



Narayana prime cordial labeling of generalized Petersen Graph

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Abstract

Graph labelling is one of the major research area in Mathematics which was introduced by A . Rosa. A new concept of labelling namely Narayana Prime cordial labelling was introduced and studied by Murali, Thirusangu and Balamurugan. In this paper, we investigate the existence of Narayana prime cordial labelling of generalized Petersen graph.

Keywords

Graph labeling, Narayana numbers, Narayana prime cordial labeling, Generalized Petersen graphs.

AMS Subject Classification

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or equal to x . For various graph theoretic undefined notations and terminology we refer to Harary [5]. A dynamic survey of the labeling theory is found in Gallian [4]. $P(n;k)$ denotes the generalized Petersen graph. We will give brief summary of definitions and other information which are useful for the present investigations.

1. Introduction

We consider only loopless, finite, undirected and connected graph $G = (V; E)$ with the vertex set V and the edge set E . The number of elements of V , denoted as $|V|$ is called the order of the graph G while the number of elements of E , denoted as $|E|$ is called the length of the graph G . A graph labelling is an assignment of integers to the vertices or edges, or both, subject to certain conditions [1, 10]. Labelled graphs serves as useful models for broad range of applications such as astronomy, circuit design, communication network addressing over finite domains. Cahit introduced the concept of cordial labelling in the year 1987. The new concept of the graph labeling called Narayana prime cordial labeling was introduced and studied by B.J. Murali et al. [2, 3, 8, 12]. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for smallest integer greater than

2. Basic Notions

Definition 2.1. Narayana Numbers, closely related to Catalan Numbers are named after Indian mathematician T.V. Narayana (1930-1987) [7, 11]. $N(n,k)$ is the k^{th} Narayana number for a given n where $n, k \in N_0$ (set of non-negative integers). Each term for a fixed n and k is defined as

$$N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}; 0 \leq k < n$$

where $N(n,k) = \frac{n!}{(n-k)!k!}$

Tabular form of $N(n,k)$

n/k	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	3	1				
4	1	6	6	1			
5	1	10	20	10	1		
6	1	15	50	50	15	1	
7	1	21	105	175	105	21	1

Properties

The divisibility of Narayana prime numbers depends on the following results [9]:

1. Let p be prime and let $n = p^m - 1$ for some $m \in \mathbb{N}_0$. Then for all $k; 0 \leq k \leq n - 1, p \nmid N(n, k)$
2. Let p be prime and let $n = p^m$ for some $m \in \mathbb{N}_0$. Then for all $k; 1 \leq k \leq n - 2, p \mid N(n, k)$.

Narayana numbers is used in multiple input and output communication systems, in RNA secondary structure configuration and partition of graphs [6].

Definition 2.2. Let $G(V, E)$ be a graph. An injective function $f : V \rightarrow \mathbb{N}_0$, is said to be a Narayana prime cordial labeling of the Graph G if the induced edge function $f^* : E \rightarrow \{0, 1\}$ satisfies the following conditions:

(i) For every $uv \in E$

- $f^*(uv) = 1$ if $p \mid N(f(u), f(v))$, where $f(u) > f(v)$ and $f(u) = p^m$ for some $m \in \mathbb{N}_0,; 1 \leq f(v) \leq f(u) - 2$ where p is a prime number
- $= 1$ if $p \mid N(f(v), f(u))$, where $f(v) > f(u)$ and $f(v) = p^m$ for some $m \in \mathbb{N}_0,; 1 \leq f(u) \leq f(v) - 2$ where p is prime number.
- $= 0$ if $p \nmid N(f(u), f(v))$, where $f(u) > f(v)$ and $f(u) = p^m - 1$ for some $m \in \mathbb{N}_0,;$
 $0 \leq f(v) \leq f(u) - 1$ where p is a prime number
- $= 0$ if $p \nmid N(f(v), f(u))$, where $f(v) > f(u)$ and $f(v) = p^m - 1$ for some $m \in \mathbb{N}_0,;$
 $0 \leq f(u) \leq f(v) - 1$ where p is prime number.

(ii) $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ where $e_{f^*}(0)$ and $e_{f^*}(1)$ denote respectively the number of edges with the label 0 and the number of edges with the label 1.

Definition 2.3. A graph $G = (V, E)$ which admits a Narayana prime cordial labeling is called a Narayana prime cordial graph.

Definition 2.4. The Generalized Petersen graph $P(n, k)$ with $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ are defined to be a graph with $V(P(n, k)) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 1 \leq i \leq n, \text{subscripts modulo } n\}$. $P(n, k)$ has $2n$ vertices and $3n$ edges. These graphs were introduced by Coxeter (1950) and named by Watkins (1969).

3. Narayana Prime Labeling of Generalized Petersen Graph

Theorem 3.1. The Generalized Petersen graph $P(n, k)$ is Narayana Prime cordial when n and k are relatively prime.

Proof. By definition the Generalized Petersen graph $P(n, k)$ with $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ consists of an outer n cycle v_1, v_2, \dots, v_n and a set of n spokes $v_i u_i$ ($1 \leq i \leq n$) and n inner edges $u_i u_{i+k}$, $1 \leq i \leq n$ with indices taken modulo n

The labelling of the Generalized Petersen graphs $P(n, k)$ when $gcd(n, k) = 1$ is done as follows

Case 1: When n in $P(n, k)$ is even

The vertices in the outer n cycle v_1, v_2, \dots, v_n are first labelled as

$$f(v_i) = \begin{cases} 2^i; & 1 \leq i \leq \frac{n}{2} + 1 \\ 2^{j+1} - 1; & \frac{n}{2} + 2 \leq j \leq n - 1 \end{cases} \quad (3.1)$$

By labelling so, the first $n/2$ edges in outer cycle receive 1 and remaining $n/2$ edges in outer cycle receive 0

By definition of $P(n, k)$, u_1 is adjacent with u_{1+k}, u_{1+2k} and so on. Thus we have a sequence $u_1, u_{(1+k)}, u_{(1+2k)}, \dots, u_{(1+(n-1)k)}$ (subscript modulo n) of inner vertices. We label these n inner vertices which forms cycle of length n as

$$f(u) = \begin{cases} 2^{(n+2)+i}; & \text{for } u = u_{1+ik}, i = 0, 2, 4, 6, \dots, n - 2 \\ 2^{(n+2)+j} - 1; & \text{for } u = u_{1+jk}, j = 1, 3, 5, 7, \dots, n - 1 \end{cases}$$

By labelling so, the $n/2$ edges in inner cycle receive 1 and remaining $n/2$ edges receive 0.

For the n spokes $v_i u_i$, ($1 \leq i \leq n$), $f(u_i) > f(v_i)$ for each i , so $n/2$ spokes are labelled 1 and remaining $n/2$ spokes are labelled 0.

That is $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ for all $3n$ edges.

Therefore $P(n, k)$ is Narayana prime cordial in this case.

Case 2: when n in $P(n, k)$ is odd

By definition of $P(n, k)$, every inner vertex u_i is adjacent with inner vertex u_{i+k} (subscript modulo n). In this case first we label these n inner vertices which forms cycle of length n as

$$f(u) = \begin{cases} 2^{2+i}; & \text{for } u = u_{1+ik}, i = 0, 2, 4, 6, \dots, n - 1 \\ 2^{2+j} - 1; & \text{for } u = u_{1+jk}, j = 1, 3, 5, 7, \dots, n - 2 \end{cases}$$

By properties of Narayana numbers, the $\lfloor \frac{n}{2} \rfloor$ edges in inner cycle receive 1 and remaining $\lfloor \frac{n}{2} \rfloor$ edges in inner cycle receive 0.

The vertices in the outer n cycle v_1, v_2, \dots, v_n are labelled as

$$f(v_i) = \begin{cases} 2^{(n+2)+i} - 1; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ 2^{(n+2)+j}; & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n - 1 \end{cases} \quad (3.2)$$

By properties of Narayana numbers, the first $\lfloor \frac{n}{2} \rfloor$ edges in outer cycle receive 0 and remaining $\lfloor \frac{n}{2} \rfloor$ edges in outer cycle receive 1.



So the $2n$ edges in inner cycle and outer cycle together have n edges as zero and n edges as one.

Also the n spokes $v_i u_i$, ($1 \leq i \leq n$), all $f(v_i) > f(u_i)$ for each i , $\lfloor \frac{n}{2} \rfloor$ spokes are labelled 1 and remaining $\lceil \frac{n}{2} \rceil$ spokes are labelled 0.

That is $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ for all $3n$ edges. Therefore $P(n, k)$ is Narayana Prime cordial in this case. Hence the theorem is proved. \square

Example 3.2. Narayana Prime Cordial Labeling for Generalized Petersen Graph $P(8, 3)$ is given in figure 1. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(9, 2)$ is shown in figure 2 and Narayana Cordial Prime Labeling of Generalized Petersen Graph $P(7, 3)$ is given in figure 3.

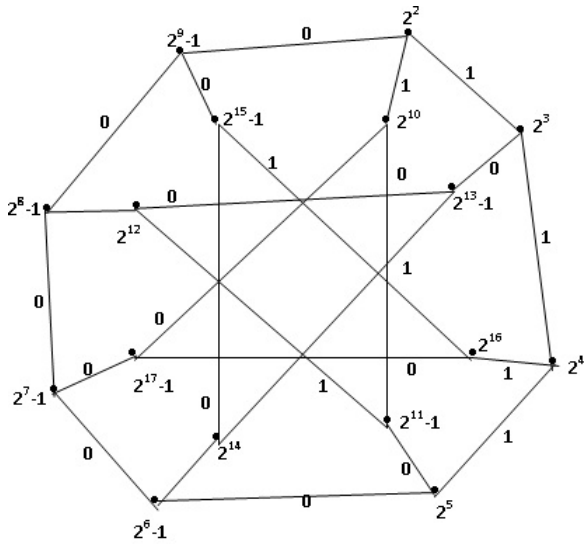


Figure 1. Narayana Cordial Prime Labeling of Generalized Petersen Graph $P(8, 3)$

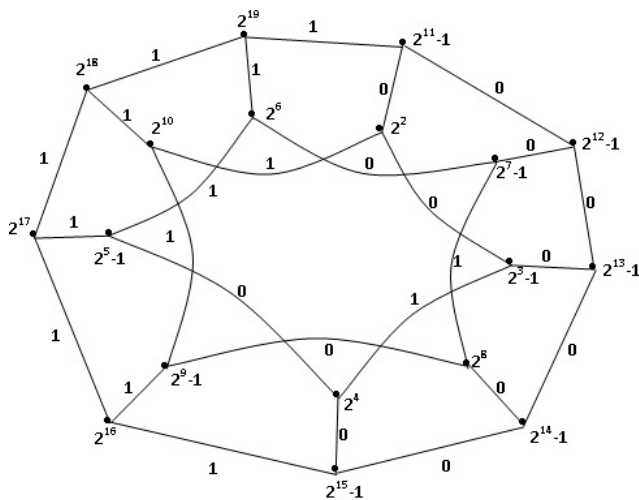


Figure 2. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(9, 2)$

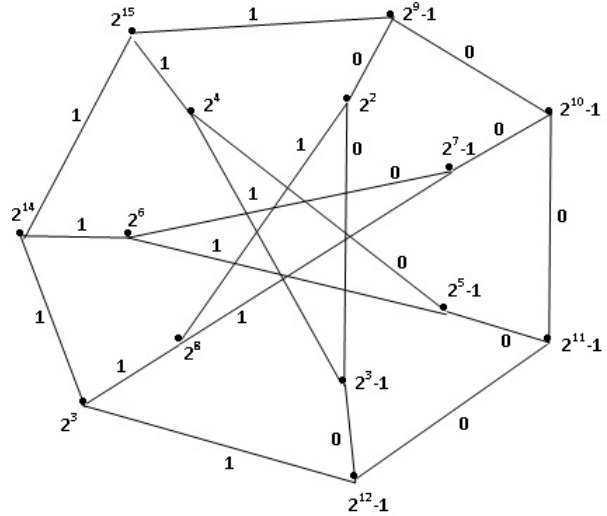


Figure 3. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(7, 3)$

Theorem 3.3. The Generalized Petersen graph $P(n, k)$ is Narayana Prime cordial, when $\gcd(n, k) = d (> 1)$.

Proof. When $\gcd(n, k) = d (> 1)$, then in $P(n, k)$ there is an outer cycle of length n and there are d disjoint cycles each of length $(\frac{n}{d})$ ($(\frac{n}{d}) > 2$), denoted by $C_{n/d}$. And for $(\frac{n}{d}) = 2$ there are d copies of K_2

Case 1: when n is even

In this case we first label the outer cycle containing n vertices as (3.1) in theorem 3.1.

The d disjoint cycles can be either of even length or of odd length. The vertices in the inner cycle are labelled as follows.

Subcase (i) when d is even

By definition of $P(n, k)$, every inner vertex u_i is adjacent with inner vertex u_{i+k} (subscript modulo n). The starting vertex u for each cycle is given by $s = 0, 2, 4, \dots, (d-2)$, For each cycle the remaining vertices in that cycle for a given s are labelled for different values of l_1 and l_2

$$f(u) = \begin{cases} 2^{(n+2)+s(\frac{n}{d})+l_1}; & u = u_{(s+1)+l_1 k}; \\ & l_1 = 0, 2, 4, \dots, ((\frac{n}{d}) - 2) \\ 2^{(n+2)+s(\frac{n}{d})+l_2} - 1; & u = u_{(s+1)+l_2 k}; \\ & l_2 = 1, 3, \dots, ((\frac{n}{d}) - 1) \end{cases}$$

And the starting vertex u for each cycle is given by $s = 1, 3, 5, \dots, (d-1)$ and remaining vertices on each cycle for a given s are labelled for various values of l_1 and l_2

$$f(u) = \begin{cases} 2^{(n+2)+s(\frac{n}{d})+l_1} - 1; & u = u_{(s+1)+l_1 k}; \\ & l_1 = 0, 2, 4, \dots, ((\frac{n}{d}) - 2) \\ 2^{(n+2)+s(\frac{n}{d})+l_2}; & u = u_{(s+1)+l_2 k}; \\ & l_2 = 1, 3, \dots, ((\frac{n}{d}) - 1) \end{cases}$$

Subcase (ii) when d is odd

By definition of $P(n, k)$, every inner vertex u_i is adjacent with inner vertex u_{i+k} (subscript modulo n). First we label these n



inner vertices which forms d disjoint cycles each of length $\frac{n}{d}$ ($\frac{n}{d} > 2$), denoted by $C_{\frac{n}{d}}$. And for $(\frac{n}{d}) = 2$, there are d copies of K_2 .

The starting vertex u for each inner cycle is given by, $s = 0, 2, \dots (d - 1)$ and remaining vertices on each cycle for a given s are labelled for various values of l_1 and l_2

$$f(u) = \begin{cases} 2^{2+n+s(\frac{n}{d})+l_1}; & u = u_{(s+1)+l_1k}; \\ & l_1 = 0, 2, 4, \dots ((\frac{n}{d}) - 2) \\ 2^{2+n+s(\frac{n}{d})+l_2} - 1; & u = u_{(s+1)+l_2k}; \\ & l_2 = 1, 3, \dots ((\frac{n}{d}) - 1) \end{cases}$$

And the starting vertex u for each inner cycle is given by, $s = 1, 3, 5 \dots (d - 2)$ and remaining vertices on each cycle for a given s are labelled for various values of l_1 and l_2

$$f(u) = \begin{cases} 2^{2+n+s(\frac{n}{d})+l_1}; & u = u_{(s+1)+l_1k}; \\ & l_1 = 0, 2, 4, \dots ((\frac{n}{d}) - 1) \\ 2^{2+n+s(\frac{n}{d})+l_2}; & u = u_{(s+1)+l_2k}; \\ & l_2 = 1, 3, \dots ((\frac{n}{d}) - 2) \end{cases}$$

For both the subcases (d is even or d is odd) when n is even, by properties of Narayana numbers $\frac{n}{2}$ edges in inner cycles receive 1 and remaining $\frac{n}{2}$ edges in inner cycles receive 0. Also first $\frac{n}{2}$ edges in outer cycle receive 1 and remaining $\frac{n}{2}$ edges in outer cycle receive 0.

So the $2n$ edges in inner cycles and outer cycle together have n edges as zero and n edges as one.

Also the n spokes $v_i u_i$, ($1 \leq i \leq n$), $\frac{n}{2}$ spokes are labelled 1 and remaining $\frac{n}{2}$ spokes are labelled 0.

That is $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ for all $3n$ edges.

Therefore $P(n, k)$ is N -prime cordial in this case.

Case 2: when n is odd

In this case we first label the d disjoint inner cycles which is of odd length $\frac{n}{d}$ as follows.

The starting vertex u for each inner cycle is given by $s = 0, 2, \dots (d - 1)$ and remaining vertices on each cycle for a given s are labelled for various values of l_1 and l_2

$$f(u) = \begin{cases} 2^{2+s(\frac{n}{d})+l_1} - 1; & u = u_{(s+1)+l_1k}; \\ & l_1 = 0, 2, 4, \dots ((\frac{n}{d}) - 1) \\ 2^{2+s(\frac{n}{d})+l_2}; & u = u_{(s+1)+l_2k}; \\ & l_2 = 1, 3, \dots ((\frac{n}{d}) - 2) \end{cases}$$

And the starting vertex for each inner cycle is given by u_s , $s = 1, 3, 5 \dots (d - 2)$ and remaining vertices on each cycle are labelled for various values of l_1 and l_2

$$f(u) = \begin{cases} 2^{2+s(\frac{n}{d})+l_1}; & u = u_{(s+1)+l_1k}; \\ & l_1 = 0, 2, 4, \dots ((\frac{n}{d}) - 1) \\ 2^{2+s(\frac{n}{d})+l_2} - 1; & u = u_{(s+1)+l_2k}; \\ & l_2 = 1, 3, \dots ((\frac{n}{d}) - 2) \end{cases}$$

In this case we label the vertices in the outer cycle as (3.2).

By properties of Narayana numbers, the $\lceil \frac{n}{2} \rceil$ edges in inner cycle receive 1 and remaining $\lfloor \frac{n}{2} \rfloor$ edges in inner cycle receive 0.

By properties of Narayana numbers, the first $\lfloor \frac{n}{2} \rfloor$ edges in outer cycle receive 0 and remaining $\lceil \frac{n}{2} \rceil$ edges in outer cycle receive 1.

So the $2n$ edges in inner cycle and outer cycle together have n edges as zero and n edges as one.

Also the n spokes $v_i u_i$, ($1 \leq i \leq n$), spokes are labelled 1 and remaining $\lfloor \frac{n}{2} \rfloor$ spokes are labelled 0.

That is $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ for all $3n$ edges.

Therefore $P(n, k)$ is Narayana prime cordial. Hence the Theorem. \square

Example 3.4. Narayana Prime Cordial Labeling for Generalized Petersen Graph $P(8, 2)$ is given in figure 4. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(10, 4)$ is shown in figure 5. Narayana Cordial Prime Labeling of Generalized Petersen Graph $P(6, 3)$ is given in figure 6 and Narayana Cordial Prime Labeling of Generalized Petersen Graph $P(9, 3)$ is given in figure 7.

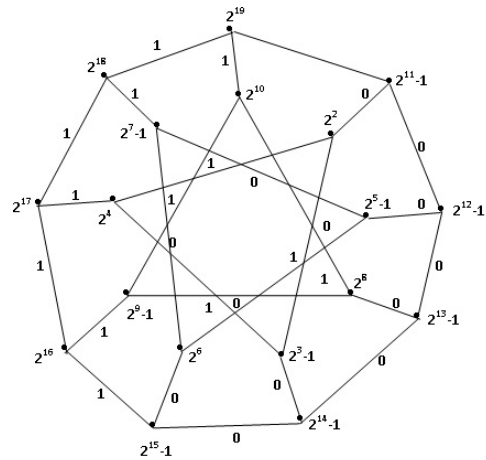


Figure 4. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(8, 2)$

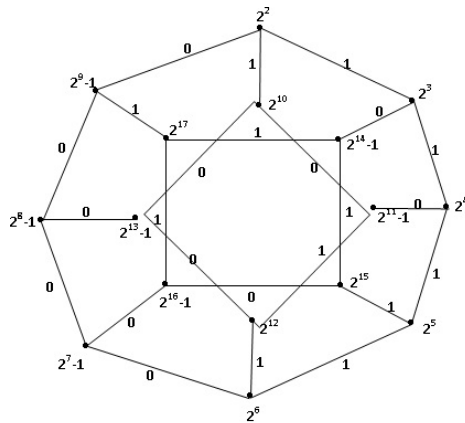


Figure 5. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(10, 4)$



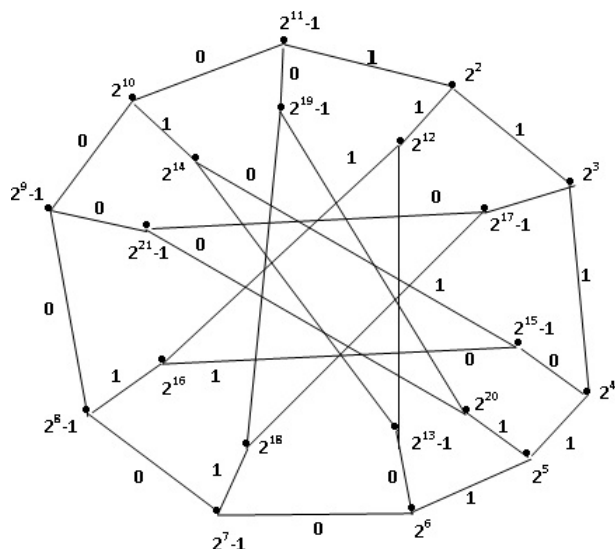


Figure 6. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(6,3)$

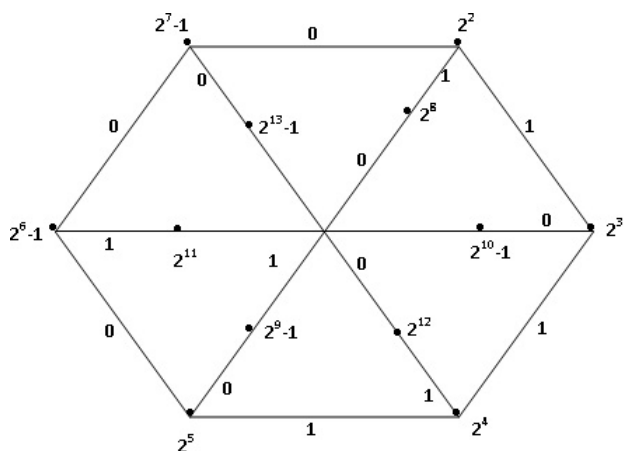


Figure 7. Narayana Prime Cordial Labeling of Generalized Petersen Graph $P(9,3)$

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