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EXISTENCE, UNIQUENESS AND STABILITY RESULTS OF SEMILINEAR FUNCTIONAL SPECIAL RANDOM IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. We deal with semilinear functional special random impulsive differential equations in this paper. Contraction mapping principle is used to study the existence and uniqueness of the mild solution of the system. Again we have established ulam stabilities, exponential stability and stability of the system, where the sufficient conditions for stability is established using continuous dependence on initial conditions. Also an example is included to verify the concepts and basic results in this paper.

Keywords. Existence, Uniqueness, Fixed point theorem, Random impulses, Stability.

AMS (MOS) subject classification: 35R12, 60H99, 34G20, 35B40.

1 Introduction

Random impulsive differential equations can be used as an essential tool to model many real life phenomena. This is why it has received attention of many researchers recently. There are plenty of literature on the topic of random impulsive differential equations. Large number of authors have considered Random impulsive differential equations [13–15, 18–20]. Moreover, random impulsive differential equations have applications in many fields such as biology, economics and neural network.

Integro differential equations are applicable in many fields like chemistry, biology and fluid dynamics [1, 2, 7, 21]. Some existence results of impulsive integro differential equations of the form

$$\begin{aligned} w^{'}(t) &= Aw(t) + f(t, w(t), Uw(t), Vw(t)), \quad 0 < t < T_0, t \neq t_0, \\ w(0) &= w_0 \\ \Delta w(t_i) &= I_i(w(t_i)) \quad i = 1, 2, 3, \ldots, \end{aligned} \tag{1}$$

has found by many researchers [5, 6, 22]. In this systems impulses are happening at fixed moments. But practically external environmental factors can affect impulses, and also need not always occur at fixed points. Impulses of this form is called random impulse. If an integro differntial equation consists of random impulse then the solution of this system become a stochastic process.

In this era study of random impulsive integro differential equations has become more popular [16]. For example, Sayooj Aby Jose et. al [14] studied existence and uniqueness of (1) with random impulses. It is well known that existence as well as stability of a system is very important. Many authers alredy established some significant results of stability on impulsive system [8, 12, 17, 23].

The inspiration by the above mentioned works, we are considering Special random impulsive semilinear differential system. We are using fixed point theorem to get the desired results. First we recall some preliminary concepts. In section 3 we study some existence and uniqueness results and section 4 deals with various type of stabilities. In section 5, an example is introduced to implement the theoretical results. Finally we are giving conclusion and future works.

2 Preliminaries

Consider a real separable Hilbert space X and a non empty set Ω. Let η_k be a random variable and η_k maps Ω to D_k, where $D_k = (0, d_k)$ for every $k \in \mathbb{N}$ (collection of natural numbers) and $0 < d_k < +\infty$. Also for $i, j = 1, 2, \ldots$ assume that if $i \neq j$ then η_i and η_j are independent with each other. Also assume η_k follow Erlang distribution. Let η be a real constant, denote $\Re_\eta = [\eta, +\infty)$, $\Re^+ = [0, +\infty]$.

Consider the semilinear functional special random impulsive differential equations of the form

$$\begin{aligned} x^{'}(t) &= Ax(t) + f(t, x_t, Ux(t), Vx(t)) \qquad t \neq \xi_k, t \geq t_0, \\ x(\xi_k) &= \alpha_k(\eta_k)x(\xi_k^-), k = 1, 2, 3, \ldots, \\ x_{t_0} &= \psi \end{aligned} \tag{2}$$

A is the infinitesimal generator of a strongly continuous semi group of bounded linear operators $\mathbb{T}(t)$, $\mathbb{T} \in X$. And $f : \Re^+ \times C \times X \times X \rightarrow X$, $C = C([-p, 0], X)$ is the set of piecewise continuous function from $[-p, 0]$

into X for some $p > 0$. $\psi : [-p, 0] \to X$. Consider a function x_t such that $x_t(r) = x(t + r)$, where t is fixed and $r \in [-p, 0]$, $\alpha_k : D_k \to X$ for every $k \in \mathbb{N}$; $\xi_0 = t_0 \in \Re_\eta$ and $\xi_k = \xi_{k-1} + \eta_k$ for all $k \in \mathbb{N}$. Also $\xi_0 < \xi_1 < \xi_2 < \xi_3 \cdots < \xi_k < \dots$, and $x(\xi_k-) = \lim_{t\uparrow\xi_k} x(t)$ according to their path with the norm $\|x\|_t = \sup_{t-p\leq r\leq t} \mid x(r) \mid$ for all $t \geq t_0$, $\|.\|$ is any given norm in X.

$$Ux(t) = \int_{t_0}^{t} \mathbb{K}(t, r)x(r)dr, \mathbb{K} \in C[\mathbb{D}, \Re^+],$$
$$Vx(t) = \int_{t_0}^{T} \mathbb{H}(t, r)x(r)dr, \mathbb{H} \in C[\mathbb{D}_{\not\vdash}, \Re^+],$$

Where $\mathbb{D} = \{(t, r) \in \Re^2 : t_0 \leq r \leq t \leq T\}, \mathbb{D}_{\not\vdash} = \{(t, r) \in \Re^2 : t_0 \leq t, r \leq T\}$. Let $\{\mathscr{B}_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$. Therefore,$\{\mathscr{B}_t \geq n\} = \{\xi_n \leq t\}$ and denote $\mathscr{F}_t$ the $\sigma-$ algebra generated by $\{\mathscr{B}_t, t \geq 0\}$. That is $(\Omega, P, \{\mathscr{F}_t\})$ is a probability space. Consider $\mathbb{L}_2 = \mathbb{L}_2(\Omega, \mathscr{F}_t, X)$ denote the Hilbert space of all $\mathscr{F}_t$ measurable square integrable random variable with values in X. Let $T > t_0$ is any fixed time and $\mathscr{B}$ is a banach space. $\mathscr{B}([t_0 - p, T], \mathbb{L}_2)$ denote the family of all $\{\mathscr{F}_t\}$-measurable , $C-$ valued random variable Ψ with the norm

$$\|\Psi\|_{\mathscr{B}} = \left(\sup_{t\in[t_0,T]} E\|\Psi\|_t^2 \right)^{\frac{1}{2}}.$$

Definition 2.1. *A stochastic process $\{x(t) \in \mathscr{B}; t \in [t_0 - p, T]\}$ is said to be a mild solution of system (2), if*

(i) $x(t) \in \mathscr{B}$ is $\mathscr{F}_t$- adapted for all $t \geq t_0$;

(ii) $x(t_0 + r) = \psi(r) \in \mathbb{L}_2^0(\Omega, \mathscr{B})$, when $r \in [-p, 0]$ and

$$\begin{aligned} x(t) =& \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} \alpha_i(\eta_i)\mathbb{T}(t - t_0)\psi(0) \\ &+ \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t - r)f(r, x_r, Ux(r), V x(r))dr \\ &+ \int_{\xi_k}^{t} \mathbb{T}(t - r)f(r, x_r, Ux(r), V x(r))dr \Bigg] I_{[\xi_k,\xi_{k+1})}(t), \quad t \in [t_0, T] \end{aligned}$$

where $T \in (t_0, +\infty)$, $\prod_{j=m}^{n}(.) = 1$ *as* $m > n$,

$$\prod_{j=i}^{k} \alpha_j(\eta_j) = \alpha_k(\eta_k)b_{k-1}(\eta_{k-1}) \dots \alpha_i(\eta_i),$$

and $I_{\mathbb{A}}(.)$ is the index function.

Main Results

In this section some hypotheses which are used for proving the main results are given below;

Hypotheses

($\mathscr{H}_1$) The function $f : [t_0, T] \times C \times X \times X \to X$ is Lipschitz. That is, for all $a_1, b_1 \in C$, $a_2, a_3, b_2, b_3 \in X$ and $t_0 \le t \le T$ there exist constants $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 > 0$ so that

$$\begin{aligned} E\|f(t, a_1, a_2, a_3) - f(t, b_1, b_2, b_3)\|^2 \le& \mathscr{L}_1 E\|a_1 - b_1\|_t^2 + \mathscr{L}_2 E\|a_2 - b_2\|_t^2 \\ &+ \mathscr{L}_3 E\|a_3 - b_3\|_t^2 \\ E\|f(t, 0, 0, 0)\|^2 \le& k, k \ge 0 \text{ a constant.} \end{aligned}$$

($\mathscr{H}_2$) $E\bigg\{ \max_{i,k} \Big\{ \prod_{j=i}^{k} \|\alpha_j(\eta_j)\| \Big\} \bigg\}$ is uniformly bounded if, for each $\eta_j \in D_j, j \in \mathbb{N}, \theta > 0$ a constant

$$E\bigg\{ \max_{i,k} \Big\{ \prod_{j=i}^{k} \|\alpha_j(\eta_j)\| \Big\} \bigg\} \le \theta,$$

($\mathscr{H}_3$) Define $\mathscr{L}, \mathbb{K}^*$ and $\mathbb{H}^*$ such that,

$$\begin{aligned} \mathscr{L} &= \max\{\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\}, \\ \mathbb{K}^* &= \sup_{t \in [t_0, T]} \int_{t_0}^{t} |\mathbb{K}(t, r)|^2 dt < \infty, \text{ and} \\ \mathbb{H}^* &= \sup_{t \in [t_0, T]} \int_{t_0}^{T} |\mathbb{H}(t, r)|^2 dt < \infty. \end{aligned}$$

3 Existence and Uniqueness

Some existence results are discussed in this section.

Theorem 3.1. *If the hypotheses ($\mathscr{H}_1$) − ($\mathscr{H}_3$) and (3) are satisfied*

$$\Gamma = N^2 \max\{1, \theta^2\}(T - t_0)^2 \mathscr{L} \Big[1 + \mathbb{K}^* + \mathbb{H}^* \Big] < 1 \tag{3}$$

then there exist a unique (local) continuous mild solution to (2) for any initial value (t_0, ψ) with $t_0 \ge 0$ and $\psi \in \mathscr{B}$.

Proof. Let T be an arbitrary number $\eta < +\infty$. Consider the nonlinear operator S maps $\mathscr{B}$ into $\mathscr{B}$ is defined as $(Sx)(t+t_0) = \psi(t)$ $t \in [-p, 0]$ and

$$(Sx)(t) = \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t-t_0)\psi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr + \int_{\xi_k}^{t} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \Big] I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [t_0, T]$$

Here S is continuous. Next we show that, S is a mapping from $\mathscr{B}$ into $\mathscr{B}$.

$$\begin{aligned}
&\|(Sx)(t)\|^2 \\
&\le \Big[\sum_{k=0}^{+\infty} \Big[\| \prod_{i=1}^{k} \alpha_i(\eta_i) \| \| \mathbb{T}(t-t_0) \| \| \psi(0) \| \\
&\quad + \sum_{i=1}^{k} \| \prod_{j=i}^{k} \alpha_j(\eta_j) \| \Big\{ \int_{\xi_{i-1}}^{\xi_i} \| \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) \| dr \Big\} \\
&\quad + \int_{\xi_k}^{t} \| \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) \| dr \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big]^2 \\
&\le 2 \Big[\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \| \alpha_i(\eta_i) \|^2 \| \mathbb{T}(t-t_0) \|^2 \| \psi(0) \|^2 I_{[\xi_k, \xi_{k+1})}(t) \Big] \\
&\quad + \Big[\sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k} \| \prod_{j=i}^{k} \alpha_j(\eta_j) \| \Big\{ \int_{\xi_{i-1}}^{\xi_i} \| \mathbb{T}(t-r) \| \| f(r, x_r, Ux(r), Vx(r)) \| dr \Big\} \\
&\quad + \int_{\xi_k}^{t} \| \mathbb{T}(t-r) \| \| f(r, x_r, Ux(r), Vx(r)) \| dr \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big]^2 \Big]
\end{aligned}$$

Since S is uniformly bounded,

$$\begin{aligned}
&E\|(Sx)(t)\|^2 \\
&\le 2N^2 E\Big[\max_k \Big\{ \prod_{i=1}^{k} \| \alpha_i(\eta_i) \|^2 \Big\} \Big] E\|\psi(0)\|^2 \\
&\quad + 2N^2 E\Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} \| \alpha_j(\eta_j) \|^2 \Big\} \Big]^2 \\
&\quad \times E\Big(\int_{t_0}^{t} \| f(r, x_r, Ux(r), Vx(r)) \| dr I_{[\xi_k, \xi_{k+1})}(t) \Big)^2 \\
&\le 2N^2\theta^2 E\|\psi(0)\|^2 + 2N^2 \max\{1, \theta^2\}
\end{aligned}$$

$$\times E\bigg(\int_{t_0}^{t}\|f(r,x_r,Ux(r),Vx(r))\|dr\bigg)^2$$

$$\begin{aligned}E\|(Sx)\|_t^2 \leq & 2N^2\theta^2E\|\psi(0)\|^2\\ & + 2N^2\max\{1,\theta^2\}(T-t_0)\int_{t_0}^{t}E\|f(r,x_r,Ux(r),Vx(r))\|^2dr\\ \leq & 2N^2\theta^2E\|\psi(0)\|^2\\ & + 4N^2\max\{1,\theta^2\}(T-t_0)^2k + 4N^2\max\{1,\theta^2\}(T-t_0)\\ & \bigg\{\mathscr{L}_1\int_{t_0}^{t}E\|x\|_r^2dr + \mathscr{L}_2\int_{t_0}^{t}E\|Ux\|_r^2dr + \mathscr{L}_3\int_{t_0}^{t}E\|Vx\|_r^2dr\bigg\}\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{t\in[t_0,T]}E\|(Sx)\|_t^2 \leq & 2N^2\theta^2E\|\psi(0)\|^2 + 4N^2\max\{1,\theta^2\}(T-t_0)^2k\\ & + 4N^2\max\{1,\theta^2\}(T-t_0)\bigg\{\mathscr{L}_1\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|x\|_r^2dr\\ & + \mathscr{L}_2\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|Ux\|_r^2dr + \mathscr{L}_3\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|Vx\|_r^2dr\bigg\}\\ \leq & 2N^2\theta^2E\|\psi(0)\|^2 + 4N^2\max\{1,\theta^2\}(T-t_0)^2k\\ & + 4N^2\max\{1,\theta^2\}(T-t_0)^2\bigg\{\mathscr{L}_1\sup_{t\in[t_0,T]}E\|x\|_t^2\\ & + \mathscr{L}_2\sup_{t\in[t_0,T]}E\|Ux\|_t^2 + \mathscr{L}_3\sup_{t\in[t_0,T]}E\|Vx\|_t^2\bigg\}\end{aligned}$$

for every $t \in [-p,T]$, thus we get $\mathscr{B}$ is mapped into $\mathscr{B}$ under S. Next we prove that S is a contraction mapping

$$\begin{aligned}&\|(Sx)(t)-(Sy)(t)\|^2\\ &\leq\bigg[\sum_{k=0}^{+\infty}\bigg\{\sum_{i=1}^{k}\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\times\int_{\xi_{i-1}}^{\xi_i}\|\mathbb{T}(t-r)\|\\ &\quad\times\|f(r,x_r,Ux(r),Vx(r))-f(r,y_r,Uy(r),Vy(r))\|dr\\ &\quad+\int_{\xi_k}^{t}\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\\ &\quad-f(r,y_r,Uy(r),Vy(r))\|dr\bigg\}I_{\xi_k,\xi_{k+1}}(t)\bigg]^2\\ &\leq N^2\bigg[\max_{i,k}\bigg\{1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\bigg\}\bigg]^2\bigg(\int_{t_0}^{t}\|f(r,x_r,Ux(r),Vx(r))\\ &\quad-f(r,y_r,Uy(r),Vy(r))\|drI_{[\xi_k,\xi_{k+1})}(t)\bigg)^2\end{aligned}$$

$$
\begin{aligned}
&\leq N^2 \max\{1,\theta^2\}\bigg(\int_{t_0}^{t} \|f(r,x_r,Ux(r),Vx(r)) \\
&\quad - f(r,y_r,Uy(r),Vy(r))dr I_{[\xi_k,\xi_{k+1})}(t)\bigg)^2 \\
E\|(Sx)-(Sy)\|_t^2 &\leq N^2 \max\{1,\theta^2\}(T-t_0)\int_{t_0}^{t} E\|f(r,x_r,Ux(r),Vx(r)) \\
&\quad - f(r,y_r,Uy(r),Vy(r))\|^2 dr \\
&\leq N^2 \max\{1,\theta^2\}(T-t_0)\bigg[\mathscr{L}_1\int_{t_0}^{t} E\|x-y\|_r^2 dr \\
&\quad + \mathscr{L}_2\int_{t_0}^{t} E\|Ux-Uy\|_r^2 dr + \mathscr{L}_3\int_{t_0}^{t} E\|Vx-Vy\|_r^2 dr\bigg] \\
E\|(Sx)-(Sy)\|_t^2 &\leq N^2 \max\{1,\theta^2\}(T-t_0)\bigg[\mathscr{L}_1\mathscr{P}+\mathscr{L}_2\mathscr{Q}+\mathscr{L}_3\mathscr{R}\bigg] \qquad (4)
\end{aligned}
$$

where

$$
\mathscr{P}=\int_{t_0}^{t} E\|x-y\|_r^2 dr,\ \mathscr{Q}=\int_{t_0}^{t} E\|Ux-Uy\|_r^2 dr,\ \mathscr{R}=\int_{t_0}^{t} E\|Vx-Vy\|_r^2 dr.
$$

Consider $\mathscr{Q}$

$$
\begin{aligned}
\int_{t_0}^{t} E\|Ux-Uy\|_r^2 dr &\leq \mathscr{L}_2 E\bigg(\int_{t_0}^{t}\int_{t_0}^{s} \|\mathbb{K}(r,\eta)\|^2\|x-y\|_r^2 d\eta dr\bigg) \\
&\leq \mathscr{L}_2 E\bigg(\int_{t_0}^{t} \|x-y\|_r^2 \int_{t_0}^{r} \|\mathbb{K}(r,\eta)\|^2 d\eta dr\bigg) \\
&\leq \mathscr{L}_2 E\|x-y\|_r^2 \mathbb{K}^*(T-t_0) \qquad (5)
\end{aligned}
$$

Similarly in $\mathscr{R}$ and $\mathscr{P}$,

$$
\int_{t_0}^{t} E\|Vx-Vy\|_r^2 dr \leq \mathscr{L}_3 E\|x-y\|_r^2 \mathbb{H}^*(T-t_0) \qquad (6)
$$

$$
\int_{t_0}^{t} E\|x-y\|_r^2 dr \leq \mathscr{L}_1 E\|x-y\|_r^2 (T-t_0) \qquad (7)
$$

Substituting (5), (6) and (7) in (4),

$$
\begin{aligned}
E\|(Sx)-(Sy)\|_t^2 \leq & N^2 \max\{1,\theta^2\}(T-t_0)^2 \\
& \times \bigg[\mathscr{L}_1 E\|x-y\|_r^2 + \mathscr{L}_2 E\|x-y\|_r^2 \mathbb{K}^* \qquad (8) \\
& + \mathscr{L}_3 E\|x-y\|_r^2 \mathbb{H}^*\bigg]
\end{aligned}
$$

From the definition of $\mathscr{L}$ and taking the supremum, we have

$$\begin{aligned}E\|(Sx)-(Sy)\|_{\mathscr{B}}^2 \leq & N^2\max\{1,\theta^2\}(T-t_0)^2\mathscr{L}\Bigg[E\|x-y\|_t^2 \\ & + E\|x-y\|_{\mathscr{B}}^2\mathbb{K}^* + E\|x-y\|_{\mathscr{B}}^2 H^*\Bigg] \\ \leq & N^2\max\{1,\theta^2\}(T-t_0)^2 \\ & \times \mathscr{L}\Bigg[1+\mathbb{K}^*+\mathbb{H}^*\Bigg]E\|x-y\|_{\mathscr{B}}^2\end{aligned}$$

From (3) we have,

$$\|(Sx)-(Sy)\|_{\mathscr{B}}^2 \leq \Gamma\|x-y\|_{\mathscr{B}}^2$$

Here $0<\Gamma<1$, where $\Gamma = N^2\max\{1,\theta^2\}(T-t_0)^2\mathscr{L}\Bigg[1+\mathbb{K}^*+\mathbb{H}^*\Bigg]$.

Hence S is a contraction mapping. Therefore we know that the operator S has a unique fixed point in $\mathscr{B}$. Hence (2) has a unique mild solution. Hence the proof. □

Remark 3.1. *Let $f : \Re^+ \times C \times X \times X \longrightarrow X$ satisfy the assumptions $(\mathscr{H}_1)-(\mathscr{H}_3)$. Then there exists a unique, global, continuous solution x to (2) for any initial value (t_0,ψ) with $t_0 \geq 0$ and $\psi \in \mathscr{B}$.*

Remark 3.2. *The above theorem is an extension of Theorem 3.1 [14]. Theorem 3.1 gives existence and uniqueness of semilinear functional special random impulsive differential equations. This solution is practically more useful than the solution of random impulsive differential equations.*

4 Stability Results

In this section first we study the stability of the system (2) using the continuous dependence of solutions on initial condition. After that we are deling Ulam stabilities and exponential stability of special random impulsive differential equations under sufficient conditions.

Theorem 4.1. *Consider $x(t)$ and $\widehat{y}(t)$ are two mild solutions of the system (2) with initial values $\psi(0)$ and $\widehat{\psi}(0) \in \mathscr{B}$ respectively. Then the mild solution of the system (2) is stable in the mean square, provided the assumptions of the Theorem 3.1 is satisfied.*

Proof. From our assumptions we get,

$$\begin{aligned}x(t)-\widehat{y}(t) = & \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t-t_0)\big[\psi(0)-\widehat{\psi}(0)\big] \\ & + \sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)\big[f(r,x_r,Ux(r),Vx(r))\end{aligned}$$

$$- f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}_r)\big]dr + \int_{\xi_k}^{t} \mathbb{T}(t-r)\big[f(r,x_r,Ux(r),Vx(r))$$
$$- f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}_r)\big]dr\bigg] I_{[\xi_k,\xi_{k+1})}(t)$$

From $(\mathscr{H}_1)-(\mathscr{H}_3)$ we have,

$$\begin{aligned}
&E\|x(t)-\widehat{y}(t)\|_t^2\\
&\le 2\sum_{k=0}^{+\infty}\bigg[\prod_{i=1}^{k}\|\alpha_i(\eta_i)\|^2\|\mathbb{T}(t-t_0)\|^2 E\big[\|\psi(0)-\widehat{\psi}(0)\|^2\big] I_{[\xi_k,\xi_{k+1})}(t)\bigg]\\
&\quad + 2E\bigg[\sum_{k=0}^{+\infty}\bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\int_{\xi_{i-1}}^{\xi_i}\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\\
&\quad - f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}_r)\|dr + \int_{\xi_k}^{t}\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\\
&\quad - f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}_r)\|dr\bigg] I_{[\xi_k,\xi_{k+1})}(t)\bigg]^2\\
&\le 2N^2E\bigg[\max_k\bigg\{\prod_{i=1}^{k}\|\alpha_i(\eta_i)\|^2\bigg\}\bigg]E\big[\|\psi(0)-\widehat{\psi}(0)\|^2\big]\\
&\quad + 2N^2E\bigg[\max_{i,k}\bigg\{1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\bigg\}\bigg]^2\\
&\quad \times E\bigg(\int_{t_0}^{t}\|f(r,x_r,Ux(r),Vx(r)) - f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}(r))\|dr I_{[\xi_k,\xi_{k+1})}(t)\bigg)^2\\
&\le 2N^2\theta^2E\|\psi(0)-\widehat{\psi}(0)\|^2\\
&\quad + 2N^2\max\{1,\theta^2\}(t-t_0)\int_{t_0}^{t}E\|f(r,x_r,Ux(r),Vx(r))\\
&\quad - f(r,\widehat{y}_r,U\widehat{y}(r),V\widehat{y}(r))\|^2dr
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[t_0,T]}E\|x(t)-\widehat{y}(t)\|_t^2\\
&\qquad\le 2N^2\theta^2E\|\psi(0)-\widehat{\psi}(0)\|^2\\
&\qquad\quad + 2N^2\max\{1,\theta^2\}(T-t_0)\bigg[\mathscr{L}_1\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|x-\widehat{y}\|_r^2dr\\
&\qquad\quad + \mathscr{L}_2\mathbb{K}^*\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|x-\widehat{y}\|_r^2dr + \mathscr{L}_3\mathbb{H}^*\int_{t_0}^{t}\sup_{r\in[t_0,T]}E\|x-\widehat{y}\|_r^2dr\bigg]
\end{aligned}$$

Using Grownwall's inequality,

$$\sup_{t\in[t_0,T]}E\|x(t)-\widehat{y}(t)\|_t^2 \le 2N^2\theta^2E\|\psi(0)-\widehat{\psi}(0)\|^2$$

$$\times\, exp\Big[2N^2\max\{1,\theta^2\}(T-t_0)^2\big[\mathscr{L}_1+\mathscr{L}_2\mathbb{K}^*+\mathscr{L}_3\mathbb{H}^*\big]\Big]$$
$$\leq \Delta E\|\psi(0)-\widehat{\psi}(0)\|^2$$

where, $\Delta = 2N^2\theta^2 exp\Big[2N^2\max\{1,\theta^2\}(T-t_0)^2\big[\mathscr{L}_1+\mathscr{L}_2\mathbb{K}^*+\mathscr{L}_3\mathbb{H}^*\big]\Big]$
For any $\varepsilon>0$ there exist a $\delta=\frac{\varepsilon}{\Delta}$ such that $E\|\psi(0)-\widehat{\psi}(0)\|^2<\delta$. Then

$$\sup_{t\in[t_0,T]} E\|x(t)-\widehat{y}(t)\|_t^2 \leq \varepsilon.$$

Hence, it is clear that the difference between the mild solution $x(t)$ and $\widehat{y}(t)$ in the interval $[t_0,T]$ tends to 0 whenever the assumptions are satisfied. Hence the proof. □

Ulam-Hyers-Rassias Stability

This section deals with Ulam-Hyers stablilty of the system (2).
Let $\varepsilon>0,\beta\geq 0$ and $\varrho:[t_0,T]\to\Re^+$ be a piecewise continuous function. Let us consider the following,

$$\begin{cases} E\|x^{'}(t)-Ax(t)-f(t,x_t,Ux(t),Vx(t))\|^2\leq\varepsilon & t\neq\xi_k, t\geq t_0,\\ E\|x(\xi_k)-\alpha_k(\eta_k)x(\xi_k^-)\|^2\leq\varepsilon, & k=1,2,3,\ldots, \end{cases} \tag{9}$$

$$\begin{cases} E\|x^{'}(t)-Ax(t)-f(t,x_t,Ux(t),Vx(t))\|^2\leq\varrho(t) & t\neq\xi_k, t\geq t_0,\\ E\|x(\xi_k)-\alpha_k(\eta_k)x(\xi_k^-)\|^2\leq\beta, & k=1,2,3,\ldots, \end{cases} \tag{10}$$

$$\begin{cases} E\|x^{'}(t)-Ax(t)-f(t,x_t,Ux(t),Vx(t))\|^2\leq\varepsilon\varrho(t) & t\neq\xi_k, t\geq t_0,\\ E\|x(\xi_k)-\alpha_k(\eta_k)x(\xi_k^-)\|^2\leq\varepsilon\beta, & k=1,2,3,\ldots, \end{cases} \tag{11}$$

Using the above inequalities we obtain some basic definitions of Ulam-Hyers-Rassias Stability Definition 5.1-5.4 [19]. And also we can derive the following results.

Proposition 1. *A function x in $\mathscr{B}$ is a mild solution of (11) if and only if there exists a function l in $\mathscr{B}$ and the sequence $l_k, k\in\mathbb{N}$(which depend on x) such that*

i $E\|l(t)\|^2\leq\varepsilon\varrho(t), t\in[t_0,T]$ *and* $E\|l_k\|^2\leq\varepsilon\beta, k=1,2,3,\ldots;$

ii $x^{'}(t)=Ax(t)+f(t,x_t,Px(t),Qx(t))+l(t) \quad t\neq\xi_k, t\geq t_0;$

iii $x(\xi_k)=\alpha_k(\eta_k)x(\xi_k^-)+l_k \quad k\in\mathbb{N}$

We will get similar inequalities for (9) and (10)

Remark 4.1. *$x \in \mathscr{B}$ is a mild solution of the following integral inequality when x is a mild solution of the inequality (11)*

$$\begin{aligned}
E\|x(t) - \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t-t_0)\psi(0) \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \\
&+ \int_{\xi_k}^{t} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \Big] I_{[\xi_k, \xi_{k+1})}(t)\|^2 \\
&\leq 2N^2 \varepsilon \Big\{ \theta^2 \beta + max\{1, \theta^2\} \int_{t_0}^{t} \varrho(r) dr \Big\}, \qquad t \in [t_0, T]
\end{aligned}$$

Proof. Using Proposition 1 we get,

$$\begin{cases} x^{'}(t) &= Ax(t) + f(t, x_t, Ux(t), Vx(t)) + l(t), \qquad t \neq \xi_k, \quad t \geq t_0, \\ x(\xi_k) &= \alpha_k(\eta_k) x(\xi_k^-) + l_k \qquad k = 1, 2, 3, \dots \end{cases} \tag{12}$$

Then $x(t + t_0) = \psi(t)$, for $t \in [-p, 0]$

$$\begin{aligned}
x(t) = &\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t-t_0)\psi(0) + \prod_{i=1}^{k} \alpha_j(\eta_j) \mathbb{T}(t-r) l_i \\
&+ \sum_{i=1}^{k} \prod_{j=1}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \\
&+ \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_k}^{t} \mathbb{T}(t-r) l(r) dr \\
&+ \int_{\xi_k}^{t} \mathbb{T}(t-r) l(r) dr \Big] I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [t_0, T]
\end{aligned}$$

Hence,

$$\begin{aligned}
E\|x(t) - \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t-t_0)\psi(0) \\
&+ \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \\
&+ \int_{\xi_k}^{t} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \Big] I_{[\xi_k, \xi_{k+1})}(t)\|^2
\end{aligned}$$

$$
\begin{aligned}
=&E\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_j(\eta_j)\mathbb{T}(t-r)l_i+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j) \\
&\times\int_{\xi_k}^{t}\mathbb{T}(t-r)l(r)dr+\int_{\xi_k}^{t}\mathbb{T}(t-r)l(r)\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^2 \\
\leq&2N^2E\Big\{\max_k\{\prod_{i=1}^{k}\|\alpha_j(\eta_j)\|^2\}\Big\}E\|l_i\|^2 \\
&+2N^2E\Big[\max_{i,k}\{1,\prod_{i=1}^{k}\|\alpha_i(\eta_i)\|\}\Big]^2(T-t_0)\int_{t_0}^{t}E\|l(r)\|^2dr \\
\leq&2N^2\varepsilon\Big\{\theta^2\beta+\max\{1,\theta^2\}(T-t_0)\int_{t_0}^{t}\varrho(r)dr\Big\}
\end{aligned}
$$

□

Similar remarks can be derived for the mild solution of the inequality (9) and (10).
Next, the main results of this Section : Ulam-Hyers-Raassias is given.

Theorem 4.2. *Assumption ($\mathscr{H}_1-\mathscr{H}_3$) holds, Suppose there exists $\lambda>0$ such that*

$$\int_{t_0}^{t}\varrho(r)dr\leq\lambda\varrho(t),\ \textit{for each } t\in[t_0,T],$$

here ϱ is a nondecreasing continuous function from $[t_0,T]$ to $\Re^+$. Then the system (2) is Ulam-Hyers-Rassias stable in the mean square.

Proof. If $x\in\mathscr{B}$ is a mild solution of (11). From Theorem 3.1 we get a unique mild solution z of the random impulsive differential system

$$
\begin{aligned}
z^{'}(t)&=Az(t)+f(t,z_t,Uz(t),Vz(t))\ t\neq\xi_k,t\geq t_0,\\
z(\xi_k)&=\alpha_k(\eta_k)z(\xi_k^-),k=1,2,3,\ldots,\\
z_{t_0}&=\psi
\end{aligned}
\tag{13}
$$

Then we get,

$$
\begin{aligned}
z(t+t_0)=&\psi(t),\ \text{for } t\in[-p,0],\\
z(t)=&\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t-t_0)\psi(0)\\
&+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)f(r,z_r,Uz(r),Vz(r))dr\\
&+\int_{\xi_k}^{t}\mathbb{T}(t-r)f(r,z_r,Uz(r),Vz(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t),\qquad t\in[t_0,T]
\end{aligned}
$$

From (11),

$$
\begin{aligned}
E\|x(t)-&\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t-t_0)\psi(0)\\
&+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)f(r,x_r,Ux(r),Vx(r))dr\\
&+\int_{\xi_k}^{t}\mathbb{T}(t-r)f(r,x_r,Ux(r),Vx(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^2\\
&=E\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\eta_i)l_i+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_k}^{t}\mathbb{T}(t-r)l(r)dr\\
&+\int_{\xi_k}^{t}\mathbb{T}(t-r)l(r)\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^2\\
\leq&2N^2E\Big\{\max_{k}\{\prod_{j=1}^{k}\|\alpha_j(\eta_j)\|^2\}\Big\}E\|l_i\|^2\\
&+2N^2E\Big[\max_{i,k}\{1,\prod_{j=1}^{k}\|\alpha_j(\eta_j)\|^2\}\Big](T-t_0)\times\varepsilon\int_{t_0}^{t}\varrho(r)dr\\
\leq&2N^2\varepsilon\Big\{\theta^2\beta+\max\{1,\theta^2\}(T-t_0)\lambda\varrho(t)\Big\}
\end{aligned}
$$

Therefore for all $t\in[t_0,T]$,

$$
\begin{aligned}
E\|x(t)-z(t)\|^2=E\|x(t)-&\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t-t_0)\psi(0)\\
&+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)f(r,z_r,Uz(r),Vz(r))dr\\
&+\int_{\xi_k}^{t}\mathbb{T}(t-r)f(r,z_r,Uz(r),Vz(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^2\\
E\|x-z\|_t^2\leq 2E\|x(t)-&\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t-t_0)\psi(0)\\
&+\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)f(r,x_r,Ux(r),Vx(r))dr\\
&+\int_{\xi_k}^{t}\mathbb{T}(t-r)f(r,x_r,Ux(r),Vx(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^2\\
&+2E\|\sum_{i=1}^{k}\prod_{j=1}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t-r)\big[f(r,x_r,Ux(r),Vx(r))
\end{aligned}
$$

$$
\begin{aligned}
&- f(r, z_r, Uz(r), Vz(r))\big]dr \\
&+ \int_{\xi_k}^{t} \mathbb{T}(t-r)\big[f(r, x_r, Ux(r), Vx(r)) \\
&- f(r, z_r, Uz(r), Vz(r))\big]dr\Bigg] I_{[\xi_k,\xi_{k+1})}(t)\|^2 \\
\leq & 4N^2\varepsilon\Bigg\{\theta^2\beta + \max\{1,\theta^2\}(T-t_0)\lambda\varrho(t)\Bigg\} \\
&+ 2N^2\max\{1,\theta^2\}(T-t_0)\Bigg[\mathscr{L}_1\int_{t_0}^{t} E\|x_r - z_r\|^2 dr \\
&+ \mathscr{L}_2\int_{t_0}^{t} E\|Ux_r - Uz_r\|^2 dr + \mathscr{L}_3\int_{t_0}^{t} E\|Vx_r - Vz_r\|^2 dr\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\sup_{t\in[t_0,T]} E\|x-z\|_t^2 \leq & 4N^2\varepsilon\Bigg\{\theta^2\beta + \max\{1,\theta^2\}(T-t_0)\lambda\varrho(t)\Bigg\} \\
&+ 2N^2\max\{1,\theta^2\}(T-t_0)^2\Bigg[\mathscr{L}_1\sup_{t\in[t_0,T]} E\|x-z\|_t^2 \\
&+ \mathscr{L}_2\sup_{t\in[t_0,T]} E\|x-z\|_t^2\mathbb{K}^* + \mathscr{L}_3\sup_{t\in[t_0,T]} E\|x-z\|_t^2\mathbb{H}^*\Bigg] \\
\leq & 4N^2\varepsilon\Bigg\{\theta^2\beta + \max\{1,\theta^2\}(T-t_0)\lambda\varrho(t)\Bigg\} \\
&+ 2N^2\max\{1,\theta^2\}(T-t_0)^2\mathscr{L}\Bigg[\sup_{t\in[t_0,T]} E\|x-z\|_t^2 \\
&+ \sup_{t\in[t_0,T]} E\|x-z\|_t^2\mathbb{K}^* + \sup_{t\in[t_0,T]} E\|x-z\|_t^2\mathbb{H}^*\Bigg]
\end{aligned}
$$

we get a constant $l = \frac{1}{1-2N^2\max\{1,\theta^2\}(T-t_0)^2\mathscr{L}\Bigg[1+\mathbb{K}^*+\mathbb{H}^*\Bigg]} > 0$ independent of $\lambda\varrho(t)$ such that

$$
\sup_{t\in[t_0,T]} E\|x-z\|_t^2 \leq 4N^2 l\varepsilon\Bigg\{\theta^2\beta + \max\{1,\theta^2\}(T-t_0)\lambda\varrho(t)\Bigg\}, \qquad t\in[t_0,T]
$$

Hence the system (2) is Ulam-Hyers-Rassias stable in the mean square. □

Exponential Stability

Here we deals about exponential stability of the second moment of a mild solution of the system (2). For any $\mathscr{F}_t$- adapted process $\Theta(t) : [-p,\infty] \to \Re$ is almost surely continuous in t. For the stability purpose, let us assume $f(t,0,0,0) \equiv 0$ for all $t \geq t_0$. Therefore (2) have a trivial solution. Furthermore, $\Theta(t) = \psi(t-t_0)$ for $t \in [t_0 - p, t_0]$ and $e^{\widehat{\alpha}(t-t_0)} E\|\Theta\|_t^2 \to 0$ as $t \to \infty$, where $\widehat{\alpha} \in (0, \gamma)$.

Theorem 4.3. *If the assumptions $(\mathscr{H}_1)$ and the following inequalities (14),(15) holds. Then the system (2) is exponentially stable in the quadratic mean.*

$$\|\mathbb{T}(t)\| \leq N e^{-\gamma(t-t_0)}, \quad t \geq t_0, \text{ where } N \geq 1 \text{ and } \gamma > 0. \tag{14}$$

$$\frac{\max\{1, \theta^2\} N^2 \mathscr{L}\big[1 + \mathbb{K}^* + \mathbb{H}^*\big]}{(\gamma - \widehat{\alpha})} < \gamma \text{ and } N\theta \geq \frac{1}{\sqrt{2}} \tag{15}$$

Proof. The nonlinear operator S from $\mathscr{B}$ into $\mathscr{B}$ as given below
$(Sx)(t + t_0) = \psi(t) \;\; t \in [-p, 0]$ and $t \geq t_0$

$$\begin{aligned}(Sx)(t) = &\sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t - t_0)\psi(0) \\ &+ \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \\ &+ \int_{\xi_k}^{t} \mathbb{T}(t-r) f(r, x_r, Ux(r), Vx(r)) dr \Bigg] I_{[\xi_k, \xi_{k+1})}(t),\end{aligned}$$

Next we are proving the continuity of S in the quadratic mean on $[t_0, \infty]$. Take $x \in \mathscr{B}, t_1 \geq t_0$ and $|\Delta|$ be sufficiently small, from $(\mathscr{H}_1)$, (14) and (15) we get

$$\begin{aligned}&(Sx)(t_1 + \Delta) - (Sx)(t_1) \\ &\quad = \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t_1 + \Delta - t_0)\psi(0) \\ &\quad + \Bigg[\sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t_1 + \Delta - r) f(r, x_r, Ux(r), Vx(r)) dr \\ &\quad + \int_{\xi_k}^{t_1+\Delta} \mathbb{T}(t_1 + \Delta - r) f(r, x_r, Ux(r), Vx(r)) dr \Bigg]\Bigg] I_{[\xi_k, \xi_{k+1})}(t_1 + \Delta) \\ &\quad - \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} \alpha_i(\eta_i) \mathbb{T}(t_1 - t_0)\psi(0) \\ &\quad + \Bigg[\sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\eta_j) \int_{\xi_{i-1}}^{\xi_i} \mathbb{T}(t_1 - r) f(r, x_r, Ux(r), Vx(r)) dr \\ &\quad + \int_{\xi_k}^{t_1} \mathbb{T}(t_1 - r) f(r, x_r, Ux(r), Vx(r)) dr \Bigg]\Bigg] I_{[\xi_k, \xi_{k+1})}(t_1)\end{aligned}$$

Hence,

$$
\begin{aligned}
&(Sx)(t_1+\Delta)-(Sx)(t_1)\\
&=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t_1+\Delta-t_0)\psi(0)\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux(r),Vx(r))dr\\
&+\int_{\xi_k}^{t_1}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux(r),Vx(r))dr\Bigg]\\
&\times\Big(I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big)\\
&+\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\alpha_i(\eta_i)\Big(\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t_1-t_0)\Big)\psi(0)\\
&+\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\Big(\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\Big)f(r,x_r,Ux(r),Vx(r))dr\\
&+\int_{\xi_k}^{t_1}\Big(\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\Big)f(r,x_r,Ux(r),Vx(r))dr\\
&+\int_{\xi_k}^{t_1+\Delta}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux(r),Vx(r))dr\Bigg]\Bigg]I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)
\end{aligned}
$$

$$
\|(Sx)(t_1+\Delta)-(Sx)(t_1)\|^2\le 2E\|\mathscr{F}_1\|^2+2E\|\mathscr{F}_2\|^2, \tag{16}
$$

where,

$$
\begin{aligned}
&\mathscr{F}_1\\
&=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\alpha_i(\eta_i)\mathbb{T}(t_1+\Delta-t_0)\psi(0)\\
&\quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux_r,Vx_r)dr\\
&\quad+\int_{\xi_k}^{t_1}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux(r),Vx(r))dr\Bigg]\\
&\quad\times\Big(I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big)\\
&\mathscr{F}_2\\
&=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\alpha_j(\eta_j)\Big(\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t_1-t_0)\Big)\psi(0)
\end{aligned}
$$

$$+\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\eta_j)\int_{\xi_{i-1}}^{\xi_i}\Big(\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\Big)f(r,x_r,Ux(r),Vx(r))dr$$
$$+\int_{\xi_k}^{t_1}\Big(\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\Big)f(r,x_r,Ux(r),Vx(r))dr$$
$$+\int_{\xi_k}^{t_1+\Delta}\mathbb{T}(t_1+\Delta-r)f(r,x_r,Ux(r),Vx(r))dr\Big]\Big]I_{[\xi_k,\xi_{k+1})}(t_1+\Delta).$$

Now

$$\begin{aligned}
&E\|\mathscr{F}_1\|^2\\
&\le\Bigg(2E\Big[\max_k\Big\{\prod_{i=1}^{k}\|\alpha_i(\eta_i)\|^2\Big\}\Big]\|\mathbb{T}(t_1+\Delta-t_0)\|^2E\|\psi(0)\|^2\\
&\quad\times\big(I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)^2\Bigg)\\
&\quad+2E\Big\{\max_{i,k}\big[1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\big]\Big\}^2\Bigg(E\sum_{k=0}^{+\infty}\int_{t_0}^{t_1}\Big(\|\mathbb{T}(t_1+\Delta-r)\|\\
&\quad\times\|f(r,x_r,Ux(r),Vx(r))\|dr\big(I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)\Bigg)^2\\
&\le\Big[2\theta^2N^2e^{-2\gamma(t_1+\Delta-t_0)}E\|\psi(0)\|^2\\
&\quad+2\max\{1,\theta^2\}(t_1-t_0)E\int_{t_0}^{t_1}e^{-2\gamma(t_1+\Delta-r)}\|f(r,x_r,Ux(r),Vx(r))\|^2dr\Big]\\
&\quad\times E\big(I_{[\xi_k,\xi_{k+1})}(t_1+\Delta)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)\to 0 \text{ as } \Delta\to 0
\end{aligned}$$

and,

$$\begin{aligned}
&E\|\mathscr{F}_2\|^2\\
&\le\Bigg(3E\Big[\max_k\Big\{\prod_{i=1}^{k}\|\alpha_i(\eta_i)\|^2\Big\}\Big]\|\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2\\
&\quad+3E\Big[\Big\{\max_{i,k}\big[1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\big]\Big\}\Big]^2E\Big[\sum_{k=0}^{+\infty}\int_{t_0}^{t_1}\|\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\|\\
&\quad\times\|f(r,x_r,Ux_r,Vx_r)\|drI_{[\xi_k,\xi_{k+1})}(t_1+\Delta)\Big]^2\\
&\quad+3E\Big[\sum_{k=0}^{+\infty}\int_{t_0}^{t_1}\|\mathbb{T}(t_1+\Delta-r)\|\|f(r,x_r,Ux(r),Vx(r))\|drI_{[\xi_k,\xi_{k+1})}(t_1+\Delta)\Big]\\
&\le 3\theta^2\|\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2dr+3\max\{1,\theta^2\}(t_1-t_0)
\end{aligned}$$

$$
\begin{aligned}
&\times E\int_{t_0}^{t_1}\|\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\|^2\|f(r,x_r,Ux_r,Vx_r)\|^2dr\\
&+3(\Delta)E\int_{t_1}^{t_1+\Delta}\|\mathbb{T}(t_1+\Delta-r)\|^2\|f(r,x_r,Ux(r),Vx(r))\|^2dr\\
\le&3\theta^2\|\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2\\
&+3\max\{1,\theta^2\}(t-t_0)\int_{t_0}^{t_1}\|\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\|^2\\
&\times\Big[\mathscr{L}_1E\|x\|_r^2+\mathscr{L}_2E\|Ux\|_r^2+\mathscr{L}_3E\|Vx\|_r^2\Big]dr\\
&+3(\Delta)\int_{t_1}^{t_1+\Delta}\|\mathbb{T}(t_1+\Delta-r)\|^2\Big[\mathscr{L}_1E\|x\|_r^2+\mathscr{L}_2E\|Ux\|_r^2+\mathscr{L}_3E\|Vx\|_r^2\Big]dr\\
\le&3\theta^2\|\mathbb{T}(t_1+\Delta-t_0)-\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2\\
&+3\max\{1,\theta^2\}(t-t_0)\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]\\
&\times\int_{t_0}^{t_1}\|\mathbb{T}(t_1+\Delta-r)-\mathbb{T}(t_1-r)\|^2E\|x\|_r^2dr\\
&+3(\Delta)\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]\int_{t_1}^{t_1+\Delta}\|\mathbb{T}(t_1+\Delta-r)\|^2E\|x\|_r^2dr\ \to 0 \text{ as } \Delta\to 0
\end{aligned}
$$

Therefore, as $\Delta\to 0$, $2E\|\mathscr{F}_1\|^2+2E\|\mathscr{F}_2\|^2\to 0$.
So, we can say that S is continuous in the quadratic mean on $[t_0,\infty)$.
Now, prove S is a mapping from $\mathscr{B}$ into $\mathscr{B}$.

$$
\begin{aligned}
&e^{\widehat{\alpha}(t-t_0)}E\|Sx\|_t^2\\
&\qquad\le 2e^{\widehat{\alpha}(t-t_0)}E\Big[\max_k\big\{\prod_{i=1}^k\|\alpha_i(\eta_i)\|^2\big\}\Big]\|\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2\\
&\qquad\quad+2e^{\widehat{\alpha}(t-t_0)}E\Big[\max_{i,k}\big\{1,\prod_{j=i}^k\|\alpha_j(\eta_j)\|\big\}\Big]^2\\
&\qquad\quad\times E\Big(\int_{t_0}^t\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\|drI_{[\xi_k,\xi_{k+1})}(t)\Big)^2
\end{aligned}
$$

$$e^{\widehat{\alpha}(t-t_0)}E\|Sx\|_t^2=\mathscr{F}_3+\mathscr{F}_4 \tag{17}$$

Here $\mathscr{F}_3$ and $\mathscr{F}_4$ are defined as follows,

$$\mathscr{F}_3=2E\Big[\max_k\big\{\prod_{i=1}^k\|\alpha_i(\eta_i)\|^2\big\}\Big]e^{\widehat{\alpha}(t-t_0)}\|\mathbb{T}(t-t_0)\|^2E\|\psi(0)\|^2$$

choose $N\theta\ge\frac{1}{\sqrt{2}}$ we get,

$$\mathscr{F}_3\le 2N^2\theta^2e^{\widehat{\alpha}(t-t_0)}e^{-2\gamma(t-t_0)}E\|\psi(0)\|^2\quad\to\quad 0\quad\text{ as }\ t\to\infty \tag{18}$$

and

$$\begin{aligned}
\mathscr{F}_4 &= 2e^{\widehat{\alpha}(t-t_0)}E\Big[\max_{i,k}\{1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\}\Big]^2 \\
&\quad\times E\bigg(\int_{t_0}^{t}\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\|drI_{[\xi_k,\xi_{k+1})}(t)\bigg)^2 \\
&\le 2\max\{1,\theta^2\}e^{\widehat{\alpha}(t-t_0)}E\bigg(\int_{t_0}^{t}Ne^{-\gamma(t-r)}\|f(r,x_r,Ux(r),Vx(r))\|dr\bigg)^2 \\
&\quad = 2\max\{1,\theta^2\}e^{\widehat{\alpha}(t-t_0)}N^2 \\
&\quad\times E\bigg(\int_{t_0}^{t}e^{\dfrac{-\gamma(t-r)}{2}}e^{\dfrac{-\gamma(t-r)}{2}}\|f(r,x_r,Ux(r),Vx(r))\|dr\bigg)^2 \\
&\le 2\max\{1,\theta^2\}N^2e^{\widehat{\alpha}(t-t_0)}\bigg(\int_{t_0}^{t}e^{-\gamma(t-r)}\bigg) \\
&\quad\times\bigg(\int_{t_0}^{t}e^{-\gamma(t-r)}E\|f(r,x_r,Ux_r,Vx_r)\|^2dr\bigg) \\
&\le \frac{2\max\{1,\theta^2\}e^{\widehat{\alpha}(t-t_0)}N^2}{\gamma}\bigg(\int_{t_0}^{t}e^{-\gamma(t-r)}E\|f(r,x_r,Ux(r),Vx(r))\|^2dr\bigg) \\
&\quad = \frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}}{\gamma} \\
&\quad\times\int_{t_0}^{t}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|f(r,x_r,Ux(r),Vx(r))\|^2dr \\
&\le \frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}}{\gamma}\int_{t_0}^{t}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)} \\
&\quad\times\Big[\mathscr{L}_1E\|x\|_r^2+\mathscr{L}_2E\|Ux\|_r^2+\mathscr{L}_3E\|Vx\|_r^2\Big]dr
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}_4 \le& \frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma} \\
&\times\int_{t_0}^{t}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2dr
\end{aligned} \tag{19}$$

For all $x \in \mathscr{B}$, $\varepsilon > 0$, there exists a $t_1 > 0$ satisfying $e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2 < \varepsilon$ for $t \ge t_1$. That is using (19) we get

$$\begin{aligned}
\mathscr{F}_4 \le& \frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma} \\
&\times\int_{t_0}^{t_1}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2dr
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t_1-t)}\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma}\\
&\times\int_{t_0}^{t_1}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2dr\\
\le&\frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma}\\
&\times\int_{t_0}^{t_1}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2dr\\
&+\frac{2\max\{1,\theta^2\}N^2\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma}\left(\frac{1}{\gamma-\widehat{\alpha}}\right)\varepsilon. \qquad (20)
\end{aligned}
$$

Here $e^{-(\gamma-\widehat{\alpha})(t-t_0)}\to 0$ as $t\to\infty$ and (15) , there exists $t_2\ge t_1$ such that for all $t\ge t_2$ we get

$$
\begin{aligned}
&\frac{2\max\{1,\theta^2\}N^2e^{-(\gamma-\widehat{\alpha})(t-t_0)}\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma} \qquad (21)\\
&\times\int_{t_0}^{t_1}e^{(\gamma-\widehat{\alpha})(r-t_0)}e^{\widehat{\alpha}(r-t_0)}E\|x\|_r^2dr\\
&\le\varepsilon-\frac{2\max\{1,\theta^2\}N^2\mathscr{L}\big[1+\mathbb{K}^*+\mathbb{H}^*\big]}{\gamma}\left(\frac{1}{\gamma-\widehat{\alpha}}\right)\varepsilon. \qquad (22)
\end{aligned}
$$

From (20),(21) we get for all $t\ge t_2, \mathscr{F}_4<\varepsilon$ then,

$$\mathscr{F}_4\to 0 \text{ as } t\to\infty \qquad (23)$$

From (17), (18), and (23) we know that

$$e^{\widehat{\alpha}(t-t_0)}E\|(Sx)\|_t^2\to 0 \text{ as } t\to\infty \qquad (24)$$

Hence we can conclude that S is mappings $\mathscr{B}$ into $\mathscr{B}$.
Next we are going to prove that S is a contraction mapping. Take $x,y\in\mathscr{B}$, we get,

$$
\begin{aligned}
E\|(Sx)-(Sy)\|_t^2\le&E\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|\alpha_j(\eta_j)\|\right\}\right]^2\\
&\times E\left(\int_{t_0}^{t}\|\mathbb{T}(t-r)\|\|f(r,x_r,Ux(r),Vx(r))\right.\\
&\left.-f(r,x_r,Uy(r),Vy(r))\|drI_{[\xi_k,\xi_{k+1})}(t)\right)^2\\
\le&\max\{1,\theta^2\}N^2E\left(\int_{t_0}^{t}e^{-\gamma(t-r)}\|f(r,x_r,Ux(r),Vx(r))\right.\\
&\left.-f(r,x_r,Uy(r),Vy(r))\|dr\right)^2
\end{aligned}
$$

$$\leq \max\{1,\theta^2\}N^2 E\bigg(\int_{t_0}^{t} e^{\dfrac{-\gamma(t-r)}{2}} e^{\dfrac{-\gamma(t-r)}{2}}$$

$$\times \|f(r,x_r,Ux(r),Vx(r)) - f(r,x_r,Uy(r),Vy(r))\|dr\bigg)^2$$

$$\leq \frac{\max\{1,\theta^2\}N^2}{\gamma}\int_{t_0}^{t} e^{-\gamma(t-r)}\Big[\mathscr{L}_1 E\|x-z\|_r^2 + \mathscr{L}_2 E\|Ux-Uy\|_r^2$$

$$+ \mathscr{L}_3 E\|Vx-Vy\|_r^2\Big]dr$$

$$E\|(Sx)-(Sy)\|_t^2$$

$$\leq \frac{\max\{1,\theta^2\}N^2}{\gamma}\int_{t_0}^{t} e^{-\gamma(t-r)}\Big[\mathscr{L}_1 E\|x-y\|_r^2$$

$$+ \mathscr{L}_2 E\|x-y\|_r^2\mathbb{K}^* + \mathscr{L}_3 E\|x-y\|_r^2\mathbb{H}^*\Big]dr$$

$$\sup_{t\in[t_0,T]} E\|(Sx)-(Sy)\|_t^2$$

$$\leq \frac{\max\{1,\theta^2\}N^2}{\gamma}\int_{t_0}^{t} e^{-\gamma(t-r)}\Big[\mathscr{L}_1 \sup_{r\in[t_0,T]} E\|x-y\|_r^2$$

$$+ \mathscr{L}_2 \sup_{r\in[t_0,T]} E\|x-y\|_r^2\mathbb{K}^* + \mathscr{L}_3 \sup_{r\in[t_0,T]} E\|x-y\|_r^2\mathbb{H}^*\Big]dr$$

$$\leq \frac{\max\{1,\theta^2\}N^2\mathscr{L}\left[1+\mathbb{K}^*+\mathbb{H}^*\right]}{\gamma}\left(\frac{1}{\gamma-\widehat{\alpha}}\right)\sup_{t\in[t_0,T]} E\|x-y\|_t^2$$

Therefore S is a contraction mapping and S has a unique fixed point. Which is the solution of (2) with $x(t_0+t)=\psi(t)$ for $[-p,0]$ and $e^{\widehat{\alpha}(t-t_0)}E\|x\|_t^2 \to 0$. This complete the proof. □

5 Application

Consider semilinear functional special random impulsive differential equations,

$$\begin{cases} z_t(x,t) & = z_{xx}(x,t) \\ & \quad + \int_{-p}^{0}\beta(\theta)z\Big(t+\theta, x, \int_0^T u\big(t,r,z(r,x)\big)dr, \int_0^a v\big(t,r,z(r,x)\big)dr\Big)d\theta, \\ & \qquad t\neq\xi_k,\ t\geq\eta \\ z(x,\xi_k) & = q(k)\eta_k z(x,\xi_k^-) \qquad as\ x\in\widehat{\triangle} \\ z(x,t) & = \psi(x,t) \qquad as\ x\in\widehat{\triangle}, -p\leq t\leq 0 \\ z(x,t) & = 0 \qquad as\ x\in\partial\widehat{\triangle} \end{cases} \tag{25}$$

Consider $\widehat{\triangle} \subset \Re^n$ be a bounded domain with smooth boundary $\partial\widehat{\triangle}$, $X = L^2(\widehat{\triangle})$, η_k be random variable defined on $D_k \equiv (0, d_k)$ for $k \in \mathbb{N}$, $d_k \in (0, +\infty)$ and $\beta : [-p, 0] \to \Re$ is a positive function. Also assume that η_k follow Erlang distribution and if $i \neq j$ then η_i and η_j are independent with each other for $i, j = 1, 2, \ldots$. Here q is a function of k, $\xi_k = \xi_{k-1} + \eta_k$ for $k \in \mathbb{N}$, $t_0 \in \Re^+$.

Let A be an operator on X by $Az = \dfrac{\partial^2 z}{\partial x^2}$ with the domain

$$D(A) = \left\{ z \in X \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial x^2} \in X, z = 0 \text{ on } \partial\widehat{\triangle} \right\}$$

Thus A generates a strongly continuous semigroup $S(t)$ which is analytic, self adjoint and compact. Furthermore the operator A can be represented as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z \in D(A)$$

Here $z_n(\vartheta) = \sqrt{\frac{2}{\pi}} Sin(n\vartheta), n = 1, 2, \ldots,$ forms the orthonormal set of eigenvectors of A. Also for every $z \in X, S(t)z = \sum_{n=1}^{\infty} e^{(-n^2 t)} < z, z_n > z_n$, which holds $\|S(t)\| \leq e^{(-\pi^2(t-t_0))}, t \geq t_0$. Therefore $S(t)$ is a contraction semigroup. Consider that the following assumptions:

(i) $\beta(.)$is a continuous function in $\Re$ with

$$\int_{-p}^{0} \beta(\theta)^2 d\theta < \infty$$

also $\int_0^t u\big(t, s, z(r, x)\big)dr$, and $\int_0^T v\big(t, r, z(r, x)\big)dr$ are continuous and finite in $\Re$.

(ii) $E\Big[\max_{i,k} \big\{ \prod_{j=i}^{k} \|q(j)(\eta_j)\| \big\} \Big] < \infty.$

Assume that assumptions (i) and (ii) are satisfied, then the problem (25) becomes an abstract semilinear functional special random impulsive differential equation (2).

$$f(t, x_t, Ux(t), Vx(t)) = \int_{-p}^{0} \beta(\theta) z\Big(t + \theta, x, \int_0^t u\big(t, r, z(r, x)\big)dr, \int_0^T v\big(t, r, z(r, x)\big)dr\Big) d\theta.$$

6 Conclusion and Further Works

In this paper, we have studied existence and various type of stabilities of semilinear functional special random impulsive differential equations by using Banach fixed point theorem and some inequality techniques. Sufficient conditions for existence as well as stability results have been constructed. An example has been provided to demonstrate the theoretical analysis of the main results. It has been shown that semilinear functional special random impulsive differential equations can be practically implemented very easily. The system we are dealing here is less restrictive than those previously presented in the literature because of Volterra and Fredholm integral term.

The methods for finding stability does not end by one method, Lyapunov method and LMI techniques can also be used for solving complicated differential equations. It is interesting, for example, to investigate the influence of delays to accuracy of results, because better accuracy can be obtained if delays are more frequent on system on which the state of the process change more rapidly. Moreover, different kinds of disturbance in continuous or impulsive control require further studies. In connection with the presented methods and comparison results we can find many applications combining with neural networks and stochastic differential equations. We can also extend the main results of this paper to semilinear fractional differential equations.

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