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Exponential Stability of Impulsive Neutral Stochastic Integrodifferential Equations Driven by a Poisson Jumps and Time-Varying Delays

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Abstract. The purpose of this work is to study the impulsive neutral stochastic integrodifferential equations driven by a Poisson jumps and time-varying delays. We use the theory of resolvent operators developed in Grimmer the prove an existence, uniqueness and we establish some conditions ensuring the exponential decay to zero in mean square for the mild solution by means of the Banach fixed point theory. Finally, an illustrative example is given to demonstrate the effectiveness of the results.

1. Introduction

In this paper, we investigate the existence, uniqueness and asymptotic behaviors of mild solutions to impulsive neutral stochastic integrodifferential equations driven by a Poisson jumps and time-varying delays of the following form:

$$\begin{aligned}
d\left[x(t)+p(t,x(t-r(t)))\right] &= A\left[x(t)+p(t,x(t-r(t)))\right]dt+\left[\int_0^t B(t-s)\left[x(s)+p(s,x(s-r(s)))\right]ds\right]dt\\
&+ f(t,x(t-\delta(t)))dt+g(t,x(t-\rho(t)))dw(t)\\
&+ \int_{\mathcal{U}} h(t,x(t-\sigma(t)),u)\widetilde{N}(dt,du), \quad t\in[0,T],\\
\Delta x(t_k) &= x(t_k^+)-x(t_k^-)=I_k(x(t_k)), \quad k\in\mathbb{N},\\
x_0(\cdot) &= \varphi, \quad t\in[-\tau,0],
\end{aligned} \tag{1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathbb{X}$ with domain $\mathcal{D}(A)$. Here $B(t)$ is a closed linear operator on $\mathbb{X}$ with domain $\mathcal{D}(B)\supset\mathcal{D}(A)$. The mappings $p,f:[0,+\infty[\times\mathbb{X}\to\mathbb{X}$, $g:[0,+\infty[\times\mathbb{X}\to\mathcal{L}_2^0(\Upsilon,\mathbb{X})$ and $h:[0,+\infty[\times\mathbb{X}\times\mathcal{U}\to\mathbb{X}$ are appropriate functions, the initial data $\varphi\in C((-\tau,0],\mathbb{X})$ the space of all continuous functions from $(-\tau,0]$ to $\mathbb{X}$ and has finite second

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moments. The impulsive moments satisfy $0 < t_1 < t_2 <, ..., \lim_{k\to\infty} t_k = \infty$, $r, \delta, \rho, \sigma : [0, +\infty[\to [0, \tau]$ $(\tau > 0)$ are continuous functions.

Stochastic delay differential equations (SDDEs) play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. In the past few decades, qualitative theory of SDDEs have been studied intensively by many scholars. Here, we refer to [10] and references therein. In recent years, existence, uniqueness, stability, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by several authors. Many important results have been reported, for instance, in [2, 7, 8, 11].

Stochastic partial differential and integrodiffenrential equations has attracted more attention because of their practical applications in various areas like physics, chemistry, economics, social sciences, finance, population dynamics, electrical engineering, medicine biology, ecology and other areas of science and engineering. Qualitative properties such as existence, uniqueness, optimality conditions, controllability and stability for various linear and nonlinear stochastic partial differential and integrodifferential equations have been extensively studied by many researchers, see for instance [2, 9, 11] and the references therein. Due to the presence of the integral term in our equation, we need to use the theory of resolvent operators as developed by [12] instead of using strongly continuous semigroups.

Furthermore, several practical systems (such as sudden price variations [jumps] due to market crashes, earthquakes, hurricanes, epidemics, and so on) experiences some jump type stochastic perturbations. The sample paths are not being continuous. Thus it is seize considering stochastic processes with jumps in describing the models. Generally, the jump models are derived from poisson random measure. The sample paths of systems being right continuous possess left limits. In the recent trend, researchers are focusing more on the theory and applications of impulsive stochastic functional differential equations with poisson jumps. Precisely, existence and stability results on impulsive stochastic functional differential equations with poisson jumps are found in [3–6] and the references therein.

Motivated by the facts stated in the above discussion, the existence and exponential stability of mild solutions for the problem (1). Here we apply the Banach fixed point principle to investigate the existence and exponential stability of mild solutions of this class of equations.

2. Preliminaries

Let $\mathbb{X}, \Upsilon$ be real separable Hilbert spaces and $\mathcal{L}(\Upsilon, \mathbb{X})$ be the space of bounded linear operators mapping Υ into $\mathbb{X}$. Let $(\Omega, \Im, \mathbb{P})$ be a complete probability space with an increasing right continuous family $\{\Im_t\}_{t\geq 0}$ of complete sub σ algebra of $\Im$. Let $\{\mathcal{W}(t) : t \geq 0\}$ denote a Υ-valued Wiener process defined on the probability space $(\Omega, \Im, \mathbb{P})$ with covariance operator Q, that is

$$\mathbf{E} < \mathcal{W}(t), x >_\Upsilon < \mathcal{W}(s), y >_\Upsilon \quad = \quad (t \wedge s) < Qx, y >_\Upsilon, \quad \text{for all } \ x, y \in \Upsilon,$$

whare Q is a positive, self-adjoint, trace class operator on Υ. In particular, we denote by $\mathcal{W}(t), t \geq 0$, a Υ-valued Q-Wienerprocess with respect to $\{\Im_t\}_{t\geq 0}$. In order to define stochastic integrals with respect to the Q-Wiener process $\mathcal{W}$, we introduce the subspace $\Upsilon_0 = Q^{\frac{1}{2}}(\Upsilon)$ to Υ which, endowed with the inner product

$$\langle u, v\rangle_{\Upsilon_0} \quad = \quad < Q^{\frac{-1}{2}} u, Q^{\frac{-1}{2}} v >_\Upsilon$$

is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in Υ, a bounded sequence of non-negative real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1, 2, ...$, and a sequence $\{\beta_i\}_{i\geq 1}$ of independent Brownian motions such that

$$\langle \mathcal{W}(t), e\rangle \quad = \quad \sum_{n=1}^{\infty} \sqrt{\lambda_i}\, \langle e_i, e\rangle\, \beta_i(t), \quad e \in \Upsilon,$$

and $\Im_t = \Im_t^w$, where $\Im_t^w$ is the sigma algebra generated by $\{\mathcal{W}(s) : 0 \leq s \leq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\Upsilon_0, \mathbb{X})$ denote the space of all Hilbert-Schmidt operators from Υ_0 into $\mathbb{X}$. It turns out to be a separable Hilbert space equipped with the norm

$$\|\zeta\|^2_{\mathcal{L}_2^0} = \mathrm{tr}\left((\zeta Q^{\frac{1}{2}})(\zeta Q^{\frac{1}{2}})^*\right)$$

for any $\zeta \in \mathcal{L}_2^0$. Clearly, for any bounded operators $\zeta \in \mathcal{L}(\Upsilon, \mathbb{X})$ this norm reduces to $\|\zeta\|^2_{\mathcal{L}_2^0} = \mathrm{tr}\,(\zeta Q \zeta^*)$. In $\widetilde{N}(dt, du) = N(dt, du) - dt(vdu)$ the Poisson measure $\widetilde{N}(dt, du)$ denotes the Poisson counting measure.

2.1. Partial Integrodifferential Equations in Banach spaces

In the present section, we recall some fundamental results needed to establish our results. Regarding the theory of resolvent operators, we refer the reader to [12, 13]. Throughout the paper, $\mathbb{X}$ is a Banach space A and $B(t)$ are closed linear operators on $\mathbb{X}$. Υ represents the Banach space $\mathcal{D}(A)$ equipped with the graph norm defined by

$$\|y\|_\Upsilon := \|Ay\| + \|y\| \quad \text{for } y \in \Upsilon.$$

The notations $C([0, +\infty); \Upsilon)$ stands for the space of all continuous functions from $[0, +\infty)$ into Υ. We consider the following Cauchy problem :

$$\begin{cases} v'(t) = Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds \quad for\ t \geq 0 \\ v(0) = v_0 \in \mathbb{X}. \end{cases} \tag{2}$$

Definition 2.1. *[12] A resolvent operator for equation (2) is a bounded linear operator-valued function $R(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$, having the following properties:*
1. $R(0) = I$ and $\|R(t)\| \leq Ne^{\eta t}$ for some constants $N > 0$ and η.
2. For each $x \in \mathbb{X}$, $R(t)x$ is strongly continuous for $t \geq 0$.
3. $R(t) \in \mathcal{L}(\Upsilon)$ for $t \geq 0$. For $x \in \Upsilon$, $R(.)x \in C^1([0, +\infty]; \mathbb{X}) \bigcap C([0, +\infty); \Upsilon)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad for\ t \geq 0. \end{aligned}$$

In the whole of this work, we assume that
(H1). A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$.
(H2). For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(\Upsilon, |\cdot|_\Upsilon)$ into $(\mathbb{X}, |\cdot|_\mathbb{X})$. Moreover, there exists an integrable function $c : [0, +\infty) \to \mathbb{R}^+$ such that for any $y \in \Upsilon$, $y \mapsto B(t)y$ belongs to $W^{1,1}([0, +\infty), \mathbb{X})$ and

$$\left|\frac{d}{dt}B(t)y\right|_\mathbb{X} \leq c(t)\left|y\right|_\Upsilon, \quad \text{for } y \in \Upsilon \text{ and } t \geq 0.$$

The resolvent operator plays an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (2) has a resolvent operator. For more details on resolvent operators, werefer to [12, 13]. Thefollowing theorem gives a satisfactory answer to this problem, and it will be used in this work to develop our main results.

Theorem 2.2. *Assume that* **(H1)**-**(H2)** *hold. Then there exists a unique resolvent operator of the Cauchy problem (2).*

In the following, we give some results for the existence of solutions for the following integrodifferential equation

$$\begin{aligned} v'(t) &= Av(t) + \int_0^t B(t-s)v(s)ds + q(t), \quad \text{for } t \geq 0 \\ v(0) &= v_0 \in \mathbb{X}, \end{aligned} \tag{3}$$

where $q : [0, +\infty[\to \mathbb{X}$ is a continuous function.

Theorem 2.3. *Assume that hypotheses* **(H1)-(H2)** *hold. Let $T(t)$ be a compact operator for $t > 0$. Then, the corresponding resolvent operator $R(t)$ in equation (3) is continuous for $t > 0$ in the operator norm, namely for all $t_0 > 0$, it holds that*

$$\lim_{h\to 0}\|R(t_0+h)-R(t_0)\| = 0.$$

Definition 2.4. *A continuous function $v : [0,+\infty) \to \mathbb{X}$ is said to be a strict solution of (3) if*

(i) $v \in C^1([0,+\infty);\mathbb{X}) \cap C([0,+\infty);\mathbb{Y})$,

(ii) v *satisfies (3), for* $t \geq 0$.

Theorem 2.5. *Assume that* **(H1)-(H2)** *hold. If v is a strict solution of (3), then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds, \quad for \quad t \geq 0. \tag{4}$$

Accordingly, we make the following definition.

Definition 2.6. *A function $v : [0,+\infty) \to \mathbb{X}$ is called a mild solution of (3) for $v_0 \in \mathbb{X}$, if v satisfies the variation of constants formula (4)*

The next theorem provides sufficient conditions for the regularity of solutions of (3).

Theorem 2.7. *Let $q \in C^1([0,+\infty);\mathbb{X})$ and v be defined by (4). If $v_0 \in \mathcal{D}(A)$, then z is a strict solution of (3).*

3. The Main Result

In this section, we first establish our main result. Our method is based on the contraction mapping principle. Let us recall the definition of mild solution for the stochastic system (1).

Definition 3.1. *A $\mathbb{X}$-valued stochastic process $\{x(t),\ t \in (-\tau,\infty)\}$ is called a mild solution of equation (1) if*

$$x(t) = \varphi(t), \quad (-\tau, 0]$$

and the following conditions hold:
(i). $x(\cdot)$ is continuous on $(0,t_1]$ and each interval $(t_k,t_{k+1}]$, $k \in \mathbb{N}$,
(ii). For each t_k, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$ exists,
(iii). For each $t \geq 0$, we have a.s.

$$\begin{aligned} x(t) &= R(t)\Big[\varphi(0) + p(0,\varphi(-r(0))) - p(t,x(t-r(t)))\Big] + \int_0^t R(t-s)f(s,x(s-\delta(s)))ds \\ &+ \int_0^t R(t-s)g(s,x(s-\rho(s)))dw(s) + \int_0^t\int_{\mathcal{U}} R(t-s)h(s,x(s-\sigma(s)),u)\widetilde{N}(ds,du) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k)). \end{aligned} \tag{5}$$

To prove the required results, we assume the following conditions:

(H3) The resolvent operator $(R(t))_{t\geq 0}$ given by theorem 2.2 satisfies the following estimation

$$\|R(t)\| \leq Me^{-\lambda t}, \ \text{for}\ t \geq 0 \ \text{and}\ \lambda > 0, M \geq 1.$$

(H4) There exist $C_f, C_g, C_h > 0$ such that, for all $t \geq 0$, $x, y \in \mathbb{X}$, the coefficients f, g and h satisfy the following conditions

$$\begin{aligned} \left\|f(t,x)-f(t,y)\right\|^2 &\leq C_f\left\|x-y\right\|^2. \\ \left\|g(t,x)-g(t,y)\right\|^2 &\leq C_g\left\|x-y\right\|^2. \\ \int_{\mathcal{U}}\left\|h(s,x,u)-h(s,y,u)\right\|^2 v(du) &\leq C_h\left\|x-y\right\|^2. \end{aligned}$$

(H5) There exist $C_p > 0$ such that, for all $t \geq 0$, $x, y \in \mathbb{X}$

$$\left\|p(t,x) - p(t,y)\right\|^2 \leq C_p \left\|x - y\right\|^2.$$

(H6) The function p is continuous in the quadratic mean sense

$$\text{For all functions } x \lim_{t \to s} \mathbf{E} \left\|p(t, x(t)) - p(s, x(s))\right\|^2 = 0.$$

(H7) There exist some positive number q_k, $k \in \mathbb{N}$ such that

$$\left\|I_k(x) - I_k(y)\right\|^2 \leq q_k \left\|x - y\right\| \text{ for all } x, y \in \mathbb{X}.$$

We further assume that $p(t, 0) = f(t, 0) = g(t, 0) = h(t, 0, u) = 0$ for all $t \geq 0$ and $u \in \mathcal{U}$, $k \in \mathbb{N}$.

Theorem 3.2. *Suppose that* **(H1)**-**(H7)** *hold. If the following conditions are satisfied.*
(i) There exists a constant $\tilde{q}$ *such that* $q_k \leq \tilde{q}(t_k - t_{k-1})$, $k = 1, 2, ...,$

$(ii) \sum_{k=1}^{\infty} (t_k - t_{k-1}) < \infty, \ k = 1, 2, ...,$

$(iii) 5\left(C_p + M^2 C_f \lambda^{-2} + M^2 C_g (2\lambda)^{-1} + M^2 C_h (2\lambda)^{-1} + \tilde{q}^2 M^2 \left(\sum_{k=1}^{\infty} (t_k - t_{k-1})\right)^2\right) < 1.$

Then the mild solution to (1) exists uniquely and is exponential decay to zero in mean square. i.e., there exists a pair of positive constants $a > 0$ *and* $M^* = M^*(\varphi, a)$ *such that*

$$\mathbf{E}\left\|x(t)\right\|^2 \leq M^* e^{-at}, \quad t \geq 0.$$

Proof. Define by S the subset of the Banach space of all stochastic processes $x(t, w) : \{(-\tau, \infty) \times \Omega \to \mathbb{X}\}$ satisfying $x(t) = \varphi(t)$, $t \in (-\tau, 0]$ and the conditions (i), (ii) in Definition 3.1 and there exist some constants $a > 0$ and $M^* = M^*(\varphi, a) > 0$ such that

$$\mathbf{E}\left\|x(t)\right\|^2 \leq M^* e^{-at}, \quad t \geq 0.$$

It is routine to check that S is Banach space endowed with a norm $\|x\|_S^2 = \sup_{t \geq 0} \mathbf{E}\left\|x(t)\right\|^2$. Define the mapping Θ on S by

$$\begin{aligned}
\Theta(x)(t) &= R(t)\Big[\varphi(0) + p(0, \varphi(-r(0))) - p(t, x(t - r(t)))\Big] + \int_0^t R(t-s) f(s, x(s - \delta(s))) ds \\
&+ \int_0^t R(t-s) g(s, x(s - \rho(s))) dw(s) + \int_0^t \int_{\mathcal{U}} R(t-s) h(s, x(s - \sigma(s)), u) \widetilde{N}(ds, du) \\
&+ \sum_{0 < t_k < t} R(t - t_k) I_k(x(t_k)).
\end{aligned}$$

To get desired results, it is enough to show that the operator Θ has a unique fixed point in S. For this purpose, we use the contraction mapping principle.
Step 1: First we show that $\Theta(S) \subset S$. Let $x \in S$. For convenience of notation, we denote by M_i^*, $i = 1, 2, ...$ the

finite positive constants depending on φ, a. we have

$$\begin{aligned}
\mathbf{E}\,\|\Theta(x)(t)\|_{\mathbb{X}}^2 &\leq 6\mathbf{E}\left\|R(t)\Big[\varphi(0)+p(0,\varphi(-r(0)))\Big]\right\|^2 + 6\mathbf{E}\Big\|p(t,x(t-r(t)))\Big\|^2 + 6\mathbf{E}\left\|\int_0^t R(t-s)f(s,x(s-\delta(s)))ds\right\|^2 \\
&+ 6\mathbf{E}\left\|\int_0^t R(t-s)g(s,x(s-\rho(s)))dw(s)\right\|^2 + 6\mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} R(t-s)h(s,x(s-\sigma(s)),u)\widetilde{N}(ds,du)\right\|^2 \\
&+ 6\mathbf{E}\left\|\sum_{0<t_k<t} R(t-t_k I_k(x(t_k)))\right\|^2 \\
&= \sum_{i=1}^{6}\mathcal{G}_i(t), \quad t\geq 0. \qquad (6)
\end{aligned}$$

Now, let us estimate the terms on the right of the inequality (6).
Without loss of generality we may assume that $0<a<\lambda$. Then, by assumption **(H3)**, we get

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_1(t)\|^2 &\leq M^2\mathbf{E}\left\|\varphi(0)+p(0,\varphi(-r(0)))\right\|^2 e^{-\lambda t} \\
&\leq M_1^* e^{-\lambda t}. \qquad (7)
\end{aligned}$$

To estimate $\mathcal{G}_i(t)$, $i=2,...,6$ we observe that for $x\in S$ and $x_1(t)=r(t),\delta(t),\rho(t)$ and $\sigma(t)$ the following estimate holds

$$\begin{aligned}
\mathbf{E}\,\|x(t-x_1(t)\|^2 &\leq \left(M^*e^{-a(t-x_1(t))}+\mathbf{E}\left\|\varphi(t-x_1(t)\right\|^2\right) \\
&\leq \left(M^*e^{-a(t-x_1(t))}+\mathbf{E}\left\|\varphi(t)\right\|_C^2 e^{-a(t-x_1(t))}\right) \\
&\leq \left(M^*+\mathbf{E}\left\|\varphi(t)\right\|_C^2\right)e^{a\tau}e^{-at},
\end{aligned}$$

where

$$\left\|\varphi\right\|_C = \sup_{-r<s<0}\left\|\varphi\right\| < \infty.$$

Then by assumption **(H5)**, we have

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_2(t)\|^2 &\leq \mathbf{E}\left\|p(t,x(t-r(t)))-p(t,0)\right\|^2 \\
&\leq C_p\mathbf{E}\,\|x(t-r(t))\|^2 \\
&\leq C_p\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}e^{-at}, \\
&\leq M_2^* e^{-at}. \qquad (8)
\end{aligned}$$

Similarly, By assumption **(H4)**, we have

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_3(t)\|^2 &= \mathbf{E}\left\|\int_0^t R(t-s)f(s,x(s-\delta(s)))ds\right\|^2 \\
&\leq M^2C_f\int_0^t e^{-\lambda(t-s)}ds\int_0^t e^{-\lambda(t-s)}\mathbf{E}\,\|x(s-\delta(s))\|^2\,ds \\
&\leq M^2C_f\lambda^{-1}\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}e^{-as}ds, \\
&\leq M_3^* e^{-at}. \qquad (9)
\end{aligned}$$

Using **(H4)** and Burkholder Davis Gundy inequality [10], we have

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_4(t)\|^2 &= \mathbf{E}\left\|\int_0^t R(t-s)g(s,x(s-\rho(s)))dw(s)\right\|^2\\
&\leq M^2C_g\int_0^t e^{-2\lambda(t-s)}\mathbf{E}\left\|x(s-\rho(s))\right\|^2 ds\\
&\leq M^2C_g\int_0^t e^{-2\lambda(t-s)}\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}e^{-as}ds,\\
&\leq M^2C_g\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}(2\lambda-a)^{-1}e^{-at}\\
&\leq M_4^*e^{-at}.
\end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_5(t)\|^2 &= \mathbf{E}\left\|\int_0^t\int_{\mathcal{U}} R(t-s)h(s,x(s-\sigma(s)))\widetilde{N}(ds,du)\right\|^2\\
&\leq M^2\mathbf{E}\int_0^t e^{-2\lambda(t-s)}\int_{\mathcal{U}}\|h(s,x(s-\sigma(s),u))\|^2\,v(du)ds\\
&\leq M^2C_h\int_0^t e^{-2\lambda(t-s)}\mathbf{E}\left\|x(s-\rho(s))\right\|^2 ds\\
&\leq M^2C_h\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}e^{-at}\int_0^t e^{(-2\lambda+a)(t-s)}ds\\
&\leq M^2C_h\left(M^*+\mathbf{E}\left\|\varphi\right\|_C^2\right)e^{a\tau}(2\lambda-a)^{-1}e^{-at}\\
&\leq M_5^*e^{-at}.
\end{aligned} \tag{11}$$

From **(H7)** and Holder's inequality, we get

$$\begin{aligned}
\mathbf{E}\,\|\mathcal{G}_5(t)\|^2 &= \mathbf{E}\left\|\sum_{0<t_k<t} R(t-t_k)I_kx(t_k)\right\|^2\\
&\leq \mathbf{E}\left(\sum_{0<t_k<t}\|R(t-t_k)\|\,\|I_k((t_k))-I_k(0)\|\right)^2\\
&\leq M^2\sum_{0<t_k<t}(t_k-t_{k-1})\sum_{0<t_k<t}(t_k-t_{k-1})\,e^{-2\lambda(t-t_k)}\mathbf{E}\,\|x(t_k)\|^2\\
&\leq \tilde{q}M^2M^*e^{-at}\sum_{k=1}^{\infty}(t_k-t_{k-1})\sum_{0<t_k<t}(t_k-t_{k-1})\,e^{(a-2\lambda)(t-t_k)}\\
&\leq \tilde{q}M^2M^*e^{-at}\left(\sum_{k=1}^{\infty}(t_k-t_{k-1})\right)^2\\
&\leq M_6^*e^{-at}.
\end{aligned} \tag{12}$$

Combining (7)-(11) and (12), we see that there exist $\bar{M}^*>0$ and $\bar{a}>0$ such that

$$\mathbf{E}\,\|(\Theta x)(t)\|^2 \leq \bar{M}^*e^{-\bar{a}t}, \quad t\geq 0.$$

we can easily see that $\mathcal{G}_i(t)$, $i=1,2,3,4,5,6$ in (6) satisfies (i), (ii) of the definition 3.1

Step 2: Now, we show that Θ is a contraction mapping. For any $x, y \in \mathbb{X}$, we have

$$\begin{aligned}
\mathbf{E}\left\|(\Theta x)(t) - (\Theta y)(t)\right\|^2 &\leq 5\mathbf{E}\left\|p(t, x(t-r(t))) - p(t, y(t-r(t)))\right\|^2 \\
&+ 5\mathbf{E}\left\|\int_0^t R(t-s)\left[f(s, x(s-\delta(s))) - f(s, y(s-\delta(s)))\right]ds\right\|^2 \\
&+ 5\mathbf{E}\left\|\int_0^t R(t-s)\left[g(s, x(s-\rho(s))) - g(s, y(s-\rho(s)))\right]dw(s)\right\|^2 \\
&+ 5\mathbf{E}\left\|\int_0^t R(t-s)\int_{\mathcal{U}}\left[h(s, x(s-\sigma(s)), u) - h(s, y(s-\sigma(s)), u)\right]\widetilde{N}(ds, du)\right\|^2 \\
&+ 5\mathbf{E}\left\|\sum_{0<t_k<t} R(t-t_k)\left[I_k(x(t_k)) - I_k(y(t_k))\right]\right\|^2 \\
&= 5\sum_{i=1}^{5} Q_i.
\end{aligned}$$

Since $x(t) = y(t) = \varphi(t)$, $t \in (-\tau, 0]$, this implies that

$$\mathbf{E}\left\|x(t-r(t)) - y(t-r(t))\right\|^2 \leq \sup_{t\geq 0}\mathbf{E}\left\|x(t) - y(t)\right\|^2.$$

Then by assumption **(H5)**, we have

$$\begin{aligned}
Q_1 &= \mathbf{E}\left\|p(t, x(t-r(t))) - p(t, y(t-r(t)))\right\|^2 \\
&\leq C_p\mathbf{E}\left\|x(t-r(t)) - y(t-r(t))\right\|^2 \\
&\leq C_p\sup_{t\geq 0}\mathbf{E}\left\|x(t) - y(t)\right\|^2.
\end{aligned} \tag{13}$$

By assumption **(H4)**, we get

$$\begin{aligned}
Q_2 &= \mathbf{E}\left\|\int_0^t R(t-s)\left[f(s, x(s-\delta(s))) - f(s, y(s-\delta(s)))\right]ds\right\|^2 \\
&\leq M^2 C_f \int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x(s-\delta(s)) - y(s-\delta(s))\right\|^2 ds \\
&\leq M^2 C_f \lambda^{-1}\int_0^t e^{-\lambda(t-s)}\mathbf{E}\left\|x(s-\delta(s)) - y(s-\delta(s))\right\|^2 ds \\
&\leq M^2 C_f \lambda^{-2}\sup_{t\geq 0}\mathbf{E}\left\|x(t) - y(t)\right\|^2.
\end{aligned} \tag{14}$$

Using **(H4)** and Burkholder Davis Gundy inequality [10], we have

$$\begin{aligned}
Q_3 &= \mathbf{E}\left\|\int_0^t R(t-s)\left[g(s, x(s-\rho(s))) - g(s, y(s-\rho(s)))\right]dw(s)\right\|^2 \\
&\leq M^2 C_g \int_0^t e^{-2\lambda(t-s)}\mathbf{E}\left\|x(s-\rho(s)) - y(s-\rho(s))\right\|^2 ds \\
&\leq M^2 C_g (2\lambda)^{-1}\sup_{t\geq 0}\mathbf{E}\left\|x(t) - y(t)\right\|^2.
\end{aligned} \tag{15}$$

By assumption **(H4)**, we get

$$\begin{aligned}
Q_4 &= \mathbf{E}\left\|\int_0^t R(t-s)\int_{\mathcal{U}}[h(s,x(s-\sigma(s)),u)-h(s,y(s-\sigma(s)),u)]\,\widetilde{N}(ds,du)\right\|^2 \\
&\le M^2\mathbf{E}\int_0^t e^{-2\lambda(t-s)}\int_{\mathcal{U}}\left\|h(s,x(s-\sigma(s)),u)-h(s,y(s-\sigma(s)),u)\right\|^2 v(du)ds \\
&\le M^2C_h(2\lambda)^{-1}\sup_{t\ge0}\mathbf{E}\left\|x(t)-y(t)\right\|^2. \qquad (16)
\end{aligned}$$

By assumption **(H7)**, we get

$$\begin{aligned}
Q_5 &= \mathbf{E}\left\|\sum_{0<t_k<t}R(t-t_k)\,[I_k(x(t_k))-I_k(y(t_k))]\right\|^2 \\
&\le M^2\left(\sum_{0<t_k<t}e^{-\lambda(t-t_k)}q_k\mathbf{E}\left\|x(t_k)-y(t_k)\right\|\right)^2 \\
&\le \tilde{q}^2M^2\left(\sum_{k=1}^{\infty}(t_k-t_{k-1})\right)^2\sup_{t\ge0}\mathbf{E}\left\|x(t)-y(t)\right\|^2. \qquad (17)
\end{aligned}$$

Thus, inequality (13)-(16) and (17) together imply

$$\begin{aligned}
\sup_{t\ge0}\mathbf{E}\left\|(\Theta x)(t)-(\Theta y)(t)\right\|^2 \le\ & 5\Bigg(C_p+M^2C_f\lambda^{-2}+M^2C_g(2\lambda)^{-2}+M^2C_h(2\lambda)^{-2} \\
&+\tilde{q}^2M^2\left(\sum_{k=1}^{\infty}(t_k-t_{k-1})\right)^2\Bigg)\times\sup_{t\ge0}\mathbf{E}\left\|x(t)-y(t)\right\|^2.
\end{aligned}$$

Therefore by the condition (iii) of the theorem 3.2 it follows that Θ is a contractive mapping. So, applying the Banach fixed point principle, the proof is complete. □

Remark 3.3. *In equation (1) provided $I_k = 0$, $k \in \mathbb{N}$ and $\Delta x(t_k) = 0$, equation (1) becomes stochastic neutral integrodifferential equations, which is investigated in [3]. In the sense, the results of this paper are generalized.*

4. Application

We consider the following model

$$\begin{aligned}
\frac{\partial}{\partial t}\Big[x(t,\zeta)+P(t,x(t-r(t),\zeta))\Big] &= A\Big[x(t,\zeta)+P(t,x(t-r(t),\zeta))\Big]+\Big[\int_0^t B(t-s)x(s,\zeta)+P(s,x(s-r(s),\zeta))ds\Big]dt \\
&+ F(t,x(t-\delta(t),\zeta))dt+G(t,x(t-\rho(t),\zeta))dw(s) \\
&+ \int_{\mathcal{U}}H(t,x(t-\sigma(t),\zeta),u)\widetilde{N}(dt,du),\ \text{for } t\ge0, t\ne t_k,\ \text{and } \zeta\in[0,\pi], \\
\Delta x(t_k,\zeta) &= x(t_k^+)-x(t_k^-)=\frac{\alpha_1}{2^k}\ \text{ for } t=t_k \text{ and } k\in\mathbb{N}, \\
(t,0)+p(t,x(t-r(t),0)) &= 0\ \text{ for } t\ge0, \\
x(t,\pi)+p(t,x(t-r(t),\pi)) &= 0\ \text{ for } t\ge0, \\
x(\theta,\zeta) &= x_0(\theta,\zeta),\varphi(s,.),\ \text{ for } \theta\in[-\tau,0] \text{ and } \zeta\in[0,\pi], \qquad (18)
\end{aligned}$$

where $\alpha_1 \geq 0$ is a constant, $w(t)$ denotes a Υ-valued Brownian motion, $P, F, G : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R}^+ \times \mathbb{R} \times \mathcal{U} \to \mathbb{R}$, $\delta, \rho, \sigma : [0, +\infty) \to [0, \tau]$ are continuous functions.
Let $\Upsilon = \mathcal{L}^2(0, \pi)$ and $e_n = \sqrt{\frac{2}{\pi}} sin(nx)$, $(n = 1, 2, 3, ...)$ denote the complete orthonormal basis in Υ. Let

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad (\lambda_n > 0),$$

where $\beta_n(t)$ are one dimensional standard Brownian motion. We suppose that
(i). For $t \geq 0$ and $u \in \mathcal{U}, P(t, 0) = F(t, 0) = G(t, 0) = H(t, 0) = 0$.
(ii). There exists a positive constant l_p such that

$$|P(t, x_1) - P(t, x_2)|^2 \leq l_p |x_1 - x_2|^2 \quad \text{for } t \geq 0 \text{ and } x_1, x_2 \in \mathbb{R}.$$

(iii). There exists a positive constant l_f, l_g, l_h, such that

$$\begin{aligned} |F(t, x_1) - F(t, x_2)|^2 &\leq l_f |x_1 - x_2|^2, \\ |G(t, x_1) - G(t, x_2)|^2 &\leq l_g |x_1 - x_2|^2, \\ \int_{\mathcal{U}} |H(t, x_1, u) - H(t, x_2, u)|^2 v(du) &\leq l_h |x_1 - x_2|^2 \quad \text{for } t \geq 0 \text{ and } x_1, x_2 \in \mathbb{R}. \end{aligned}$$

For $t \geq 0$ and $\zeta \in [0, \pi]$ and $\phi \in \mathbb{X}$, define the operator $p, f, g : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ and $h : \mathbb{R}^+ \times \mathbb{X} \times \mathcal{U} \to \mathbb{X}$ by

$$\begin{aligned} p(t, \phi)(\zeta) &= P\Big(t, \phi(t - r(t))(\zeta)\Big), \\ f(t, \phi)(\zeta) &= F\Big(t, \phi(t - \delta(t))(\zeta)\Big), \\ g(t, \phi)(\zeta) &= G\Big(t, \phi(t - \rho(t))(\zeta)\Big), \\ h(t, \phi, u)(\zeta) &= H\Big(t, \phi(t - \sigma(t))(\zeta), u\Big), \end{aligned}$$

If we put

$$\begin{cases} x(t) = x(t, \zeta) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \zeta \in [0, \pi] \\ \varphi(0)(\zeta) = x_0(\theta, \zeta) \quad \text{for} \quad \theta \in [-\tau, 0] \quad \text{and} \quad \zeta \in [0, \pi], \end{cases}$$

Then equation (18) takes the following abstract form equation (1).
As a consequence of the continuity of P, F, G, H and assumption (i), it follows that p, f, g, h are continuous.
By assumption (ii), we have

$$\begin{aligned} \Big|f(t, x_1) - f(t, x_2)\Big|^2 &\leq l_f |x_1 - x_2|^2, \\ \Big|g(t, x_1) - g(t, x_2)\Big|^2 &\leq l_g |x_1 - x_2|^2, \\ \int_{\mathcal{U}} \Big|h(t, x_1, u) - f(t, x_2, u)\Big|^2 v(du) &\leq l_h |x_1 - x_2|^2. \end{aligned}$$

Furthermore, by assumption (iii), we obtain

$$\Big|p(t, x_1) - p(t, x_2)\Big|^2 \leq l_p |x_1 - x_2|^2.$$

The remaining conditions can be verified similarly. Thus, all the assumptions of theorem 3.2 are fulfilled.
Therefore, the existence of a unique mild solution of equation (1) follows and this solution is exponential decay to zero in mean square.

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