.$$

Here $k(\cdot)$ is a concave, continuous and nondecreasing function from $\mathfrak{R}^+$ to $\mathfrak{R}^+$ s.t $k(0) = 0$, $k(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{ds}{k(s)} = \infty$.

(H5) $I_k, J_k : \mathcal{B} \to \mathfrak{X}$ are continuous and there are positive constants $p_k, q_k > 0$ such that for each $\varphi, \phi \in \mathcal{B}$,

$$\|I_k(\varphi) - I_k(\phi)\|^2 \leq p_k \|\varphi - \phi\|_{\mathcal{B}}^2, \quad \|J_k(\varphi) - J_k(\phi)\|^2 \leq q_k \|\varphi - \phi\|_{\mathcal{B}}^2,\ (k = 1, 2, ..., m).$$

(H6) $\|\mathfrak{g}(t, 0)\|^2 \vee \|\mathfrak{f}(t, 0)\|^2 \vee \|\sigma(t, 0)\|^2 \vee \int_{\mathcal{U}} \|\mathfrak{h}(t, o, u)\|^2 \leq k_0$, where k_0 is a positive constant, $I_k(0) = 0$, $J_k(0) = 0$, $(k = 1, 2, ..., m)$.

Now, The successive approximations are considered as follows,

$$
\begin{aligned}
y^0(t) &= S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(s)\left[y_1 - \mathfrak{g}(0, \varphi)\right] ds, \quad t \in [0, a], && (3.1)\\
y^n(t) &= S_\alpha(t)\varphi(0) + \int_0^t S_\alpha(s)\left[y_1 - \mathfrak{g}(0, \varphi)\right] ds + \sum_{t_k < t} \mathfrak{S}_\alpha(t - t_k) I_k(y_{t_k}^{n-1}) \\
&+ \sum_{t_k < t} S_\alpha(t - t_k)\left[J_k(y_{t_k}^{n-1}) - \mathfrak{g}(t_k, y_{t_k}^{n-1} + I_k(y_{y_{t_k}}^{n-1})) + \mathfrak{g}(t_k, y_{t_k}^{n-1})\right] ds \\
&+ \int_0^t S_\alpha(t - s)\mathfrak{g}(s, y_s^{n-1}) ds + \int_0^t T_\alpha(t - s)\mathfrak{f}(s, y_s^{n-1}) ds + \int_0^t T_\alpha(t - s)\sigma(s, y_s^{n-1}) dw(s) \\
&+ \int_0^t \int_{\mathcal{U}} T_\alpha(t - s)\mathfrak{h}(s, y_s^{n-1}, u)\widetilde{\mathfrak{N}}(ds, du), \quad t \in J, && (3.2)\\
y^n(t) &= \varphi(t), \quad -\infty < t \leq 0, \quad n \geq 1. && (3.3)
\end{aligned}
$$

Lemma 3.1. *Suppose that* **(H3)** – **(H6)** *hold, and*

$$8mL^2M_a \sum_{t_k < t} p_k + 16mL^2a^2M_a \sum_{t_k < t} q_k < 1,$$

then $y^n(t) \in \mathcal{M}^2((-\infty, a]; \mathfrak{X})$, $\forall\ t \in (-\infty, a]$, $n \geq 0$*,*

$$\mathbb{E}\left\|y^n(t)\right\|^2 \leq \widetilde{M}, \quad n = 1, 2, \cdots, \tag{3.4}$$

here $\widetilde{M} > 0$.

Proof. Let $\mathrm{y}^0(\mathrm{t}) \in \mathcal{M}^2((-\infty, a]; \mathfrak{X})$ and

$$\begin{aligned}
\mathbb{E}\left\|\mathrm{y}^n(\mathrm{t})\right\|^2 &\leq 8\mathbb{E}\left\|\mathrm{S}_\alpha(\mathrm{t})\varphi(0)\right\|^2 + 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{s})\left[\mathrm{y}_1 - \mathfrak{g}(0,\varphi)\right]d\mathrm{s}\right\|^2 + 8\mathbb{E}\left\|\sum_{\mathrm{t_k}<\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{t_k})I_{\mathrm{k}}(\mathrm{y}^{n-1}_{\mathrm{t_k}})\right\|^2 \\
&+ 8\mathbb{E}\left\|\sum_{\mathrm{t_k}<\mathrm{t}}\int_{\mathrm{t_k}}^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\left[J_{\mathrm{k}}(\mathrm{y}^{n-1}_{\mathrm{t_k}}) - \mathfrak{g}(\mathrm{t_k}, \mathrm{y}^{n-1}_{\mathrm{t_k}} + I_{\mathrm{k}}(\mathfrak{y}^{n-1}_{\mathrm{t_k}})) + \mathfrak{g}(\mathrm{t_k}, \mathrm{y}^{n-1}_{\mathrm{t_k}})\right]d\mathrm{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{g}(\mathrm{s}, \mathrm{y}^{n-1}_{\mathrm{s}})d\mathrm{s}\right\|^2 + 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{f}(\mathrm{s}, \mathrm{y}^{n-1}_{\mathrm{s}})d\mathrm{s}\right\|^2 \\
&+ 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\sigma(\mathrm{s}, \mathrm{y}^{n-1}_{\mathrm{s}})dw(\mathrm{s})\right\|^2 + 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{h}(\mathrm{s}, \mathrm{y}^{n-1}_{\mathrm{s}}, \mathrm{u})\widetilde{\mathfrak{N}}(d\mathrm{s}, d\mathrm{u})\right\|^2 \\
&= 8\sum_{i=1}^{8}\mathcal{G}_i.
\end{aligned}$$

It's easy to get the estimations

$$\mathcal{G}_1 \leq 8\mathrm{L}^2\mathbb{E}\left\|\varphi(0)\right\|^2.$$

Next,

$$\mathcal{G}_2 \leq 24\mathrm{L}^2a^2\left(\left\|\mathrm{y}_1\right\|^2 + \mathrm{k}(\|\varphi\|^2_{\mathcal{B}}) + \mathrm{k}_0\right),$$

and

$$\begin{aligned}
\mathcal{G}_3 &\leq 8m\mathrm{L}^2\sum_{\mathrm{t_k}<\mathrm{t}}\mathbb{E}\left\|I_{\mathrm{k}}(\mathrm{y}^{n-1}_{\mathrm{t_k}})\right\|^2_{\mathcal{B}} \\
&\leq 8m\mathrm{L}^2\sum_{\mathrm{t_s}<\mathrm{t}} p_{\mathrm{k}}\mathbb{E}\left\|\mathrm{y}^{n-1}_{\mathrm{t_k}}\right\|^2_{\mathcal{B}}.
\end{aligned}$$

By (**H3**) − (**H6**) and $\|\mathrm{S}_\alpha(\mathrm{t})\|_{\mathcal{L}(\mathfrak{X},\mathfrak{X})} \leq \mathrm{L}$, we have

$$\begin{aligned}
\mathcal{G}_4 &\leq 16\mathbb{E}\left\|\sum_{\mathrm{t_k}<\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})J_{\mathrm{k}}(\mathrm{y}^n_{\mathrm{t_k}})d\mathrm{s}\right\|^2 \\
&+ 16\mathbb{E}\left\|\sum_{\mathrm{t_k}<\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\left[\mathfrak{g}(\mathrm{t_k}, \mathrm{y}^{n-1}_{\mathrm{t_k}} + I_{\mathrm{k}}(\mathrm{y}^{n-1}_{\mathrm{t_k}})) - \mathfrak{g}(\mathrm{t_k}, \mathrm{y}^{n-1}_{\mathrm{t_k}})\right]ds\right\|^2 \\
&\leq 16m\mathrm{L}^2a^2\sum_{\mathrm{t_k}<\mathrm{t}} q_{\mathrm{k}}\mathbb{E}\left\|\mathrm{y}^{n-1}_{\mathrm{t_k}}\right\|^2_{\mathcal{B}} + 16m\mathrm{L}^2a\sum_{\mathrm{t_k}<\mathrm{t}}\int_{\mathrm{t_k}}^{\mathrm{t}} \mathrm{k}(\mathbb{E}p_{\mathrm{k}}\left\|\mathrm{y}^{n-1}_{\mathrm{t_k}}\right\|^2_{\mathcal{B}})d\mathrm{s}.
\end{aligned}$$

and

$$\mathcal{G}_5 \leq 8\mathbb{E}\left\|\int_0^{\mathrm{t}} \mathrm{S}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{g}(\mathrm{s}, \mathrm{y}^{n-1}_{\mathrm{s}})d\mathrm{s}\right\|^2$$

$$
\begin{aligned}
&\leq 16\mathrm{L}^2 a\mathbb{E}\int_0^{\mathrm{t}}\left[\left\|\mathfrak{g}(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1})-\mathrm{g}(\mathrm{s},0)\right\|^2+\|\mathfrak{g}(\mathrm{s},0)\|^2\right]ds\\
&\leq 16\mathrm{L}^2 a\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}+16\mathrm{L}^2a^2\mathrm{k}_0.
\end{aligned}
$$

By (**H3**) – (**H6**), we have

$$
\begin{aligned}
\mathcal{G}_6 &\leq 8\mathbb{E}\left\|\int_0^{\mathrm{t}}\mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{f}(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1})d\mathrm{s}\right\|^2\\
&\leq 8\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}\mathbb{E}\int_0^{\mathrm{t}}\left\|\mathfrak{f}(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1})-\mathfrak{f}(\mathrm{s},0)+\mathfrak{f}(\mathrm{s},0)\right\|^2ds\\
&\leq 16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}+16\mathrm{L}_a^2\frac{a^{2\alpha}}{2\alpha-1}\mathrm{k}_0.
\end{aligned}
$$

Next, by (**H3**) – (**H6**), Holder inequality and B-D-G inequality, we obtain

$$
\begin{aligned}
\mathcal{G}_7 &\leq 8\mathbb{E}\left\|\int_0^{\mathrm{t}}\mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\sigma(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1})dw(\mathrm{s})\right\|^2\\
&\leq 8\mathrm{L}_a^2a^{2\alpha-2}\mathbb{E}\int_0^{\mathrm{t}}\left\|\sigma(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1})-\sigma(\mathrm{s},0)+\sigma(\mathrm{s},0)\right\|_{\mathcal{L}_2^0}^2d\mathrm{s}\\
&\leq 16\mathrm{L}_a^2a^{2\alpha-2}\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}+16\mathrm{L}_a^2a^{2\alpha-1}\mathrm{k}_0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{G}_8 &\leq 8\mathbb{E}\left\|\int_0^{\mathrm{t}}\int_{\mathcal{U}}\mathrm{T}_\alpha(\mathrm{t}-\mathrm{s})\mathfrak{h}(\mathrm{s},\mathrm{y}_{\mathrm{s}}^{n-1},\mathrm{u})\widetilde{\mathfrak{N}}(d\mathrm{s},d\mathrm{u})\right\|^2\\
&\leq 8\mathrm{L}_a^2a^{2\alpha-2}\left[\mathbb{E}\int_0^{\mathrm{t}}\int_{\mathcal{U}}\left\|\mathfrak{h}(\mathrm{s},\mathrm{y}^{n-1},\mathrm{u})\right\|^2\mathrm{v}(d\mathrm{u})d\mathrm{s}+\mathbb{E}\left(\int_0^{\mathrm{t}}\int_{\mathcal{U}}\left\|\mathfrak{h}(\mathrm{s},\mathrm{y}^{n-1},\mathrm{u})\right\|^4\mathrm{v}(d\mathrm{u})d\mathrm{s}\right)^{1/2}\right]\\
&\leq 8\mathrm{L}_a^2a^{2\alpha-2}\Big[\mathbb{E}\int_0^{\mathrm{t}}\int_{\mathcal{U}}\left\|\mathfrak{h}(\mathrm{s},\mathrm{y}^{n-1},\mathrm{u})-\mathfrak{h}(\mathrm{s},0,\mathrm{u})+\mathfrak{h}(\mathrm{s},0,\mathrm{u})\right\|^2\mathrm{v}(d\mathrm{u})d\mathrm{s}\\
&+ \mathbb{E}\left(\int_0^{\mathrm{t}}\int_{\mathcal{U}}\left\|\mathfrak{h}(\mathrm{s},\mathrm{y}^{n-1},\mathrm{u})\right\|^4\mathrm{v}(d\mathrm{u})d\mathrm{s}\right)^{1/2}\Big]\\
&\leq 16\mathrm{L}_a^2a^{2\alpha-2}\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}+16\mathrm{L}_a^2a^{2\alpha-1}\mathrm{k}_0+8\mathrm{L}_a^2a^{2\alpha-2}\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}\\
&\leq 24\mathrm{L}_a^2a^{2\alpha-2}\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^2)d\mathrm{s}+16\mathrm{L}_a^2a^{2\alpha-1}\mathrm{k}_0.
\end{aligned}
$$

Let

$$
Q_1 = 8\mathrm{L}^2\mathbb{E}\|\varphi(0)\|^2+24\mathrm{L}^2a^2\left(\left\|\mathrm{y}_1\right\|^2+\mathrm{k}(\|\varphi\|_{\mathcal{B}}^2)+\mathrm{k}_0\right)+16\mathrm{L}^2a^2\mathrm{k}_0+16\mathrm{L}_a^2\frac{a^{2\alpha}}{2\alpha-1}\mathrm{k}_0+16\mathrm{L}_a^2a^{2\alpha-1}\mathrm{k}_0.
$$

From the above estimations, together yields

$$
\mathbb{E}\left\|\mathrm{y}^n(\mathrm{t})\right\|^2 \leq Q_1+8m\mathrm{L}^2\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}\mathbb{E}\left\|\mathrm{y}_{\mathrm{t_k}}^{n-1}\right\|_{\mathcal{B}}^2+16m\mathrm{L}^2a^2\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\mathbb{E}\left\|\mathrm{y}_{\mathrm{t_k}}^{n-1}\right\|_{\mathcal{B}}^2
$$

$$
\begin{aligned}
&+ \quad 16m\mathrm{L}^2a\sum_{\mathrm{t_k}<\mathrm{t}}\int_{\mathrm{t_k}}^{\mathrm{t}}\mathrm{k}(\mathbb{E}p_{\mathrm{k}}\left\|\mathrm{y}_{\mathrm{t_k}}^{n-1}\right\|_{\mathcal{B}}^{2})ds \\
&+ \quad \left(16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2}\right)\int_0^{\mathrm{t}}\mathrm{k}(\mathbb{E}\left\|\mathrm{y}_{\mathrm{s}}^{n-1}\right\|_{\mathcal{B}}^{2})ds.
\end{aligned}
$$

By using Lemma 2.1 and $\mathrm{k}(\cdot)$, we have to show that a pair of +ve constants β and λ s.t $\mathrm{k}(u)\le\beta+\lambda u$, $\forall\ u\ge 0$. Then

$$
\begin{aligned}
\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2 \quad &\le \quad Q_1+8m\mathrm{L}^2\mathrm{N}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}\mathbb{E}\left\|\varphi\right\|_{\mathcal{B}}+16m\mathrm{L}^2a^2\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\mathbb{E}\left\|\varphi\right\|_{\mathcal{B}}+16m^2\mathrm{L}^2a^2a \\
&+ \quad \left(16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2}\right)b\beta \\
&+ \quad \left(8m\mathrm{L}^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}+16m\mathrm{L}^2a^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\right)\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\|y^n(\mathrm{s})\|^2+\Bigg(16m\mathrm{L}^2a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}} \\
&+ \quad 16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2}\Bigg)\lambda\mathbb{E}\int_0^{\mathrm{t}}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\max_{1\le n\le\tilde{\mathrm{k}}}\left\{\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2\right\} \quad &\le \quad Q_1+8m\mathrm{L}^2\mathrm{N}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}\mathbf{E}\left\|\varphi\right\|_{\mathcal{B}}+16m\mathrm{L}^2a^2\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\mathbb{E}\left\|\varphi\right\|_{\mathcal{B}}+16m^2\mathrm{L}^2a^2\beta \\
&+ \quad \left(16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2}\right)b\beta \\
&+ \quad \left(8m\mathrm{L}^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}+16m\mathrm{L}^2a^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\right)\max_{1\le n\le\tilde{\mathrm{k}}}\left\{\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2\right\} \\
&+ \quad \Bigg(16m\mathrm{L}^2a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}+16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1} \\
&+ \quad 24\mathrm{L}_a^2a^{2\alpha-2}\Bigg)\lambda\int_0^{\mathrm{t}}\max_{1\le n\le\tilde{\mathrm{k}}}\left\{\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2\right\}ds,
\end{aligned}
$$

where $\tilde{\mathrm{k}}>0$. Let

$$
\begin{aligned}
Q_2 \quad &= \quad \frac{Q_1+8m\mathrm{L}^2\mathrm{N}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}\mathbb{E}\left\|\varphi\right\|_{\mathcal{B}}+16m\mathrm{L}^2a^2\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}\mathbb{E}\left\|\varphi\right\|_{\mathcal{B}}}{1-8m\mathrm{L}^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}-16m\mathrm{L}^2a^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}} \\
&+ \quad \frac{16m^2\mathrm{L}^2a^2\beta+(16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2})}{1-8m\mathrm{L}^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}-16m\mathrm{L}^2a^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}} \\
Q_3 \quad &= \quad \frac{(16m\mathrm{L}^2a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}+16\mathrm{L}^2a+16\mathrm{L}_a^2a^{2\alpha-2}+16\mathrm{L}_a^2\frac{a^{2\alpha-1}}{2\alpha-1}+24\mathrm{L}_a^2a^{2\alpha-2})\lambda}{1-8m\mathrm{L}^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}p_{\mathrm{k}}-16m\mathrm{L}^2a^2\mathrm{M}_a\sum_{\mathrm{t_k}<\mathrm{t}}q_{\mathrm{k}}}.
\end{aligned}
$$

Then

$$
\max_{1\le n\le\tilde{\mathrm{k}}}\left\{\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2\right\} \quad \le \quad Q_2+Q_3\int_0^{t}\max_{1\le n\le\tilde{\mathrm{k}}}\left\{\mathbb{E}\sup_{0\le\mathrm{s}\le\mathrm{t}}\left\|\mathrm{y}^n(\mathrm{s})\right\|^2\right\}ds.
$$

Using Gronwall inequality,

$$\max_{1\le n\le \tilde{k}}\left\{\mathbb{E}\sup_{0\le s\le t}\left\|y^n(s)\right\|^2\right\} \;\le\; Q_2 e^{Q_3}.$$

Due to the arbitrary of $\tilde{k}$, we have

$$\mathbb{E}\sup_{0\le s\le t}\left\|y^n(s)\right\|^2 \;\le\; Q_2 e^{Q_3} = M', \;\; 0\le t\le a, \;\; n\ge 1.$$

Consequently,

$$\left\|y^n(s)\right\|^2 \;\le\; \mathbb{E}\|\varphi\|^2_{\mathcal{B}} + \mathbb{E}\left(\sup_{0\le s\le t}\left\|y^n(s)\right\|^2\right) \le \mathbb{E}\|\varphi\|^2_{\mathcal{B}} + M' < \infty,$$

let $\widetilde{M} = \mathbb{E}\|\varphi\|^2_{\mathcal{B}} + M'$. Hence the proof. □

Theorem 3.2. *If* (**H3**) − (**H6**) *and*

$$\max\left\{8mL^2M_a\sum_{t_k<t}p_k + 16mL^2a^2M_a\sum_{t_k<t}q_k, 8mL^2\sum_{t_k<t}p_k + 16mL^2a\sum_{t_k<t}q_k\right\} < 1 \tag{3.5}$$

hold, then the system (1.1) has a unique mild solution of $(-\infty, a]$.

Proof. Let

$$\begin{aligned}
\mathbb{E}\left\|y^{n+m}(t)-y^n(t)\right\|^2 \;=\;& 8\mathbb{E}\left\|\sum_{t_k<t}S_\alpha(t-t_k)\left[I_k(y^{m+n-1}_{t_k}) - I_k(y^{n-1}_{t_k})\right]\right\|^2\\
&+\; 8\mathbb{E}\left\|\sum_{t_k<t}\int_{t_k}^{t}S_\alpha(t-s)\left[J_k(y^{m+n-1}_{t_k}) - J_k(y^{n-1}_{t_k})\right]\right\|^2\\
&+\; 8\mathbb{E}\left\|\sum_{t_k<t}\int_{t_k}^{t}S_\alpha(t-s)\left[\mathfrak{g}(t_k, y^{m+n-1}_{t_k} + I_k(y^{m+n-1}_{t_k})) - \mathfrak{g}(t_k, y^{n-1}_{t_k} + I_k(y^{n-1}_{t_k}))\right]ds\right\|^2\\
&+\; 8\mathbb{E}\left\|\sum_{t_k<t}\int_{t_k}^{t}S_\alpha(t-s)\left[\mathfrak{g}(t_k, y^{m+n-1}_{t_k}) - \mathfrak{g}(t_k, y^{n-1}_{t_k})\right]ds\right\|^2\\
&+\; 8\mathbb{E}\left\|\int_0^{t}S_\alpha(t-s)\left[\mathfrak{g}(t_k, y^{m+n-1}_{t_k}) - \mathfrak{g}(t_k, y^{n-1}_{t_k})\right]ds\right\|^2\\
&+\; 8\mathbb{E}\left\|\int_0^{t}T_\alpha(t-s)\left[\mathfrak{f}(t_k, y^{m+n-1}_{t_k}) - \mathfrak{f}(t_k, y^{n-1}_{t_k})\right]ds\right\|^2\\
&+\; 8\mathbb{E}\left\|\int_0^{t}T_\alpha(t-s)\left[\sigma(t_k, y^{m+n-1}_{t_k}) - \sigma(t_k, y^{n-1}_{t_k})\right]dw(s)\right\|^2\\
&+\; 8\mathbb{E}\left\|\int_0^{t}\int_{\mathcal{U}}T_\alpha(t-s)\left[\mathfrak{h}(t_k, y^{m+n-1}_{t_k}, u) - \mathfrak{h}(t_k, y^{n-1}_{t_k}, u)\widetilde{\mathfrak{N}}(ds, du)\right]ds\right\|^2
\end{aligned}$$

By the fact (**H3**) − (**H6**), we have

$$
\begin{aligned}
\mathbb{E}\left\|y^{n+m}(t)-y^{n}(t)\right\|^{2} \leq\ & 8mL^{2}\mathbb{E}\sum_{t_k<t} p_k \sup_{0\leq s\leq t}\left\|y^{m+n-1}(s)-y^{n-1}(s)\right\|^{2} \\
&+ 8mL^{2}a\mathbb{E}\sum_{t_k<t} q_k \sup_{0\leq s\leq t}\left\|y^{m+n-1}(s)-y^{n-1}(s)\right\|^{2} \\
&+ 8mL^{2}\sum_{t_k<t}\int_{t_k}^{t} k(p_k\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(s)-y^{n-1}(s)\right\|^{2})ds \\
&+ 8mL^{2}\sum_{t_k<t}\int_{t_k}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(s)-y^{n-1}(s)\right\|^{2})ds \\
&+ 8L^{2}a\int_{t_k}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
&+ 8mL^{2}\frac{a^{2\alpha-1}}{2\alpha-1}\int_{0}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
&+ 8mL^{2}a^{2\alpha-2}\int_{0}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
&+ 32mL^{2}a^{2\alpha-2}\int_{0}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
=\ & (8mL^{2}\sum_{t_k<t} p_k + 8mL^{2}a\sum_{t_k<t} q_k)\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
&+ 8mL^{2}\sum_{t_k<t}\int_{t_k}^{t} k((1+p_k)\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds) \\
&+ (8L^{2}a + 8mL^{2}\frac{a^{2\alpha-1}}{2\alpha-1} + 8mL^{2}a^{2\alpha-2} + 32mL^{2}a^{2\alpha-2}) \\
&\qquad\qquad \times\int_{0}^{t} k(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds
\end{aligned}
$$

Let $\bar{p} = \max_{1\leq k\leq m} p_k$. Since

$$
\begin{aligned}
\int_{t_k}^{t} k((1+p_k)\mathbb{E}\sup_{0\leq r\leq t_k}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds) &\leq \int_{t_k}^{t} k((1+p_k)\mathbb{E}\sup_{0\leq r\leq s}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds) \\
&\leq \int_{t_k}^{t} k((1+\bar{p})\mathbb{E}\sup_{0\leq r\leq s}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds)
\end{aligned}
$$

and $\tilde{k}\circ c(\cdot) = k(c(\cdot))$ is a concave function, we get

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{n+m}(s)-y^{n}(s)\right\|^{2} &\leq (8mL^{2}\sum_{t_k<t} p_k + 8mL^{2}a\sum_{t_k<t} q_k)\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds \\
&+ (16L^{2}a + 8mL^{2}\frac{a^{2\alpha-1}}{2\alpha-1} + 8mL^{2}a^{2\alpha-2} \\
&+ 32mL^{2}a^{2\alpha-2})\int_{0}^{t}\tilde{k}(\mathbb{E}\sup_{0\leq s\leq t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^{2})ds. \qquad (3.6)
\end{aligned}
$$

Hence,

$$\mathbb{E}\sup_{0\le s\le t}\left\|y^{n+m}(s)-y^{n}(s)\right\|^2-(8mL^2\sum_{t_k<t}p_k+8mL^2a\sum_{t_k<t}q_k)\mathbb{E}\sup_{0\le s\le t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^2)ds$$
$$\le(16L^2a+8mL^2\frac{a^{2\alpha-1}}{2\alpha-1}+8mL^2a^{2\alpha-2}+32mL^2a^{2\alpha-2})\int_0^t\tilde{k}(\mathbb{E}\sup_{0\le s\le t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^2)ds$$

Using Lemma 3.1, we get

$$\mathbb{E}\sup_{0\le s\le t}\left\|y^{n+m}(s)-y^{n}(s)\right\|^2-(8mL^2\sum_{t_k<t}p_k+8mL^2a\sum_{t_k<t}q_k)\mathbb{E}\sup_{0\le s\le t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^2)ds$$
$$\le(16L^2a+8mL^2\frac{a^{2\alpha-1}}{2\alpha-1}+8mL^2a^{2\alpha-2}+32mL^2a^{2\alpha-2})\int_0^t\tilde{k}(2M')ds$$
$$\le Q_4\tilde{k}(2M')t=Q_5t.$$

Define

$$\varphi_1(t)=Q_5t,\quad \varphi_{n+1}(t)=Q_4\int_0^t\tilde{k}(\varphi_n(s))ds,\quad n\ge1.$$

Choose $a_1\in[0,a)$ such that $Q_4\tilde{k}(Q_5t)\le Q_5$, $\forall\ 0\le t\le a_1$.

For any $t\in[0,a_1)$, $\{\varphi_n(t)\}$ is a decreasing sequence. In fact

$$\varphi_2(t)=Q_4\int_0^t\tilde{k}(\varphi_1(s))ds=Q_4\int_0^t\tilde{k}(Q_5(s))ds\le\int_0^t Q_5ds=\varphi_1(t).$$

By induction, we get

$$\varphi_{n+1}(t)=Q_4\int_0^t\tilde{k}(\varphi_n(s))ds\le Q_4\int_0^t\tilde{k}(Q_{n-1}(s))ds=\varphi_n(t),\ \forall 0\le t\le a_1. \tag{3.7}$$

Therefore, the statement is true and we can define the function $\phi(t)$ as

$$\phi(t)=\lim_{n\to\infty}\varphi_n(t)=\lim_{n\to\infty}Q_4\int_0^t\tilde{k}(\varphi_{n-1}(s))ds=\lim_{n\to\infty}Q_4\int_0^t\tilde{k}(\phi(s))ds,\ 0\le t\le a_1.$$

By the Bihari's inequality, we have $\phi(t)=0\ \forall\ 0\le t\le a_1$. It means that for all $0\le t\le a_1$,

$$\lim_{n\to\infty}\Big[\mathbb{E}\sup_{0\le s\le t}\left\|y^{n+m}(s)-y^{n}(s)\right\|^2-(8mL^2\sum_{t_k<t}p_k$$
$$+8mL^2a\sum_{t_k<t}q_k)\mathbb{E}\sup_{0\le s\le t}\left\|y^{m+n-1}(r)-y^{n-1}(r)\right\|^2\Big]=0. \tag{3.8}$$

By using the condition $8mL^2\sum_{t_k<t}p_k+8mL^2a\sum_{t_k<t}q_k<1$ and (3.8), we obtain

$$\lim_{n\to\infty}\mathbb{E}\left\|y^{n+m}(s)-y^{n}(s)\right\|^2=0,\quad 0\le t\le a_1, \tag{3.9}$$

this implies that $\{y^n(t)\}$ is a Cauchy sequence in $\mathcal{L}_2(\Omega, \mathfrak{X})$. Let $\lim_{n\to\infty} y^n(t) = y(t)$, obviously,

$$\left\|y(t)\right\|^2 \leq \widetilde{M}, \ 0 \leq t \leq a_1. \tag{3.10}$$

Taking limits on both side of equation (3.2), $\forall t \in [0, a_1]$, we have

$$y(t) = \begin{cases} S_\alpha(t)\varphi(0) + \int_0^t S_\alpha\left[y_1 - g(0,\varphi)\right] ds + \int_0^t S_\alpha(t-s)g(s, y_s)ds \\ + \sum_{t_k<t} S_\alpha(t-t_k)I_k(y_{t_k}) + \sum_{t_k<t} \int_{t_k}^t S_\alpha(t-s)\left[J_k(y_{t_k}) - g(t_k, y_{t_k} + I_k(y_{t_k}))\right] ds \\ + \int_0^t T_\alpha(t-s)\mathfrak{f}(s, y_s)ds + \int_0^t T_\alpha(t-s)\sigma(s, y_s)dw(s) \\ + \int_0^t \int_{\mathcal{U}} T_\alpha(t-s)\mathfrak{h}(s, y_s, u)\widetilde{\mathfrak{N}}(ds, du), \end{cases} \tag{3.11}$$

so we have presented the existence of the mild solution of (1.1) on $[0, a_1]$. By iteration we can get the existence of the mild solution of (1.1) on $[0, a]$.

Suppose that y_1, y_2 are two solutions of (1.1). Using the similar discussion as (3.8), we get

$$\begin{aligned} &\left[1 - (8mL^2 \sum_{t_k<t} p_k + 8mL^2 a \sum_{t_k<t} q_k)\right] \mathbb{E} \sup_{0\leq s\leq t} \left\|y_1 - y_2\right\|^2)ds \\ &\leq (16L^2 a + 8mL^2 \frac{a^{2\alpha-1}}{2\alpha-1} + 8mL^2 a^{2\alpha-2} + 32mL^2 a^{2\alpha-2}) \int_0^t \tilde{k}(\mathbb{E} \sup_{0\leq s\leq t} \left\|y_1 - y_2\right\|^2)ds \end{aligned} \tag{3.12}$$

the Bihari's inequality implies $\mathbb{E}\left\|y_1 - y_2\right\|^2 = 0$, and we have shown the existence and the uniqueness of the mild solution of (1.1). □

4. Stability results

In this section, we have given the continuous dependence of solutions on the initial values by means of the Bihari's inequality. We first propose the following assumption on g instead of (**H4**),

(**H7**) $g : [0, a] \times \mathscr{B} \to \mathfrak{X}$ satisfy

$$\|g(t,\varphi) - g(t,\phi)\|^2 \leq k_1 \|\varphi - \phi\|^2_{\mathcal{B}}, \ \ k_1 > 0.$$

Definition 4.1. *A mild solution $y^{\varphi,y_1}(t)$ of Cauchy problem (1.1) with initial value (φ, y_1) is known as stable in square if $\forall\ \epsilon > 0,\ \ni \delta > 0$ s.t*

$$\mathbb{E} \sup_{0\leq s\leq a} \left\|y^{\varphi,y_1}(s) - z^{\varphi,z_1}(s)\right\| \leq \epsilon, \quad \mathbb{E}\|\varphi - \phi\|^2_{\mathcal{B}} + \mathbb{E}\left\|y_1 - z_1\right\|^2 < \delta, \tag{4.1}$$

where $z^{\varphi,z_1}(t)$ is further solution of (1.1) with initial condition (φ, z_1).

Theorem 4.2. *Assume $3(8mL^2 \sum_{t_k<t} p_k + 8mL^2 a \sum_{t_k<t} q_s) < 1$, by Theorem 3.2 and* (**H7**)*, are satisfied, then the mild solution of (1.1) is stable.*

Proof. Using similar argument of Theorem 3.2, we get $\forall 0 \le t \le a$,

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}\left\|y^{\varphi,y_1}(s)-z^{\varphi,z_1}(s)\right\|^2 &\le 3\mathbb{E}\left\|S_\alpha(t)\varphi(0)-S_\alpha(t)\phi(0)\right\|^2+3\mathbb{E}\left\|\int_0^t S_\alpha(s)(\left\|y_1-z_1\right\|_{\mathfrak{X}}+\|\varphi-\phi\|_{\mathcal{B}})ds\right\|^2\\ &+ 3(8mL^2\sum_{t_k<t}p_k+8mL^2a\sum_{t_k<t}q_k)\mathbb{E}\sup_{0\le s\le t}\left\|y^{\varphi,y_1}(s)-z^{\varphi,z_1}(s)\right\|^2\\ &+ (16L^2a+16m^2L^2+16mL^2+8mL\frac{a^{2\alpha-1}}{2\alpha-1}+8mL^2a^{2\alpha-2}\\ &+ 32mL^2a^{2\alpha-2})\int_0^t\tilde{k}(\mathbb{E}\sup_{0\le r\le s}\left\|y^{\varphi,y_1}(s)-z^{\varphi,y_1}(s)\right\|^2)ds\end{aligned}$$

Then we get

$$\mathbb{E}\sup_{0\le s\le t}\left\|y^{\varphi,y_1}(s)-z^{\varphi,y_1}(s)\right\|^2 \le \frac{\nu}{\mathcal{G}}(\left\|y_1-z_1\right\|^2+\|\varphi-\phi\|_{\mathcal{B}})+\frac{\tilde{\nu}}{\mathcal{G}}\int_0^t\tilde{k}(\mathbb{E}\sup_{0\le r\le s}\left\|y^{\varphi,y_1}(s)-z^{\varphi,y_1}(s)\right\|^2)ds$$

where $\nu = \max\left\{6L^2, 6L^2a^2k_1+3L^2\right\}$, $\tilde{\nu} = 3(16L^2a+16m^2L^2+16mL^2+8mL\frac{a^{2\alpha-1}}{2\alpha-1}+8mL^2a^{2\alpha-2}+32mL^2a^{2\alpha-2})$, and $\mathcal{G} = 1-3(8mL^2\sum_{t_k<t}p_k+8mL^2a\sum_{t_k<t}q_k)$. The function $\tilde{k}(u)$ is defined in (3.6) which has the Lemma 2.4 in [19]. For $\epsilon>0$, letting $\epsilon_1=\frac{1}{2}\epsilon$, we have $\lim_{s\to 0}\int_s^{\epsilon_1}\frac{1}{\tilde{k}(u)}du=\infty$. $\exists\,\delta$ and $\delta<\epsilon_1$ such that $\int_s^{\epsilon_1}\frac{1}{\tilde{k}(u)}du\ge\mathfrak{T}$. Let $u_0=\frac{\nu}{\Lambda}(\left\|y_1-y_1\right\|^2+\|\varphi-\phi\|_{\mathcal{B}})$, $u(t)=\mathbb{E}\sup_{0\le s\le t}\left\|y^{\varphi,y_1}(s)-z^{\varphi,y_1}(s)\right\|$, $v(t)=1$. If $u_0\le\delta\le\epsilon_1$, the Lemma 2.5 in [19], shows that $\int_{u_0}^{\epsilon_1}\frac{1}{\tilde{k}(u)}du\ge\int_\delta^{\epsilon_1}\frac{1}{\tilde{k}(u)}du\ge\mathfrak{T}=\int_0^a v(s)ds$. So for any $t\in[0,a]$, the estimate $u(t)\le\epsilon_1\le\epsilon$ holds. Hence the proof. □

5. Conclusions

In this manuscript, we investigate a class of second-order impulsive fractional neutral stochastic differential equations (IFNSDEs) driven by Poisson jumps in Banach space. Firstly, sufficient conditions of the existence and the uniqueness of the mild solution for this type of equations are drived by means of the successive approximation and the Bihari's inequality. Next we get the stability in mean square of the mild solution via continuous dependence on initial value are established.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. Anguraj, K. Ramkumar, E. M. Elsayed, Existence, uniqueness and stability of impulsive stochastic partial neutral functional differential equations with infinite delays driven by a fractional Brownian motion, *Discontinuity, Nonlinearity, and Complexity*, **9** (2020), 327–337.

2. A. Anguraj, K. Ravikumar, Existence and stability results of impulsive stochastic partial neutral functional differential equations with infinite delays and Poisson jumps, *Journal of Applied Nonlinear Dynamics*, **9** (2020), 245–255.

3. A. Anguraj, K. Ravikumar, Existence and stability results for impulsive stochastic functional integrodifferential equations with Poisson jumps, *Journal of Applied Nonlinear Dynamics*, **8** (2019), 407–417.

4. P. Balasubramaniam, S. Saravanakumar, K. Ratnavelu, Study a class of Hilfer fractional stochastic integrodifferential equations with Poisson jumps, *Stoch. Anal. Appl.,*, **36** (2018), 1021–1036.

5. A. Chadha, D. N. Pandey, Existence results for an impulsive neutral stochastic fractional integrodifferential equation with infinite delay, *Nonlinear Analysis*, **128** (2015), 149–175.

6. K. Dhanalakshmi, P. Balasubramaniam, Stability result of higher-order fractional neutral stochastic differential system with infinite delay driven by Poisson jumps and Rosenblatt process, *Stoch. Anal. Appl.*, **38** (2019), 352–372.

7. E. Hausenblas, SPDEs driven by Poisson random measure with non Lipschitz coefficients:existence results, *Probab. Theory Rel.*, **137** (2006), 161–200.

8. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier Science, Vol 204, 2006.

9. J. Luo, T. Taniguchi, The existence and uniqueness for non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps, *Stoch. Dyn.*, **9** (2009), 135–152.

10. X. Mao, *Stochastic Differential Equations and Applications*, Chichester: Horwood, 1997.

11. K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, New York: John Wiley, 1993.

12. M. A. Ouahra, B. Boufoussi, E. Lakhel, Existence and stability for stochastic impulsive neutral partial differential equations driven by Rosenblatt process with delay and Poisson jumps, *Commun. Stoch. Anal.*, **11** (2017), 99–117.

13. I. Podlubny, *Fractional Differential Equations*, London: Academic Press, 1999.

14. G. D. Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, London: Cambridge University Press, 2014.

15. Y. Ren, R. Sakthivel, Existence, uniqueness and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps, *J. Math. Phys.*, **53** (2012), 073517.

16. F. A. Rihan, C. Rajivganthi, P. Muthukumar, Fractional stochastic differential equations with Hilfer fractional derivative:Poisson jumps and optimal control, *Disc. Dyn. Nat. Soc.*, **2017** (2017).

17. P. Tamilalagan, P. Balasubramaniam, Existence results for semilinear fractional stochastic evolution inclusions driven by Poisson jumps, In *Mathematical Analysis and Its Applications*, Springer, (2014), 477–487.

18. S. Xie, Existence results of mild solutions for impulsive fractional integro-differential evolution equations with infinite delay, *Fract. Calc. Appl. Anal.*, **17** (2014), 1158–1174.

19. S. Zhao, M. Song, Stochastic impulsive fractional differential evolution equations with infinite delay, *Filomat*, **31** (2017), 4261–4274.