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Existence results for an impulsive neutral functional integro-differential equations in Banach spaces

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Abstract. In this paper, we study the existence of mild solutions for a first order impulsive neutral functional integro-differential equations in Banach spaces. The results are obtained by using Krasnoselski-Schaefer fixed point theorem and semigroup theory.

1 Introduction

The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems from mechanics, electrical engineering, medicine, biology, ecology, etc. For more details on impulsive differential equations and on its applications, we refer the reader to [1, 2, 3, 11, 17, 18] and the references therein.

The theory of impulsive neutral functional differential and integrodifferential equations have been studied by many authors, see for instance [4, 5, 6, 7, 20]. The authors treated in these works [4, 5, 7, 8, 9] are really ordinary and not partial differential equations. Partial differential equations with impulses are studied, for instance, by Liu [18], Hernandez [14, 15, 16] and Ntouyas [21]. In [12], J.P. Dauer et al. studied

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the existence of mild solutions to semilinear neutral evolution equations with nonlocal conditions is proved. The result is obtained by using the Krasnoselski-Schaefer type fixed point theorem. Recently Hernandez [13], investigated the existence results for partial neutral functional integrodifferential equations with unbounded delay by using the Leray-Schauder nonlinear alternative fixed point theorem. More recently Ntouyas [21], proved the existence of solutions of impulsive partial neutral functional differential inclusions under the mixed Lipschitz and Caratheodory conditions. The results generalize those of [15, 21].

This paper is divided into three sections. Frist two sections are introduction and preliminaries. Sections 3 is devoted to study the existence of solutions for first order impulsive neutral functional integro-differential equations in Banach spaces and also, we study the same problem with nonlocal conditions.

In this paper, first we study the existence of solutions for first order neutral functional integrodifferential equations with impulsive effects as

$$\frac{d}{dt}[x(t)-g(t,x_t)] = Ax(t)+f\left(t,x_t,\int_0^t h(t,s,x_s)\mathrm{d}s\right), \tag{1.1}$$

$$t \in J=[0,b], \quad t \neq t_k, \quad k=1,2,\ldots,m,$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k=1,2,\ldots,m, \tag{1.2}$$

$$x(t)=\phi(t), \quad t\in[-r,0], \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t),\ t\geq 0\}$, on a Banach space X, $g: J\times D \to X, h: J\times J\times D\to X$ and $f: J\times D\times X\to X$ are given functions, where $D=\{\psi:[-r,0]\to X$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exists and $\psi(s^-)=\psi(s)\}$, $\phi\in D(0<r<\infty), 0=t_0<t_1<\cdots<t_m<t_{m+1}=b, I_k\in C(X,X)(k=1,2,\ldots,m)$ are bounded functions, $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t=t_k$, respectively.

For any continuous functions x defined on the interval $[-r,b]-\{t_1,t_2\ldots t_m\}$ and any $t\in J$, we denote by x_t the element of D defined by

$$x_t(\theta)=x(t+\theta), \quad \theta\in[-r,0].$$

Here $x_t(\cdot)$ represents the history of the time $t-r$, upto the present time t.
For $\psi\in D$, then

$$\|\psi\|_D=\sup\{|\psi(\theta)|:\ \theta\in[-r,0]\}.$$

2 Preliminaries and Hypotheses

Let X be a Banach space provided with norm $\|\cdot\|$. Let $A: D(A)\to X$ is the infinitesimal generator of an analytic semigroup $\{T(t),\ t\geq 0\}$, of bounded linear operators on

X. If $\{T(t),\ t \geq 0\}$, is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^{\alpha}$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^{\alpha}$. Further more, the subspace $D(-A)^{\alpha}$ is dense in X, and the expression

$$\|x\|_{\alpha} = \|(-A)^{\alpha}x\|, \quad x \in D(-A)^{\alpha}$$

defines a norm on $D(-A)^{\alpha}$. For more details of fractional power of operators and semigroup theory, we refer [22].

From this theory, we define the following Lemma.

Lemma 1. The following properties hold:

(i) If $0 < \beta < \alpha \leq 1$, then $X_{\alpha} \subset X_{\beta}$ and the imbedding is compact whenever the resolvent operator of A is compact.
(ii) For every $0 < \alpha \leq 1$ there exists $C_{\alpha} > 0$ such that

$$\|(-A)^{\alpha}T(t)\| \leq \frac{C_{\alpha}}{t^{\alpha}}, \quad 0 < t \leq b.$$

Lemma 2. [13] Let $v(\cdot), w(\cdot) : [0,b] \to [0,\infty)$ be continuous functions. If $w(\cdot)$ is non-decreasing and there are constants $\theta > 0, 0 < \alpha < 1$ such that

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t \in J,$$

then

$$v(t) \leq e^{\theta^n \Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} (\frac{\theta b^{\alpha}}{\alpha})^j w(t),$$

for every $t \in [0,b]$ and every $n \in N$ such that $n\alpha > 1$, and $\Gamma(\cdot)$ is the Gamma function.

We need the following Krasnoselski-Schaefer type fixed point theorem to prove our existence theorem.

Theorem 1. [10] Let Φ_1, Φ_2 be two operators satisfying:

(a) Φ_1 is contraction, and
(b) Φ_2 is completely continuous.

Then either

(i) the operator equation $\Phi_1 x + \Phi_2 x = x$ has a solution, or
(ii) the set $\zeta = \{u \in X : \lambda\Phi_1(\frac{u}{\lambda}) + \lambda\Phi_2 u = u\}$ is unbounded for $\lambda \in (0,1)$.

Now we list the following hypotheses:

(H1) There exist constants $0<\beta<1, c_1, c_2, L_g$ such that g is X_β -valued, $(-A)^\beta g$ is continuous, and
$(i)\|(-A)^\beta g(t,x)\| \leq c_1\|x\|_D + c_2, \quad (t,x)\in J\times D,$
$(ii)\|(-A)^\beta g(t,x_1)-(-A)^\beta g(t,x_2)\| \leq L_g\|x_1-x_2\|_D, (t,x_i)\in J\times D, i=1,2.$ with

$$L_g\{\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\} < 1.$$

(H2) A is the infinitesimal generator of an analytic semigroup $\{T(t), t\geq 0\}$, of bounded linear operators on X, and $0\in\rho(A)$ such that

$$\|T(t)\| \leq M,\ t\geq 0 \quad \text{and} \quad \|(-A)^{1-\beta}T(t)\| \leq \frac{C_{1-\beta}}{t^{1-\beta}}, \quad 0<t\leq b,$$

for some constants $M, C_{1-\beta}$ and every $t\in J=[0,b]$.
(H3) There exists a constant d_k such that $\|I_k(x)\| \leq d_k, k=1,2,\ldots,m$ for each $x\in X$.
(H4) (i) For each $(t,s)\in J\times J$, the function $h(t,s,\cdot): D\to X$ is continuous, and for each $x\in D$, the function $h(\cdot,\cdot,x): J\times J\to X$ is strongly measurable.
(ii) For each $t\in J$, the function $f(t,\cdot,\cdot): D\times X\to X$ is continuous, and for each $(x,y)\in D\times X$, the function $f(\cdot,x,y): J\to X$ is strongly measurable.
(iii) For every positive integer k there exists $\alpha_k\in L^1(0,b)$ such that

$$\sup_{\{\|x\|,\|y\|\}\leq k} \|f(t,x,y)\| \leq \alpha_k(t), \quad \text{for } t\in J \quad a.e.$$

(iv) There exists an integrable function $m:[0,b]\to[0,\infty)$ and a constant $\alpha>0$ such that

$$\|h(t,s,x)\| \leq \alpha m(s)\Omega_0(\|x\|_D), \quad 0\leq s<t\leq b,\ x\in D$$

where $\Omega_0:[0,\infty)\to(0,\infty)$ is a continuous and nondecreasing function.
(H5) $\|f(t,x,y)\| \leq p(t)\Omega(\|x\|_D+\|y\|)$ for almost all $t\in J$ and all $x\in D, y\in X$, where $p\in L^1(J,R^+)$ and $\Omega: R^+\to(0,\infty)$ is continuous and increasing with

$$\int_0^b m^*(s)ds < \int_{C_0}^\infty \frac{ds}{\Omega(s)+\Omega_0(s)},$$

where

$$C_0 = \frac{F}{1-c_1\|(-A)^{-\beta}\|}, \quad C_2 = \frac{1}{1-c_1\|(-A)^{-\beta}\|},$$
$$B_0 = e^{C_1^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1}(\frac{C_1 b^\beta}{\beta})^j,$$
$$m^*(t) = \max\{B_0C_2Mp(t), \alpha m(t)\}$$

and $F = M\|\phi\|_D\{1+c_1\|(-A)^{-\beta}\|\} + \{M+1\}\{c_2\|(-A)^{-\beta}\|\} + \frac{c_2C_{1-\beta}b^\beta}{\beta} + M\sum_{k=1}^m d_k$.

(H6) ζ is completely continuous and there exists a constant Q such that

$$\|\zeta(u_1,\ldots,u_p)(t)\| \le Q, \quad \text{for} \quad (u_1,\ldots,u_p) \in D^p,\ t \in [-r,0].$$

(H7) (i) The function $h(t,s,\cdot) : D \to X$ is continuous for almost all $(t,s) \in J \times J$ and for each $x \in D$, the function $h(\cdot,\cdot,x) : J \times J \to X$ is strongly measurable.
(ii) There exists an integrable function $q : [0,b] \to [0,\infty)$ and a constant $L > 0$ such that

$$\|h(t,s,x)\| \le Lq(s)\psi(\|x\|_D), \quad 0 \le s < t \le b,\ x \in D,$$

where $\psi : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function.

(H8) (i) The function $f(t,\cdot,\cdot) : D \times X \to X$ is continuous for almost all $t \in J$, and for each $(x,y) \in D \times X$, the function $f(\cdot,x,y) : J \to X$ is strongly measurable.
(ii) For every positive integer ρ there exists $\alpha_\rho \in L^1(J,R_+)$ such that

$$\sup_{\{\|x\|,\|y\|\}\le\rho} \|f(t,x,y)\| \le \alpha_\rho(t), \quad \text{for } t \in J \quad a.e.$$

(iii) $\|f(t,x,y)\| \le p(t)\omega(\|x\|_D + \|y\|)$ for almost all $t \in J$ and all $x \in D, y \in X$, where $p \in L^1(J,R_+)$ and $\omega : R_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^b m^*(s)\mathrm{d}s < \int_{C_0}^\infty \frac{d\tau}{\omega(\tau)+\psi(\tau)},$$

where

$$C_0 = \frac{F}{1 - c_1\|(-A)^{-\beta}\|}, \quad C_2 = \frac{1}{1 - c_1\|(-A)^{-\beta}\|},$$

$$H = e^{C_1^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} (\frac{C_1 b^\beta}{\beta})^j,$$

$$m^*(t) = \max\{HMC_2 p(t), Lq(t)\}$$

and $F = M\|\phi\|_D[1 + c_1\|(-A)^{-\beta}\|] + \{M+1\}\{c_2\|(-A)^{-\beta}\|\} + MQ + \frac{c_2 C_{1-\beta} b^\beta}{\beta} + M\sum_{k=1}^m d_k$.

3 Existence results

In order to define the solution of problems (1.1)-(1.3) and (3.1)-(3.3) we consider the following space
$PC([-r,b],X) = \{x : [-r,b] \to X$ such that $x(t)$ is continuous almost everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+), k = 1,2,\ldots,m$ exists and $x(t_k^-) = x(t_k)\}$.

For any $t \in J$, we have $x_t \in D$ and

$$\|x\|_{PC} = \sup\{|x(t)| : t \in [-r,b]\}.$$

Let $AC(J,X)$ is the space of all absolutely continuous functions $x : J \to X$.

Definition 1. A function $x \in PC([-r,b],X) \cap AC((t_k,t_{k+1}),X), k = 1,2,...,m$, is said to be solution of (1.1)-(1.3) if $x(t) - g(t,x_t)$ is absolutely continuous on $J - \{t_1,t_2,...,t_m\}$ and (1.1)-(1.3) are satisfied.

Theorem 1. If the assumptions $(H1)-(H5)$ are satisfied, then IVP (1.1)-(1.3) has at least one solution on $[-r,b]$.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $\Phi : PC([-r,b],X) \to PC([-r,b],X)$ defined by

$$\Phi x(t) = \begin{cases} \phi(t) & \text{if} \quad t \in [-r,0], \\ T(t)[\phi(0) - g(0,\phi(0))] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds & \\ \quad + \int_0^t T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds & \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) & t \in J. \end{cases}$$

From hypothesis $(H1)$ the following inequality holds.

$$\begin{aligned} \|AT(t-s)g(s,x_s)\| &\le \|(-A)^{1-\beta}T(t-s)\| \|(-A)^\beta g(s,x_s)\| \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}[c_1\|x_s\|_D + c_2] \end{aligned}$$

Then from Bochner theorem [19], it follows that $AT(t-s)g(s,x_s)$ is integrable on $[0,t)$.

Now we decompose Φ as $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1 x(t) = \begin{cases} 0 & \text{if } t \in [-r,0], \\ -T(t)g(0,\phi) + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds & \text{if } t \in J. \end{cases}$$

$$\Phi_2 x(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \int_0^t T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)d\tau)ds & \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) & \text{if } t \in J. \end{cases}$$

Now, we will show that the operators Φ_1 and Φ_2 satisfy all the conditions of Theorem 2.1 on $[-r,b]$.

First we show that Φ_1 is contraction on $PC([-r,b],X)$. Let $x,y \in X$. From hypothesis $(H1)$, we have

$$\begin{aligned} \|\Phi_1 x(t) - \Phi_1 y(t)\| &\le \|g(t,x_t) - g(t,y_t)\| + \|\int_0^t AT(t-s)[g(s,x_s) - g(s,y_s)]ds\| \\ &\le \|(-A)^{-\beta}\| L_g \|x_t - y_t\|_D + L_g\|x_t - y_t\|_D \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} ds \\ &\le L_g\|x_t - y_t\|_D \{\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\} \end{aligned}$$

Taking supremum over t, $\|\Phi_1 x - \Phi_1 y\| \leq L_0\|x-y\|_D$, $L_0 = L_g\{\|(-A)^{-\beta}\| + \frac{C_{1-\beta}b^{\beta}}{\beta}\}$. Since $L_0 < 1$, this shows that Φ_1 is contraction on $PC([-r,b],X)$.

Next, we show that Φ_2 is completely continuous on $PC([-r,b],X)$. First we prove that Φ_2 maps bounded sets into bounded sets in $PC([-r,b],X)$. Let B be a bounded set in $PC([-r,b],X)$. Now for each $u(t) \in \Phi_2 x(t)$, then for each $t \in J, x \in B$,

$$\begin{aligned} u(t) &= T(t)\phi(0) + \int_0^t T(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k^-)) \end{aligned}$$

Then

$$\|u(t)\| \leq M\|\phi\|_D + M\int_0^t \left\|f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\right\|\mathrm{d}s + M\sum_{k=1}^m d_k.$$

From hypothesis $(H4)(iii)$ we have

$$\|u(t)\| \leq M\|\phi\|_D + M\int_0^t \alpha_k(s)\mathrm{d}s + M\sum_{k=1}^m d_k.$$

for all $u \in \Phi_2(x) \subset \Phi_2(B)$. Hence $\Phi_2(B)$ is bounded.

Next, we show that Φ_2 maps bounded sets into equicontinuous sets. Let B be bounded, as above, and $h \in \Phi_2 x$ for some $x \in B$, then for each $t \in J$, we have

$$\begin{aligned} h(t) &= T(t)\phi(0) + \int_0^t T(t-s) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k^-)) \end{aligned}$$

Let $r_1, r_2 \in J - \{t_1, t_2, \dots, t_m\}, r_1 < r_2$. Then we have

$$\begin{aligned} &\|h(r_2) - h(r_1)\| \\ &\leq \|T(r_2) - T(r_1)\|\|\phi(0)\| \\ &\quad + \int_0^{r_1-\varepsilon} \|T(r_2-s) - T(r_1-s)\| \left\|f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\right\|\mathrm{d}s \\ &\quad + \int_{r_1-\varepsilon}^{r_1} \|T(r_2-s) - T(r_1-s)\| \left\|f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\right\|\mathrm{d}s \\ &\quad + \int_{r_1}^{r_2} \|T(r_2-s)\| \left\|f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\right\|\mathrm{d}s \\ &\quad + M\sum_{0<t_k<r_2-r_1} d_k + \sum_{0<t_k<r_1} \|T(r_2-t_k) - T(r_1-t_k)\| d_k. \end{aligned}$$

As $r_2 \to r_1$ and ε be small the right hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t), t > 0$ implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \neq t_i, i = 1,2,...,m$. Similarly one can prove that at $t = t_i$. The equicontinuity for the other cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ are obvious.

Next, we show that Φ_2 is continuous. Let $\{x_n\} \subset B$ and $x_n \to x$ in $PC([-r,b],X)$. Then by hypothesis $(H4)(iii)$, we have

$$f\left(s, x_{n_s}, \int_0^t h(t,s,x_{n_s})\mathrm{d}s\right) \to f\left(s, x_s, \int_0^t h(t,s,x_s)\mathrm{d}s\right), \quad n \to \infty$$

and

$$\|f\left(s, x_{n_s}, \int_0^t h(t,s,x_{n_s})\mathrm{d}s\right) - f\left(s, x_s, \int_0^t h(t,s,x_s)\mathrm{d}s\right)\| \leq 2\alpha_k(s).$$

By dominated convergence theorem, we obtain the continuity of Φ_2:

$$\begin{aligned}\|\Phi_2 x_n - \Phi_2 x\| \leq \sup_{t\in J}\Big[\|\int_0^t T(t-s)[f\left(s, x_{n_s}, \int_0^s h(s,\tau,x_{n_\tau})\mathrm{d}\tau\right) \\ - f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)]\mathrm{d}s \\ + \sum_{0<t_k<t} T(t-t_k)[I_k(x_n(t_k)) - I_k(x(t_k))]\|\Big] \\ \to 0 \quad as \quad n \to \infty\end{aligned}$$

Thus Φ_2 is continuous. From the Arzela-Ascoli theorem it suffices to show that Φ_2 maps B into a precompact set in X. Let $0 < t \leq b$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $x \in B$ we define

$$\begin{aligned}(\Phi_2^\varepsilon x)(t) = T(t)\phi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon) f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ + T(\varepsilon)\sum_{0<t_k<t-\varepsilon} T(t-t_k-\varepsilon) I_k(x(t_k)).\end{aligned}$$

Then from the compactness of $T(t), t > 0$, the set $V_\varepsilon(t) = \{(\Phi_2^\varepsilon x)(t) : x \in B\}$ is precompact in X for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $x \in B$, we have

$$\begin{aligned}|(\Phi_2 x)(t) - (\Phi_2^\varepsilon x)(t)| &\leq \int_{t-\varepsilon}^t \|T(t-s)\|\|f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\|\mathrm{d}s \\ &\quad + \sum_{t-\varepsilon<t_k<t} \|T(t-t_k)\| d_k \\ &\leq \int_{t-\varepsilon}^t M\alpha_k(s)\mathrm{d}s + \sum_{t-\varepsilon<t_k<t} M d_k\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\Phi_2 x)(t) : x \in B\}$. Hence the set $\Phi_2(B)$ is precompact in X. Hence the operator Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$\zeta(\Phi) = \{x(\cdot) : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$$

is bounded for $\lambda \in (0,1)$. To this end let $x(\cdot) \in \zeta(\Phi)$. Then $\lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x$ for some $\lambda \in (0,1)$ and

$$\begin{aligned}
\|x(t)\| &= \|\lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x\| \\
&= \lambda\| - T(t)g(0, \frac{x(0)}{\lambda}) + g(t, \frac{x_t}{\lambda}) + \int_0^t AT(t-s)g(s, \frac{x_s}{\lambda})\mathrm{d}s \\
&\quad + T(t)\phi(0) + \int_0^t T(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \sum_{0<t_k<t} T(t-t_k)I_K(x(t_k^-))\| \\
&\leq M\|(-A)^{-\beta}\|[c_1\|\phi\|_D + c_2] + \|(-A)^{-\beta}\|[c_1\|x_t\|_D + c_2] + \frac{c_2 C_{1-\beta} b^\beta}{\beta} \\
&\quad + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s + M\|\phi\|_D \\
&\quad + M\int_0^t p(s)\Omega\left(\|x_s\|_D + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s + M\sum_{k=1}^m d_k \\
&\leq M\|(-A)^{-\beta}\|c_1\|\phi\|_D + Mc_2\|(-A)^{-\beta}\| + \|(-A)^{-\beta}\|c_1\|x_t\|_D \\
&\quad + c_2\|(-A)^{-\beta}\| + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s + \frac{c_2 C_{1-\beta} b^\beta}{\beta} + M\|\phi\|_D \\
&\quad + M\int_0^t p(s)\Omega\left(\|x_s\|_D + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s + M\sum_{k=1}^m d_k \\
&\leq F + c_1\|(-A)^{-\beta}\|\|x_t\|_D + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s \\
&\quad + M\int_0^t p(s)\Omega\left(\|x_s\|_D + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in J,
\end{aligned}$$

where

$$F = M\|\phi\|_D[1 + c_1\|(-A)^{-\beta}\|] + \{M+1\}\{c_2\|(-A)^{-\beta}\|\} + \frac{c_2 C_{1-\beta} b^\beta}{\beta} + M\sum_{k=1}^m d_k.$$

Let $\mu(t) = \max\{\|x(s)\| : -r \leq s \leq t\}$, $t \in J$. Then $\|x_t\|_D \leq \mu(t)$ for all $t \in J$ and there is a point $t^* \in [-r, t]$ such that $\mu(t) = \|x(t^*)\|$. Hence we have

$$\begin{aligned}\mu(t) &= \|x(t^*)\| \\ &\leq F + c_1\|(-A)^{-\beta}\|\|x_{t^*}\|_D + c_1C_{1-\beta}\int_0^{t^*}\frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s \\ &\quad + M\int_0^{t^*}p(s)\Omega\left(\|x_s\|_D + \int_0^s \alpha m(\tau)\Omega_0(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}&\leq F + c_1\|(-A)^{-\beta}\|\mu(t) + c_1C_{1-\beta}\int_0^t\frac{\mu(s)}{(t-s)^{1-\beta}}\mathrm{d}s \\ &\quad + M\int_0^t p(s)\Omega\left(\mu(s) + \alpha\int_0^s m(\tau)\Omega_0(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned}$$

or

$$\begin{aligned}\mu(t) &\leq \frac{F}{1-c_1\|(-A)^{-\beta}\|} + \frac{1}{1-c_1\|(-A)^{-\beta}\|}c_1C_{1-\beta}\int_0^t\frac{\mu(s)}{(t-s)^{1-\beta}}\mathrm{d}s \\ &\quad + \frac{M}{1-c_1\|(-A)^{-\beta}\|}\left\{\int_0^t p(s)\Omega\left(\mu(s)+\alpha\int_0^s m(\tau)\Omega_0(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right\} \\ &\leq C_0 + C_1\int_0^t\frac{\mu(s)}{(t-s)^{1-\beta}}\mathrm{d}s \\ &\quad + MC_2\int_0^t p(s)\Omega\left(\mu(s)+\alpha\int_0^s m(\tau)\Omega_0(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s,\ t\in J.\end{aligned}$$

where

$$C_0 = \frac{F}{1-c_1\|(-A)^{-\beta}\|},\quad C_1 = \frac{c_1C_{1-\beta}}{1-c_1\|(-A)^{-\beta}\|}\quad \text{and}\quad C_2 = \frac{1}{1-c_1\|(-A)^{-\beta}\|}.$$

From Lemma 2.2, we have

$$\mu(t) \leq B_0\left(C_0 + MC_2\int_0^t p(s)\Omega\left(\mu(s)+\alpha\int_0^s m(\tau)\Omega_0(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right),$$

where

$$B_0 = e^{C_1^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)}\sum_{j=0}^{n-1}\left(\frac{C_1b^\beta}{\beta}\right)^j.$$

Let us take the right hand side of the above inequality as $v(t)$. Then $v(0) = B_0C_0, \mu(t) \leq v(t), 0 \leq t \leq b$ and

$$\begin{aligned}v'(t) &\leq B_0C_2Mp(t)\Omega\left(\mu(t)+\alpha\int_0^t m(s)\Omega_0(\mu(s))\mathrm{d}s\right) \\ &\leq B_0C_2Mp(t)\Omega\left(v(t)+\alpha\int_0^t m(s)\Omega_0(v(s))\mathrm{d}s\right)\end{aligned}$$

Let $w(t) = v(t) + \alpha\int_0^t m(s)\Omega_0(v(s))ds$.Then $w(0) = v(0) = B_0C_0, v(t) \le w(t)$ for all $t \in J$, and

$$\begin{aligned} w'(t) &= v'(t) + \alpha m(t)\Omega_0(v(t)) \\ &\le B_0C_2Mp(t)\Omega(w(t)) + \alpha m(t)\Omega_0(w(t)) \\ &\le m^*(t)\{\Omega(w(t)) + \Omega_0(w(t))\}. \end{aligned}$$

Integrating from 0 to t, we obtain

$$\int_0^t \frac{w'(s)}{\Omega(w(s)) + \Omega_0(w(s))} ds \le \int_0^t m^*(s)ds$$
$$\int_{w(0)}^{w(t)} \frac{ds}{\Omega(s) + \Omega_0(s)} \le \int_0^b m^*(s)ds < \int_{B_0C_0}^{\infty} \frac{ds}{\Omega(s) + \Omega_0(s)}.$$

Hence there exists a constant M such that $v(t) \le M$ for all $t \in J$, and $\mu(t) \le v(t) \le M$ for all $t \in J$. Therefore

$$\|x\| = \sup_{t\in[-r,b]} \|x(t)\| = \mu(b) \le w(b) \le M \quad \text{for all } x \in \zeta(\Phi).$$

This shows that the set ζ is bounded in $PC([-r,b],X)$. Consequently, by Theorem 2.1, the operator Φ has a fixed point in $PC([-r,b],X)$. Thus the IVP (1.1)-(1.3) has a solution on $[-r,b]$. This completes the proof. □

Finally, we study the same problem (1.1)-(1.3) with nonlocal conditions of the form

$$\frac{d}{dt}[x(t) - g(t,x_t)] = Ax(t) + f\left(t, x_t, \int_0^t h(t,s,x_s)ds\right), \tag{3.1}$$
$$t \in J = [0,b], \quad t \neq t_k, \quad k = 1,2,\ldots,m,$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m, \tag{3.2}$$
$$x(t) + (\zeta(x_{\eta_1},\ldots,x_{\eta_p}))(t) = \phi(t), \quad t \in [-r,0], \tag{3.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t), t \ge 0\}$, on a Banach space $X, g : J \times D \to X, h : J \times J \times D \to X$ and $f : J \times D \times X \to X$ are given functions, where $D = \{\psi : [-r,0] \to X$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s^-)$ and $\psi(s^+)$ exists and $\psi(s^-) = \psi(s)\}$, $\phi \in D(0 < r < \infty), 0 < \eta_1 < \cdots < \eta_p \le b, p \in N, \zeta : D^p \to D, (D^p = D \times D \times D \times \cdots \times D,\ p$ times), $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b, I_k \in C(X,X)(k = 1,2\ldots m)$, are bounded functions. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

Now, we state and prove the existence result for the problem (3.1)-(3.3).

Theorem 2. If the assumptions $(H1)-(H3)$ and $(H6)-(H8)$ are satisfied, then IVP (3.1)-(3.3) has at least one solution on $[-r,b]$.

Proof. Transform the problem (3.1)-(3.3) into a fixed point problem. Consider the operator $\Phi : PC([-r,b],X) \to PC([-r,b],X)$ defined by

$$\Phi x(t) = \begin{cases} \phi(t) - (\zeta(x_{\eta_1},\dots,x_{\eta_p}))(t) & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - (\zeta(x_{\eta_1},\dots,x_{\eta_p}))(0) - g(0,\phi(0))] + g(t,x_t) & \\ \quad + \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s & \\ \quad + \int_0^t T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau)\mathrm{d}s & \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) & \text{if } t \in J. \end{cases}$$

From hypothesis $(H1)$ the following inequality holds.

$$\begin{aligned} \|AT(t-s)g(s,x_s)\| &\le \|(-A)^{1-\beta}T(t-s)\|\|(-A)^{\beta}g(s,x_s)\| \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\beta}}[c_1\|x_s\|_D + c_2] \end{aligned}$$

Now we decompose Φ as $\Phi = \Phi_1 + \Phi_2$ where

$$\Phi_1 x(t) = \begin{cases} 0 & \text{if } t \in [-r,0], \\ -T(t)g(0,\phi) + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)\mathrm{d}s & \text{if } t \in J. \end{cases}$$

$$\Phi_2 x(t) = \begin{cases} \phi(t) - (\zeta(x_{\eta_1},\dots,x_{\eta_p}))(t) & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - (\zeta(x_{\eta_1},\dots,x_{\eta_p}))(0)] & \\ \quad + \int_0^t T(t-s)f(s,x_s,\int_0^s h(s,\tau,x_\tau)\mathrm{d}\tau)\mathrm{d}s & \\ \quad + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)) & \text{if } t \in J. \end{cases}$$

Now, we will show that the operators Φ_1 and Φ_2 satisfy all the conditions of Theorem 2.1 on $[-r,b]$. From Theorem 3.1 easily we can prove that Φ_1 is contraction and Φ_2 is completely continuous.

To apply the Krasnoselski-Schaefer theorem, it remains to show that the set

$$G(\Phi) = \{x(\cdot) : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$$

is bounded for $\lambda \in (0,1)$. To this end let $x(\cdot) \in G(\Phi)$. Then $\lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x$ for some $\lambda \in (0,1)$ and

$$
\begin{aligned}
\|x(t)\| &= \|\lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x\| \\
&\leq M[\|\phi\|_D + Q] + M\|(-A)^{-\beta}\|[c_1\|\phi\|_D + c_2] + \|(-A)^{-\beta}\|[c_1\|x_t\|_D + c_2] \\
&\quad + \int_0^t \|(-A)^{1-\beta}T(t-s)\|\|(-A)^{\beta}g(s,x_s)\|\mathrm{d}s \\
&\quad + M\int_0^t p(s)\omega\left(\|x_s\|_D + \int_0^s Lq(\tau)\psi(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s + M\sum_{k=1}^m d_k \\
&\leq M[\|\phi\|_D + Q] + M\|(-A)^{-\beta}\|c_1\|\phi\|_D + Mc_2\|(-A)^{-\beta}\| + \frac{c_2 C_{1-\beta} b^{\beta}}{\beta} \\
&\quad + c_1\|(-A)^{-\beta}\|\|x_t\|_D + c_2\|(-A)^{-\beta}\| + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s \\
&\quad + M\int_0^t p(s)\omega\left(\|x_s\|_D + \int_0^s Lq(\tau)\psi(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s + M\sum_{k=1}^m d_k \\
&\leq F + c_1\|(-A)^{-\beta}\|\|x_t\|_D + c_1 C_{1-\beta}\int_0^t \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s \\
&\quad + M\int_0^t p(s)\omega\left(\|x_s\|_D + \int_0^s Lq(\tau)\psi(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in J.
\end{aligned}
$$

where

$$
\begin{aligned}
F &= M\|\phi\|_D[1 + c_1\|(-A)^{-\beta}\|] + \{M+1\}\{c_2\|(-A)^{-\beta}\|\} + \frac{c_2 C_{1-\beta} b^{\beta}}{\beta} \\
&\quad + MQ + M\sum_{k=1}^m d_k.
\end{aligned}
$$

Let $\mu(t) = \max\{\|x(s)\| : -r \leq s \leq t\}$, $t \in J$. Then $\|x_t\|_D \leq \mu(t)$ for all $t \in J$ and there is a point $t^* \in [-r,t]$ such that $\mu(t) = \|x(t^*)\|$. Hence we have

$$
\begin{aligned}
\mu(t) &= \|x(t^*)\| \\
&\leq F + c_1\|(-A)^{-\beta}\|\|x_{t^*}\|_D + c_1 C_{1-\beta}\int_0^{t^*} \frac{\|x_s\|_D}{(t-s)^{1-\beta}}\mathrm{d}s \\
&\quad + M\int_0^{t^*} p(s)\omega\left(\|x_s\|_D + \int_0^s Lq(\tau)\psi(\|x_\tau\|_D)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq F + c_1\|(-A)^{-\beta}\|\mu(t) + c_1 C_{1-\beta}\int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}}\mathrm{d}s \\
&\quad + M\int_0^t p(s)\omega\left(\mu(s) + L\int_0^s q(\tau)\psi(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s,
\end{aligned}
$$

or

$$
\begin{aligned}
\mu(t) \leq & \frac{F}{1-c_1\|(-A)^{-\beta}\|} + \frac{1}{1-c_1\|(-A)^{-\beta}\|} c_1 C_{1-\beta} \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} \mathrm{d}s \\
& + \frac{M}{1-c_1\|(-A)^{-\beta}\|} \left\{ \int_0^t p(s)\omega\left(\mu(s) + L\int_0^s q(\tau)\psi(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s \right\} \\
\leq & C_0 + C_1 \int_0^t \frac{\mu(s)}{(t-s)^{1-\beta}} \mathrm{d}s + MC_2 \int_0^t p(s)\omega\left(\mu(s) + L\int_0^s q(\tau)\psi(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s,
\end{aligned}
$$

where

$$
C_0 = \frac{F}{1-c_1\|(-A)^{-\beta}\|}, \quad C_1 = \frac{c_1 C_{1-\beta}}{1-c_1\|(-A)^{-\beta}\|} \quad \text{and} \quad C_2 = \frac{1}{1-c_1\|(-A)^{-\beta}\|}.
$$

From Lemma 2.2, we have

$$
\mu(t) \leq H\left(C_0 + MC_2 \int_0^t p(s)\omega\left(\mu(s) + L\int_0^s q(\tau)\psi(\mu(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right),
$$

where

$$
H = e^{C_1^n(\Gamma(\beta))^n b^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \left(\frac{C_1 b^\beta}{\beta}\right)^j.
$$

Let us take the right hand side of the above inequality as $v(t)$. Then $v(0) = HC_0$, $\mu(t) \leq v(t), 0 \leq t \leq b$ and

$$
\begin{aligned}
v'(t) &= HC_2 M p(t)\omega\left(\mu(t) + L\int_0^t q(s)\psi(\mu(s))\mathrm{d}s\right) \\
&\leq HC_2 M p(t)\omega\left(v(t) + L\int_0^t q(s)\psi(v(s))\mathrm{d}s\right)
\end{aligned}
$$

Let $w(t) = v(t) + L\int_0^t q(s)\psi(v(s))\mathrm{d}s$. Then $w(0) = v(0) + 0 = HC_0$, $v(t) \leq w(t)$ for all $t \in J$, and

$$
\begin{aligned}
w'(t) &= v'(t) + Lq(t)\psi(v(t)) \\
&\leq HC_2 M p(t)\omega(w(t)) + Lq(t)\psi(w(t)) \\
&\leq m^*(t)\{\omega(w(t)) + \psi(w(t))\}
\end{aligned}
$$

$$
\frac{w'(t)}{\omega(w(t)) + \psi(w(t))} \leq m^*(t).
$$

This implies that

$$\int_{w(0)}^{w(t)} \frac{d\tau}{\omega(\tau)+\psi(\tau)} \leq \int_0^b m^*(s)\mathrm{d}s < \int_{HC_0}^{\infty} \frac{d\tau}{\omega(\tau)+\psi(\tau)}.$$

Hence there exists a constant M such that $v(t) \leq M$ for all $t \in J$, and $\mu(t) \leq v(t) \leq M$ for all $t \in J$. Therefore

$$\|x\| = \sup_{t\in[-r,b]} \|x(t)\| = \mu(b) \leq w(b) \leq M \quad \text{for all } x \in G(\Phi).$$

This shows that the set G is bounded in $PC([-r,b],X)$. Consequently, by Theorem 3.2.3, the operator Φ has a fixed point in $PC([-r,b],X)$. Thus the IVP (3.4.1)-(3.4.3) has a solution on $[-r,b]$. This completes the proof. □

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References

[1] A. Anguraj and M. Mallika Arjunan, Existence and uniqueness of mild and classical solutions of impulsive evolution equations, *Electronic Journal Of Differential Equations*, **2005:111** (2005), 1-8.

[2] D.D. Bainov and P.S. Simeonov, *Systems with impulsive effect*, Ellis Horwood Ltd., England, 1989.

[3] D.D. Bainov and P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical Group, England, 1993.

[4] M. Benchohra and S.K. Ntouyas, Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces, *J. Math. Anal. Appl.* **263** (2001), 763-780.

[5] M. Benchohra, J. Henderson and S.K. Ntouyas, Semilinear impulsive neutral functional differential inclusions in Banach spaces, *Appl. Anal.* **81:4** (2002), 951-963.

[6] M. Benchohra, J. Henderson and S.K. Ntouyas, Existence results for impulsive semilinear neutral functional differential equations in Banach spaces, *Mem. Differ. Equ. Math. Phys.* **25** (2002), 105-120.

[7] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive neutral functional differential inclusions in Banach spaces, *Appl. Math. Lett.* **15:8** (2002), 917-924.

[8] M. Benchohra and A. Ouahab, Impulsive neutral functional differential equations with variable times, *Nonlinear Analysis*, **55:6** (2003), 679-693.

[9] M. Benchohra and A. Ouahab, Impulsive neutral functional differential inclusions with variable times, *Electronic J. Differential Equations*, **2003:67** (2003), 1-12.

[10] T.A. Burton and C.Kirk, A fixed point theorem of Krasnoselski-Schaefer type, *Math. Nachr.* **189** (1998), 23-31.

[11] Y.-K. Chang, A. Anguraj and M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Analysis: Hybrid Systems*, **2:1** (2008), 209-218.

[12] J.P. Dauer and N.I. Mahmdov, Integral inequalities and mild solutions of semilinear neutral evolution equations, *J. Math. Anal. Appl.* **300** (2004), 189-202.

[13] E. Hernández, Existence results for partial neutral functional integrodifferential equations with unbounded delay, *J. Math. Anal. Appl.* **292** (2004), 194-210.

[14] E. Hernández, A second order impulsive Cauchy problem, *Int. J. Math. Math. Sci.* **31:8** (2002), 451-461.

[15] E. Hernández and R. Henríquez, Impulsive partial neutral differential equations, *Appl. Math. Lett.* **19** (2006), 215-222.

[16] E. Hernández, H.R. Henríquez and R. Marco, Existence of solutions for a class of impulsive partial neutral functional differential equations, *J. Math. Anal. Appl.* **331:2** (2007), 1135-1158.

[17] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[18] J.H. Liu, Nonlinear impulsive evolution equations, *Dynam. Contin. Discrete Impuls. Systems*, **6:1** (1999), 77-85.

[19] C.M. Marle, *Measures et Probabilities*, Hermam, Paris, 1974.

[20] G. Marina, P. Pietramala and L. Muglia, Impulsive neutral integrodifferential equations on unbounded intervals, *Mediterr. j. math.* **1** (2004), 93-108.

[21] S.K. Ntouyas, Existence results for impulsive partial neutral functional differential inclusions, *Electronic J. Differential Equations*, **2005:30** (2005), 1-11.

[22] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, Newyork, 1983.