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Existence and Stability Results of Stochastic Differential Equations with Non-instantaneous Impulse and Poisson jumps

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Abstract: This paper focuses on a new class of non-instantaneous impulsive stochastic differential equations generated by mixed fractional Brownian motion with poisson jump in real separable Hilbert space. A set of sufficient conditions are generated based on the stochastic analysis technique, analytic semigroup theory of linear operators, fractional power of operators, and fixed point theory to obtain existence and uniqueness results of mild solutions for the considered system. Furthermore, the asymptotic behaviour of the system is investigated. Finally, an example is proposed to validate the obtained results.

Keywords: Stability; Stochastic Differential System; Fractional Brownian motion; non-instantaneous impulse

MSC: 37H30, 37H10, 60G22

1 Introduction

Because of environmental noise, deterministic models frequently fluctuate. As a result, it is essential to transition from the deterministic to the stochastic case. Stochastic Differential Equations (SDEs) have recently become popular in many fields of science and technology. For more references on SDEs one may refer the books [14, 16] and the scholarly articles [1, 4, 19, 20, 22, 23]. Fractional Brownian motion (fBm) is a generalization of Brownian motion established by Kolmogrov in 1940. The fBm of Hurst parameter $H \in (0, 1)$ is a centered Guassian random process with continuous sample paths. When $H \neq 1/2$, fBm is a generalization of classical Brownian motion with stationary increments or decrements and it reduces to Brownian motion when $H = 1/2$. These properties of long-range dependence and self-similarity, makes fBm a prospect model for noise in the fields of biological physics, financial markets, telecommunication and traffic networks. As a matter of fact, fBm has been made a natural modeling in many authentic circumstances such as describing economical background for the stock price, level of water in a dam as a function of time etc.
In certain scenarios, a dynamical system requires both Weiner process and fBm to model its dynamics. For more details refer [3, 5, 6, 11] and the references cited therein. Impulsive differential equations (IDEs) is an eminent tool to model processes with sudden discontinuities. Almost all physical systems evolving with respect

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to time experiences abrupt change called impulses. IDEs are classified as two classes subject to the length of the impulsive actions.
(i)Instantaneous Impulsive Differential Equations (IIDEs),
(ii) Non-instantaneous Impulsive Differential Equations (NIDEs).
IIDEs are impulses with the duration insignificant compared to the total duration of the entire phenomena. On the other hand, NIDEs are impulsive disturbances starting at a particular time and it remains active on a finite time period for instance, the process of inducing a vaccine and absorption of the drug by the body since being gradual, the drug can be considered to be a non-instantaneous impulse since it starts with a swift and remains active for a finite interval of time. This process can be modelled mathematically by non-instantaneous impulsive differential equations. Recently, many authors have established results on NIDEs see [7, 12, 13, 18]. Hernandez and O'Regan [10] introduced a new class of non-instantaneous impulsive differential equations of the form

$$\begin{aligned} \mathsf{w}'(t) &= \mathsf{A}\mathsf{w}(t) + \mathsf{Q}(t, \mathsf{w}(t)), \quad t \in (s_l, t_{l+1}], \ l = 0, 1, \cdots, \mathcal{K}, \\ \mathsf{w}(t) &= \mathcal{D}_l(t, \mathsf{w}(t)), \quad t \in (s_l, t_l], \ l = 1, 2, \cdots, \mathcal{K}, \\ \mathsf{w}(0) &= \mathsf{w}_0. \end{aligned} \tag{1.1}$$

Zhou.et.al [24] established the existence and exponential stability in the pth moment for impulsive stochastic integro-differential equations driven by mixed fBm. The existence and stability results of mild solutions for non-instantaneous impulsive SDEs driven by mixed fBm and poisson jumps are hardly available in the literature thereby motivating our work in this paper.
Influenced by the above facts, let us consider the following non-instantaneous impulsive SDEs driven by mixed fBm with poisson jump.

$$\begin{aligned} d[\mathsf{w}(t) + \mathsf{Q}_1(t, \mathsf{w}(t))] + \mathsf{A}\mathsf{w}(t)dt &= \mathsf{Q}_2(t, \mathsf{w}(t))dt + \mathsf{Q}_3(t, \mathsf{w}(t))d\mathsf{Z}(t) + \mathsf{G}(t)d\mathsf{Z}^{\mathsf{H}}(t) \\ &\quad + \int_{\mathcal{U}} \sigma(t, \mathsf{w}_t, \mathfrak{u})\tilde{\mathcal{N}}(dt, d\mathfrak{u}), \quad t \in \cup_{l=1}^{\mathcal{K}}(t_l, s_l] \\ \mathsf{w}(t) &= \mathcal{D}_l(t, \mathsf{w}(t_l^-)), \quad t \in \cup_{l=1}^{\mathcal{K}}(t_l, s_l], \\ \mathsf{w}(0) &= \mathsf{w}_0, \end{aligned} \tag{1.2}$$

where $\mathsf{w}(.)$ takes values in a real separable Hilbert space W, $-\mathsf{A}$ is the generator of an analytic semigroup $\{\mathsf{P}(t)\}_{t\geq 0}$ on W and $0 = s_0 = t_0 < t_1 < s_1 < t_2 < \cdots < t_{\mathcal{K}} < s_{\mathcal{K}} < t_{\mathcal{K}+1} = \mathfrak{b} < \infty$, $\mathcal{J}_3 = [0, \mathfrak{b}]$. The functions $\mathcal{D}_l(t, \mathsf{w}(t_l^-))$ denotes the non-instantaneous impulses during the time intervals $(t_l, s_l]$, $l = 1, 2, \cdots, \mathcal{K}$. $\{\mathsf{Z}(t)\}_{t\geq 0}$ is a Weiner process in a real separable space S_1. $\mathsf{Z}^{\mathsf{H}} = \{\mathsf{Z}^{\mathsf{H}}(t)\}_{t\geq 0}$ is a fBm in a real separable Hilbert space S_2 with Hurst index $\mathsf{H} \in (1/2, 1)$. The process Z^{H} and Z are independent. The functions $\mathsf{Q}_1 : \mathcal{J}_3 \times W \to W$, $\mathsf{Q}_2 : \mathcal{J}_3 \times W \to W$, $\mathsf{Q}_3 : \mathcal{J}_3 \times W \to \mathcal{L}_2^1(\mathsf{S}_1, W)$, $\mathsf{G} : \mathcal{J}_3 \to \mathcal{L}_2^2(\mathsf{S}_2, W)$ and $\mathcal{D}_l : (t_l, s_l] \times W \to W$, $l = 1, 2, \cdots \mathcal{K}$ are certain conditions to be used later.
The novelty of the paper is to study on the wellposedness results of non-instantaneous impulsive SDEs with mixed Brownian motion and poisson jumps in Hilbert space of the form (1.2). Here, the impulses start abrubtly at the point t_l and the action continue on a finite time interval $(t_l, s_l]$. Precisely, the function w acquires an impulse t_l thereby following two subintervals $(t_l, s_l]$ and $(s_l, t_{l+1}]$ of the interval $(t_l, t_{l+1}]$. In certain phenomena arising in financial markets, economics etc., the Weiner process and fractional Brownian motion are used to model the processes. Thus it is vital to explore the stability criteria for the considered non-instantaneous impulsive stochastic differential equations driven by mixed fractional Brownian motion and poisson jumps. The structure of this article is as follows: Section 2 depicts the sufficient preliminaries needed to interpret the main results. The existence and uniqueness of the proposed stochastic system are investigated in section 3. In the later section the asymptotic behaviour of the mild solution of the proposed system are established.

2 Preliminaries

[8] Let $\mathcal{L}(\mathsf{S}_k, W)$ be a space of all bounded linear operators from S_k to W, $k = 1, 2$. $\|.\|$ denote the norms of $W, \mathsf{S}_k, \mathcal{L}(\mathsf{S}_k, W)$. Let $(\Omega, \Im, \{\Im\}_{t\geq 0}, \mathcal{P})$ be a complete probability space, where $\{\Im_t, t \in \mathcal{J}_3\}$ is a family of right continuous increasing σ-algebras with $\Im_t \subset \Im$. Let $\mathsf{Q}_k \in \mathcal{L}(\mathsf{S}_k, \mathsf{S}_k)$ be two operators defined by $\mathsf{Q}_k e_j^k = \lambda_j^k e_j^k$ with finite trace $Tr(\mathsf{Q}_k) = \sum_{j=1}^{\infty} \lambda_j^k < \infty$, where $\{e_j^k\}_{j\geq 1}$ is a complete orthonormal basis in S_k and $\{\lambda_j^k\}_{j\geq 1}$ are non-negative real numbers. $\mathsf{Z}(t)$ be a S_1- valued Brownian motion defined as

$$\mathsf{Z}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j^1} \mathcal{F}_j(t) e_j^1,$$

where $\mathcal{F}_j(t)$ are real independent Brownian motions. $\mathsf{Z}^{\mathsf{H}}(t)$ be infinite dimensional S_2-valued fBm be defined as

$$\mathsf{Z}^{\mathsf{H}}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j^2} \mathcal{F}_j^{\mathsf{H}}(t) e_j^2,$$

where $\mathcal{F}^{\mathsf{H}}(t)$ are real independent fBm's.
Let $\mathcal{F} = \{\mathcal{F}(t, t \in \mathcal{J}_3)\}$ be Brownian motions and $\mathcal{F}^{\mathsf{H}} = \{\mathcal{F}^{\mathsf{H}}(t, t \in \mathcal{J}_3)\}$ be one-dimensional fBm with Hurst index $\mathsf{H} \in (1/2, 1)$. The fBm $\mathcal{F}^{\mathsf{H}}(t)$ can be expressed in the integral form:

$$\mathsf{Z}^{\mathsf{H}}(t) = \int_0^t \mathcal{V}_{\mathsf{H}}(t, q) d\mathcal{F}(q),$$

where $\mathcal{V}_{\mathsf{H}}(t, q)$ be given by

$$\mathcal{V}_{\mathsf{H}}(t, q) = \mathcal{X}_{\mathsf{H}} q^{1/2-\mathsf{H}} \int_q^t (\tau - q)^{\mathsf{H}-3/2} \tau^{\mathsf{H}-1/2} d\tau \quad \text{for} \quad t > q.$$

We put $\mathcal{V}_{\mathsf{H}}(t, q) = 0$ if $t \leq q$. We note that $\frac{\partial \mathcal{V}_{\mathsf{H}}}{\partial t}(t, q) = \mathcal{X}_{\mathsf{H}}(t/q)^{\mathsf{H}-1/2}(t - q)^{\mathsf{H}-3/2}$, here, $\mathcal{X}_{\mathsf{H}} = [\mathsf{H}(2\mathsf{H} - 1)/\xi(2 - 2\mathsf{H}, \mathsf{H} - 1/2)]^{1/2}$ and $\xi(., .)$ be beta function. By [15] the Weiner integral of the function $\omega \in \mathcal{L}^2([0, b])$ w.r.t fBm Z^{H} be given by

$$\int_0^b \omega(q) d\mathsf{Z}^{\mathsf{H}}(q) = \int_0^b \mathcal{V}_{\mathsf{H}}^{\star} \omega(q) d\mathsf{Z}(q),$$

where $\mathcal{V}_{\mathsf{H}}^{\star}\omega(q) = \int_q^b \omega(t) \frac{\partial \mathcal{V}_{\mathsf{H}}}{\partial t}(t, q) dt$.
Let $\varphi_k \in \mathcal{L}(\mathsf{S}_k, W)$ and define

$$\|\varphi_k\|_{\mathcal{L}_2^k} = \left[\sum_{j=1}^{\infty} \left\| \sqrt{\lambda_j^k} \varphi_k e_j^k \right\|^2 \right]^{1/2}.$$

When $\|\varphi_k\|_{\mathcal{L}_2^k} < \infty$, φ_k is said to be Q_k-Hilbert-Schmidt operator and $\mathcal{L}_2^k(\mathsf{S}_k, W)$ is a real separable Hilbert space of all Q-Hilbert-Schmidt operators with inner product

$$\langle \varphi^1, \varphi^2 \rangle_{\mathcal{L}_2^k} = \sum_{j=1}^{\infty} \langle \varphi^1 e_j^k, \varphi^2 e_j^k \rangle.$$

The stochastic integral of function $\chi : \mathcal{J}_3 \to \mathcal{L}_2^2(\mathsf{S}_2, W)$ w.r.t fBm Z^{H} being defined by

$$\begin{aligned} \int_0^t \chi(q) d\mathsf{Z}^{\mathsf{H}}(q) &= \sum_{j=1}^{\infty} \int_0^t \sqrt{\lambda_j^2} \chi(q) e_j^2 d\mathcal{F}^{\mathsf{H}}(q) \\ &= \sum_{j=1}^{\infty} \int_0^t \sqrt{\lambda_j^2} \mathcal{V}_{\mathsf{H}}^{\star}(\chi e_j^2)(q) d\mathcal{F}^(q). \end{aligned} \tag{2.1}$$

Lemma 2.1. *[2] If $\chi : \mathcal{J}_3 \to \mathcal{L}_2^2(\mathsf{S}_2, W)$ satisfies $\int_0^b \|\chi(q)\|^2_{\mathcal{L}_2^2} dq < \infty$, then (2.1) is well defined and W-valued random variable and we obtain*

$$\mathbb{E}\left\|\int_0^t \chi(q) d\mathsf{Z}^{\mathsf{H}}(q)\right\|^2 \leq 2\mathsf{H}t^{2\mathsf{H}-1}\int_0^t \|\chi(q)\|^2_{\mathcal{L}_2^2} dq.$$

Lemma 2.2. *[9] For any $\alpha \geq 1$ and for arbitrary $\mathcal{L}_2^1$-valued predictable process $Y(.)$ such that*

$$\sup_{s\in[0,t]} \mathbb{E}\left\|\int_0^s Y(\tau)d\mathsf{Z}(\tau)\right\|^{2\alpha} \leq (\alpha(2\alpha-1))^{\alpha}\left(\int_0^t \left(\mathbb{E}\|Y(q)\|^{2\alpha}_{\mathcal{L}_2^1}\right)^{1/\alpha} dq\right)^{\alpha}, \quad t \in \mathcal{J}_3.$$

For $\alpha = 1$, we obtain $\sup_{s\in[0,t]} \mathbb{E}\|\int_0^s Y(\tau)d\mathsf{Z}(\tau)\|^2 \leq \int_0^t \mathbb{E}\|Y(q)\|^2_{\mathcal{L}_2^1}dq.$

If $-\mathsf{A}$ generates an analytic semigroup $\{\mathsf{P}(t)\}_{t\geq 0}$ on W. Without loss of generality we may assume that the semigroup is uniformly bounded and $0 \in \rho(-\mathsf{A})$ which means $-\mathsf{A}$ is invertible. For $0 \leq q \leq 1$, fractional power operator A^q can be defined as a closed linear invertible operator with domain $\mathcal{D}(\mathsf{A}^q)$ being dense in W. It is proved that $W_q = \mathcal{D}(\mathsf{A}^q)$ is a banach space with the norm

$$\|\mathsf{w}\|_q = \|\mathsf{A}^q\mathsf{w}\|$$

We may define the space $\mathcal{PC}(W)$ formed by all $\Im$-adapted measurable, W-valued stochastic process $\{\mathsf{w}(t) : t \in \mathcal{J}_3\}$ such that w is continuous at $t \neq t_l$, $\mathsf{w}(t_l^-) = \mathsf{w}(t_l)$ and $\mathsf{w}(t_l^+)$ exists $\forall\, l = 1, 2, \cdots, \mathcal{K}$, endowed with the norm

$$\|\mathsf{w}\|_{\mathcal{PC}} = \left(\sup_{0\leq t\leq b} \mathbb{E}\|\mathsf{w}(t)\|^2\right)^{1/2}.$$

Then $(\mathcal{PC}(W), \|.\|_{\mathcal{PC}})$ is a Banach space.

Definition 2.1. *An $\Im_t$-adapted stochastic process $\mathsf{w} : \mathcal{J}_3 \to W$ is considered to be a mild solution of the stochastic system (1.2) if for every $t \in \mathcal{J}_3$, $\mathsf{w}(t)$ fulfils $\mathsf{w}(0) = \mathsf{w}_0$, $\mathsf{w}(t) = \mathcal{D}_l(t, \mathsf{w}(t_l^-))$, $t \in (t_l, s_l]$, $l = 1, 2, \cdots, \mathcal{K}$ and $\mathsf{w}(t)$ also fulfils the consecutive integral equations*

$$\begin{aligned}\mathsf{w}(t) \quad = \quad & \mathsf{P}(t)[\mathsf{w}_0 + \mathsf{Q}_1(0, \mathsf{w}(0))] - \mathsf{Q}_1(t, \mathsf{w}(t)) + \int_0^t \mathsf{A}\mathsf{P}(t-s)\mathsf{Q}_1(s, \mathsf{w}(s))ds + \int_0^t \mathsf{P}(t-s)\mathsf{Q}_2(s, \mathsf{w}(s))ds \\ & + \int_0^t \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s) + \int_0^t \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^{\mathsf{H}}(s) + \int_0^t\int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathrm{u})\tilde{\mathcal{N}}(ds, d\mathrm{u})\end{aligned}$$

$\forall\, t \in [0, t_1]$
and

$$\begin{aligned}\mathsf{w}(t) \quad = \quad & \mathsf{P}(t-s_l)[\mathcal{D}_l(s_l, \mathsf{w}(t_l^-)) + \mathsf{Q}_1(s_l, \mathsf{w}(s_l))] - \mathsf{Q}_1(t, \mathsf{w}(t)) + \int_{s_l}^t \mathsf{A}\mathsf{P}(t-s)\mathsf{Q}_1(s, \mathsf{w}(s))ds \\ & + \int_{s_l}^t \mathsf{P}(t-s)\mathsf{Q}_2(s, \mathsf{w}(s))ds + \int_{s_l}^t \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s) + \int_{s_l}^t \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^{\mathsf{H}}(s) \\ & + \int_{s_l}^t\int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathrm{u})\tilde{\mathcal{N}}(ds, d\mathrm{u}) \qquad (2.2)\end{aligned}$$

3 Main Results

This segment regards to prove the existence and uniqueness results of mild solutions for the system (1.2). The following hypotheses are imposed to prove our results.

(H1)The operator $-\mathsf{A}$ is the generator of an analytic semigroup $\{\mathsf{P}(t)\}_{t\geq 0}$ on W. $0 \in \rho(-\mathsf{A})$ and there exists $\mathbb{N}, \mathbb{N}_{1-p} > 0$ such that

$$\|\mathsf{P}(t)\| \leq \mathbb{N} \quad \text{and} \quad \|\mathsf{A}^{1-p}\mathsf{P}(t)\| \leq \frac{\mathbb{N}_{1-p}}{t^{1-p}}$$

(H2)The map $\mathsf{Q}_2 : \mathcal{J}_3 \times W \to W$ is a continuous function and there exists constants $\mathbb{N}_{\mathsf{Q}_2}, \mathsf{S}_{\mathsf{Q}_2} > 0$ such that

$$\mathbb{E}\left\|\mathsf{Q}_2(t,\mathsf{w})\right\|^2 \leq \mathsf{S}_{\mathsf{Q}_2}, \ \forall\ t \in \mathcal{J}_3, \ \mathsf{w} \in W.$$

$$\mathbb{E}\left\|\mathsf{Q}_2(t,\mathsf{w}_1) - \mathsf{Q}_2(t,\mathsf{w}_2)\right\|^2 \leq \mathbb{N}_{\mathsf{Q}_2}\mathbb{E}\left\|\mathsf{w}_1 - \mathsf{w}_2\right\|^2, \ \forall\ t \in \mathcal{J}_3, \ \mathsf{w}_1, \mathsf{w}_2 \in W.$$

(H3)The function $\mathsf{G} : \mathcal{J}_3 \to \mathcal{L}_2^2(\mathsf{S}_2, W)$ satisfies

$$\int_0^t \left\|\mathsf{G}(s)\right\|^2_{\mathcal{L}_2^2} ds < \infty, \quad t \in \mathcal{J}_3$$

there exists $\Lambda_{\mathsf{G}} > 0$ such that $\left\|\mathsf{G}(s)\right\|^2_{\mathcal{L}_2^2} \leq \Lambda_{\mathsf{G}}$ uniformly in $\mathcal{J}_3$.

(H4)there exists $0 < p < 1$ and $\mathbb{N}_{\mathsf{Q}_1}, \mathsf{S}_{\mathsf{Q}_1} > 0$ such that the W_p-valued continuous function $\mathsf{A}^p\mathsf{Q}_1$ satisfies

$$\mathbb{E}\left\|\mathsf{A}^p\mathsf{Q}_1(t,\mathsf{w})\right\|^2 \leq \mathsf{S}_{\mathsf{Q}_1} \ \forall\ t \in \mathcal{J}_3, \ \mathsf{w} \in W.$$

$$\mathbb{E}\left\|\mathsf{A}^p\mathsf{Q}_1(t,\mathsf{w}_1) - \mathsf{A}^p\mathsf{Q}_1(t,\mathsf{w}_2)\right\|^2 \leq \mathbb{N}_{\mathsf{Q}_1}\mathbb{E}\left\|\mathsf{w}_1 - \mathsf{w}_2\right\|^2, \ t \in \mathcal{J}_3, \ \mathsf{w}_1, \mathsf{w}_2 \in W$$

(H5)The maps $\mathcal{D}_l : (t_l, s_l] \times W \to W$, $l = 1, 2, \cdots, \mathcal{K}$ are continuous functions and there exists $\mathbb{N}_{\mathcal{D}_l}, \mathsf{S}_{\mathcal{D}_l} > 0$, $l = 1, 2, \cdots, \mathcal{K}$ such that

$$\mathbb{E}\left\|\mathcal{D}_l(t,\mathsf{w})\right\|^2 \leq \mathsf{S}_{\mathcal{D}_l}, \ \forall\ \mathsf{w} \in W, \ t \in (t_l, s_l]$$

$$\mathbb{E}\left\|\mathcal{D}_l(t,\mathsf{w}_1) - \mathcal{D}_l(t,\mathsf{w}_2)\right\|^2 \leq \mathbb{N}_{\mathcal{D}_l}\mathbb{E}\left\|\mathsf{w}_1 - \mathsf{w}_2\right\|^2, \ \forall\ \mathsf{w}_1, \mathsf{w}_2 \in W, \ t \in (t_l, s_l].$$

(H6)The map $\mathsf{Q}_3 : \mathcal{J}_3 \times W \to \mathcal{L}_2^1(\mathsf{S}_1, W)$ is a continuous function and there exists $\mathbb{N}_{\mathsf{Q}_3}, \mathsf{S}_{\mathsf{Q}_3} > 0$ such that

$$\mathbb{E}\left\|\mathsf{Q}_3(t,\mathsf{w})\right\|^2_{\mathcal{L}_2^1} \leq \mathsf{S}_{\mathsf{Q}_3} \ \forall\ \mathsf{w} \in W, \ t \in \mathcal{J}_3.$$

$$\mathbb{E}\left\|\mathsf{Q}_3(t,\mathsf{w}_1) - \mathsf{Q}_3(t,\mathsf{w}_2)\right\|^2_{\mathcal{L}_2^1} \leq \mathbb{N}_{\mathsf{Q}_3}\mathbb{E}\left\|\mathsf{w}_1 - \mathsf{w}_2\right\|^2, \ \forall\ \mathsf{w}_1, \mathsf{w}_2 \in W, \ t \in \mathcal{J}_3$$

(H7)The map $\sigma : \mathcal{J}_3 \times \mathcal{C} \times \mathcal{U} \to W$ is a continuous function and there exists constants $\mathbb{N}_\sigma, \mathsf{S}_\sigma > 0$ such that

$$\int_{\mathcal{U}} \left\|\sigma(t,\mathsf{w},\mathfrak{u}) - \sigma(t,\mathsf{w}_2,\mathfrak{u})\right\|^2 \nu(d\mathfrak{u}) \vee \left(\int_{\mathcal{U}} \left\|\sigma(t,\mathsf{w},\mathfrak{u}) - \sigma(t,\mathsf{w}_2,\mathfrak{u})\right\|^4 \nu(d\mathfrak{u})\right)^{1/2} \leq \mathbb{N}_\sigma\mathbb{E}\left\|\mathsf{w}_1 - \mathsf{w}_2\right\|^2$$

$$\left(\int_{\mathcal{U}} \left\|\sigma(t,\mathsf{w}_1,\mathfrak{u})\right\|^4 \nu(d\mathfrak{u})\right)^{1/2} \leq \mathsf{S}_\sigma, \ \forall\ \mathsf{w}_1, \mathsf{w}_2 \in W, \ t \in \mathcal{J}_3.$$

Let a positive constant $\mathfrak{R}$ satisfies the following estimate

$$\mathfrak{R} \geq \max_{1\leq l\leq \mathcal{K}}[\Delta_1, \mathcal{C}_{\mathcal{D}_l}\widehat{\Delta}_l],$$

where,

$$\begin{aligned}
\Delta_1 &= 7\left[2\mathbb{N}^2\mathbb{E}\|\mathrm{w}_0\|^2 + (2\mathbb{N}^2+1)\lambda \mathrm{S}_{\mathrm{Q}_1} + \mathbb{N}_{1-p}^2 \mathrm{S}_{\mathrm{Q}_1}\frac{t_1^{2p}}{2p-1} + \mathbb{N}^2\left[t_1^2\mathrm{S}_{\mathrm{Q}_1} + t_1\mathrm{SQ}_3 + 2\mathrm{H}t_1^{2\mathrm{H}}\Lambda_{\mathrm{G}} + \mathrm{S}_\sigma\right]\right]\\
\widehat{\Delta_l} &= 7\left[2\mathbb{N}^2\mathrm{S}_{\mathrm{P}_l} + (2\mathbb{N}^2+1)\lambda \mathrm{S}_{\mathrm{Q}_1} + \mathbb{N}_{1-p}^2 \mathrm{S}_{\mathrm{Q}_1}\frac{t_{l+1}^{2p}}{2p-1} + \mathbb{N}^2\left[t_{l+1}^2\mathrm{S}_{\mathrm{Q}_1} + t_{l+1}\mathrm{SQ}_3 + 2\mathrm{H}t_{1+1}^{2\mathrm{H}}\Lambda_{\mathrm{G}} + \mathrm{S}_\sigma\right]\right]
\end{aligned}$$

Theorem 3.1. *If the hypotheses (H1)-(H7) gets satisfied, the system (1.2) has a unique mild solution provided,*

$$\chi = \max_{1\le l\le \mathcal{K}}\left[\chi_1, \mathbb{N}_{\mathcal{D}_l}, \widehat{\chi_l}\right] < 1.$$

where,

$$\begin{aligned}
\chi_1 &= 5\left[\lambda\mathbb{N}_{\mathrm{Q}_1} + \mathbb{N}_{1-p}^2\mathbb{N}_{\mathrm{Q}_1}\frac{t_1^{2p}}{2p-1} + \mathbb{N}^2 t_1[\mathbb{N}_{\mathrm{Q}_2}t_1 + \mathbb{N}_{\mathrm{Q}_3} + \mathbb{N}_\sigma t_1]\right],\\
\widehat{\chi_1} &= 12\mathbb{N}^2[\mathbb{N}_{\mathcal{D}_l} + \lambda\mathbb{N}_{\mathrm{Q}_1}] + 6\left[\mathbb{N}_{\mathrm{Q}_1}\left(\lambda + \mathbb{N}_{1-p}^2\frac{t_{l+1}^{2p}}{2p-1}\right) + \mathbb{N}^2 t_{l+1}\left(\mathbb{N}_{\mathrm{Q}_2}t_{l+1} + \mathbb{N}_{\mathrm{Q}_3} + \mathbb{N}_\sigma t_{l+1}\right)\right].
\end{aligned}$$

Proof. For $\mathfrak{R} > 0$, let us define,

$$\varpi_{\mathfrak{R}} = \{\mathrm{w} \in \mathcal{PC}(W) : \|\mathrm{w}\|_{\mathcal{PC}}^2 \le \mathfrak{R}\}.$$

It is obvious that $\varpi_{\mathfrak{R}}$ is a bounded and closed subset of $\mathcal{PC}(W)$. Defining an operator Φ on $\varpi_{\mathfrak{R}}$

$$(\Phi\mathrm{w})(t) = \begin{cases}
\mathrm{P}(t)[\mathrm{w}_0 + \mathrm{Q}_1(0,\mathrm{w}(0))] - \mathrm{Q}_1(t,\mathrm{w}(t)) + \int_0^t \mathrm{AP}(t-s)\mathrm{Q}_1(s,\mathrm{w}(s))ds\\
+\int_0^t \mathrm{P}(t-s)\mathrm{Q}_2(s,\mathrm{w}(s))ds + \int_0^t \mathrm{P}(t-s)\mathrm{Q}_3(s,\mathrm{w}(s))d\mathrm{Z}(s)\\
+\int_0^t \mathrm{P}(t-s)\mathrm{G}(s)d\mathrm{Z}^{\mathrm{H}}(s) + \int_0^t\int_{\mathcal{U}} \mathrm{P}(t-s)\sigma(s,\mathrm{w}_s,\mathrm{u})\tilde{\mathbb{N}}(ds,d\mathrm{u}), \quad t\in[0,t_1],\ l=0,\\
\mathcal{D}_l(t,\mathrm{w}(t_l^-)), \qquad t\in(t_l,s_l],\ \ l\ge 1,\\
\mathrm{P}(t-s_l)[\mathcal{D}_l(s_l,\mathrm{w}(t_l^-)) + \mathrm{Q}_1(s_l,\mathrm{w}(s_l))] - \mathrm{Q}_1(t,\mathrm{w}(t))\\
+\int_{s_l}^t \mathrm{AP}(t-s)\mathrm{Q}_1(s,\mathrm{w}(s))ds + \int_{s_l}^t \mathrm{P}(t-s)\mathrm{Q}_2(s,\mathrm{w}(s))ds\\
+\int_{s_l}^t \mathrm{P}(t-s)\mathrm{Q}_3(s,\mathrm{w}(s))d\mathrm{Z}(s) + \int_{s_l}^t \mathrm{P}(t-s)\mathrm{G}(s)d\mathrm{Z}^{\mathrm{H}}(s)\\
+\int_{s_l}^t\int_{\mathcal{U}} \mathrm{P}(t-s)\sigma(s,\mathrm{w}_s,\mathrm{u})\tilde{\mathbb{N}}(ds,d\mathrm{u}), \qquad t\in(s_l,t_{l+1}],\ \ l\ge 1.
\end{cases}$$

STEP 1:

To prove Φ is well defined. For any $\mathrm{w}\in\varpi_{\mathfrak{R}}$, for $t\in[0,t_1]$, we have
$\mathbb{E}\left\|(\Phi\mathrm{w})(t)\right\|^2$

$$\begin{aligned}
&\le 7\mathbb{E}\left\|\mathrm{P}(t)[\mathrm{w}_0+\mathrm{Q}_1(0,\mathrm{w}(0))]\right\|^2 + 7\mathbb{E}\left\|\mathrm{Q}_1(t,\mathrm{w}(t))\right\|^2 + 7\mathbb{E}\left\|\int_0^t \mathrm{AP}(t-s)\mathrm{Q}_1(s,\mathrm{w}(s))ds\right\|^2\\
&+ 7\mathbb{E}\left\|\int_0^t \mathrm{P}(t-s)\mathrm{Q}_2(s,\mathrm{w}(s))ds\right\|^2 + 7\mathbb{E}\left\|\int_0^t \mathrm{P}(t-s)\mathrm{Q}_3(s,\mathrm{w}(s))d\mathrm{Z}(s)\right\|^2\\
&+ 7\mathbb{E}\left\|\int_0^t \mathrm{P}(t-s)\mathrm{G}(s)d\mathrm{Z}^{\mathrm{H}}(s)\right\|^2 + 7\mathbb{E}\left\|\int_0^t\int_{\mathcal{U}} \mathrm{P}(t-s)\sigma(s,\mathrm{w}_s,\mathrm{u})\tilde{\mathbb{N}}(ds,d\mathrm{u})\right\|^2\\
&\le 14\mathbb{N}^2\mathbb{E}\left\|\mathrm{w}_0\right\|^2 + 14\lambda\mathbb{N}^2\mathrm{S}_{\mathrm{Q}_1} + 7\lambda\mathrm{S}_{\mathrm{Q}_1} + 7\mathbb{N}_{1-p}^2\mathrm{S}_{\mathrm{Q}_1}\frac{t_1^{2p}}{2p-1} + 7\mathbb{N}^2 t_1\mathrm{S}_{\mathrm{Q}_3}\\
&+ 14\mathrm{H}\mathbb{N}^2 t_1^{2\mathrm{H}}\Lambda_{\mathrm{G}} + 7\mathbb{N}^2\mathrm{S}_\sigma\\
&\le 7\left[2\mathbb{N}^2\mathbb{E}\left\|\mathrm{w}_0\right\|^2 + (2\mathbb{N}^2+1)\lambda\mathrm{S}_{\mathrm{Q}_1} + \mathbb{N}_{1-p}^2\mathrm{S}_{\mathrm{Q}_1}\frac{t_1^{2p}}{2p-1} + \mathbb{N}^2[t_1^2\mathrm{S}_{\mathrm{Q}_1} + t_1\mathrm{S}_{\mathrm{Q}_3} + 2\mathrm{H}t_1^{2\mathrm{H}}\Lambda_{\mathrm{G}} + \mathrm{S}_\sigma]\right]\\
&= \Delta_1 \qquad (3.1)
\end{aligned}$$

Now, for $t \in (t_l, s_l]$, $l = 1, 2, \cdots, \mathcal{K}$, we obtain

$$\mathbb{E}\left\|(\Phi \mathrm{w})(t)\right\|^2 = \mathbb{E}\left\|\mathcal{D}_l(t, \mathrm{w}(t_l^-))\right\|^2 \le \mathrm{S}_{\mathcal{D}_l} \tag{3.2}$$

Similarly, for $t \in (s_l, t_{l+1}]$, $l = 1, 2, \cdots \mathcal{K}$,
$\mathbb{E}\left\|(\Phi \mathrm{w})(t)\right\|^2$

$$\begin{aligned}
&\le 7\mathbb{E}\left\|\mathrm{P}(t-s)[\mathcal{D}_l(s_l, \mathrm{w}(t_l^-)) + \mathrm{Q}_1(s_l, \mathrm{w}(s_l))]\right\|^2 + 7\left\|\mathrm{Q}_1(t, \mathrm{w}(t))\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t} \mathrm{AP}(t-s)\right. \\
&\times \left.\mathrm{Q}_1(s, \mathrm{w}(s))ds\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t} \mathrm{P}(t-s)\mathrm{Q}_2(s, \mathrm{w}(s))ds\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t} \mathrm{P}(t-s)\mathrm{Q}_3(s, \mathrm{w}(s))d\mathrm{Z}(s)\right\|^2 \\
&+ +7\mathbb{E}\left\|\int_{s_l}^{t} \mathrm{P}(t-s)\mathrm{G}(s)d\mathrm{Z}^{\mathrm{H}}(s)\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t}\int_{\mathcal{U}} \mathrm{P}(t-s)\sigma(s, \mathrm{w}_s, \mathfrak{u})\tilde{\mathbb{N}}(ds, d\mathfrak{u})\right\|^2 \\
&\le 14\mathbb{N}^2\mathrm{S}_{\mathcal{D}_l} + 14\lambda\mathbb{N}^2\mathrm{S}_{\mathrm{Q}_1} + 7\lambda\mathrm{S}_{\mathrm{Q}_1} + 7\mathbb{N}_{1-p}^2\mathrm{S}_{\mathrm{Q}_1}\frac{t_{l+1}^{2p}}{2p-1} + 7\mathbb{N}^2 t_{l+1}^2\mathrm{S}_{\mathrm{Q}_2} + 7\mathbb{N}^2 t_{l+1}\mathrm{S}_{\mathrm{Q}_3} \\
&+ 14\mathrm{H}\mathbb{N}^2 t_{l+1}^{2\mathrm{H}}\Lambda_{\mathrm{G}} + 7\mathbb{N}^2\mathrm{S}_\sigma \\
&\le 7\left[2\mathbb{N}^2\mathrm{S}_{\mathcal{D}_l} + (2\mathbb{N}^2+1)\lambda\mathrm{S}_{\mathrm{Q}_1} + \mathbb{N}_{1-p}^2\mathrm{S}_{\mathrm{Q}_1}\frac{t_{l+1}^{2p}}{2p-1} + \mathbb{N}^2\left(t_{l+1}^2\mathrm{S}_{\mathrm{Q}_2} + t_{l+1}\mathrm{SQ}_3 + 2\mathrm{H}t_{l+1}^{2\mathrm{H}}\Lambda_{\mathrm{G}} + \mathrm{S}_\sigma\right)\right] \\
&= \widehat{\Delta}_l
\end{aligned} \tag{3.3}$$

$\therefore$ from (3.1)-(3.3),

$$\mathbb{E}\left\|(\Phi \mathrm{w})(t)\right\|^2 \le \mathfrak{R}.$$

Thus, $\left\|(\Phi \mathrm{w})\right\|_{\mathcal{PC}}^2 \le \mathfrak{R}$. Hence, $\Phi : \varpi_{\mathfrak{R}} \to \mathfrak{R}$.

STEP 2:

To depict Φ is a contraction mapping on $\varpi_{\mathfrak{R}}$. For all $\mathrm{w}_1, \mathrm{w}_2 \in \varpi_{\mathfrak{R}}$, if $t \in [0, t_1]$, we have
$\mathbb{E}\left\|(\Phi \mathrm{w}_1)(t) - (\Phi \mathrm{w}_2)(t)\right\|^2$

$$\begin{aligned}
&\le 5\mathbb{E}\left\|\mathrm{Q}_1(t, \mathrm{w}(t)) - \mathrm{Q}_1(t, \mathrm{w}_2(t))\right\|^2 + 5\mathbb{E}\left\|\int_0^t \mathrm{AP}(t-s)\left[\mathrm{Q}_1(s, \mathrm{w}_1(s)) - \mathrm{Q}_1(s, \mathrm{w}_2(s))\right]ds\right\|^2 \\
&+ 5\mathbb{E}\left\|\int_0^t \mathrm{P}(t-s)\left[\mathrm{Q}_2(s, \mathrm{w}_1(s)) - \mathrm{Q}_2(s, \mathrm{w}_2(s))\right]ds\right\|^2 + 5\left\|\int_0^t \mathrm{P}(t-s)\left[\mathrm{Q}_3(s, \mathrm{w}_1(s))\right.\right. \\
&- \left.\left.\mathrm{Q}_3(s, \mathrm{w}_2(s))\right]d\mathrm{Z}(s)\right\|^2 + 5\mathbb{E}\left\|\int_0^t\int_{\mathcal{U}} \mathrm{P}(t-s)\left[\sigma(s, \mathrm{w}_{1_s}, \mathfrak{u}) - \sigma(s, \mathrm{w}_{2_s}, \mathfrak{u})\tilde{\mathbb{N}}(ds, d\mathfrak{u})\right]\right\|^2 \\
&\le 5\lambda\mathbb{N}_{\mathrm{Q}_1}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 + 5\mathbb{N}_{1-p}^2\frac{t_1^{2p-1}}{2p-1}\int_0^t \mathbb{E}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|^2 ds + 5\mathbb{N}^2\mathbb{N}_{\mathrm{Q}_2}\int_0^t \mathbb{E}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|^2 ds \\
&+ 5\mathbb{N}^2\mathbb{N}_{\mathrm{Q}_3}\int_0^t \mathbb{E}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|^2 d\mathrm{Z}(s) + 5\mathbb{N}^2\left(\mathbb{E}\left\|\int_0^t\int_{\mathcal{U}}\left[\sigma(s, \mathrm{w}_{1_s}, \mathfrak{u}) - \sigma(s, \mathrm{w}_{2_s}, \mathfrak{u})\right]\tilde{\mathbb{N}}(ds, d\mathfrak{u})\right\|^4\right)^{1/2} \\
&\le 5\lambda\mathbb{N}_{\mathrm{Q}_1}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 + 5\mathbb{N}_{1-p}^2\mathbb{N}_{\mathrm{Q}_1}\frac{t_1^{2p}}{2p-1}\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 + 5\mathbb{N}^2\mathbb{N}_{\mathrm{Q}_2}t_1^2\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 \\
&+ 5\mathbb{N}^2\mathbb{N}_{\mathrm{Q}_3}t_1\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 + 5\mathbb{N}^2\mathbb{N}_\sigma t_1^2\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2 \\
&\le \chi_1\left\|\mathrm{w}_1 - \mathrm{w}_2\right\|_{\mathcal{PC}}^2
\end{aligned} \tag{3.4}$$

where

$$\chi_1 = 5\lambda\mathbb{N}_{Q_1} + 5\mathbb{N}^2_{1-p}\mathbb{N}_{Q_1}\frac{t_1^{2p}}{2p-1} + 5\mathbb{N}^2\mathbb{N}_{Q_2}t_1^2 + 5\mathbb{N}^2\mathbb{N}_{Q_3}t_1 + 5\mathbb{N}^2\mathbb{N}_\sigma t_1^2$$

For, $t \in (t_l, s_l]$, $l = 1, 2, \cdots, \mathcal{K}$, then

$$\mathbb{E}\left\|(\Phi w_1)(t) - (\Phi w_2)(t)\right\|^2 \leq \mathbb{N}_{\mathcal{D}_l}\|w_1 - w_2\|^2_{\mathcal{PC}} \tag{3.5}$$

Correspondingly, for $t \in (s_l, t_l]$, $l = 1, 2, \cdots, \mathcal{K}$, we have
$\mathbb{E}\left\|(\Phi w_1)(t) - (\Phi w_2)(t)\right\|^2$

$$\begin{aligned}
&\leq 12\mathbb{E}\left\|P(t-s_l)\left[\mathcal{D}_l(s_l, w_1(t_l^-)) - \mathcal{D}_l(s_l, w_2(t_l^-))\right]\right\|^2 + 12\mathbb{E}\Bigg\|P(t-s_l)\Bigg[Q_1(s_l, w_1(s_l)) \\
&- Q_1(s_l, w_2(s_l))\Bigg]\Bigg\|^2 + 6\mathbb{E}\left\|Q_1(t, w_1(t)) - Q_1(t, w_2(t))\right\|^2 + 6\mathbb{E}\Bigg\|\int_{s_l}^{t} AP(t-s)\Bigg[Q_1(s, w_1(s)) \\
&- Q_1(s, w_2(s))\Bigg]ds\Bigg\|^2 + 6\mathbb{E}\left\|\int_{s_l}^{t} P(t-s)\left[Q_2(s, w_1(s)) - Q_2(s, w_2(s))\right]ds\right\|^2 + 6\mathbb{E}\Bigg\|\int_{s_l}^{t} P(t-s) \\
&\times \left[Q_3(s, w_1(s)) - Q_3(s, w_2(s))\right]dZ(s)\Bigg\|^2 + 6\mathbb{E}\left\|\int_{s_l}^{t}\int_{\mathcal{U}} P(t-s)\left[\sigma(s, w_{1_s}, u) - \sigma(s, w_{2_s}, u)\right]\tilde{\mathbb{N}}(ds, du)\right\|^2 \\
&\leq \left\{12\mathbb{N}^2[\mathbb{N}_{\mathcal{D}_l} + \lambda\mathbb{N}_{Q_1}] + 6\left[\lambda\mathbb{N}_{Q_1} + \mathbb{N}^2_{1-p}\mathbb{N}_{Q_1}\frac{t_{l+1}^{2p}}{2p-1} + \mathbb{N}^2 t_{l+1}[\mathbb{N}_{Q_2}t_{l+1} + \mathbb{N}_{Q_3} + \mathbb{N}_\sigma t_{l+1}]\right]\right\}\|w_1 - w_2\|^2_{\mathcal{PC}} \\
&\leq \widehat{\chi_l}\|w_1 - w_2\|^2_{\mathcal{PC}}
\end{aligned} \tag{3.6}$$

where,

$$\widehat{\chi_l} = 12\mathbb{N}^2[\mathbb{N}_{\mathcal{D}_l} + \lambda\mathbb{N}_{Q_1}] + 6\left[\lambda\mathbb{N}_{Q_1} + \mathbb{N}^2_{1-p}\mathbb{N}_{Q_1}\frac{t_{l+1}^{2p}}{2p-1} + \mathbb{N}^2 t_{l+1}[\mathbb{N}_{Q_2}t_{l+1} + \mathbb{N}_{Q_3} + \mathbb{N}_\sigma t_{l+1}]\right]$$

Encapsulating the inequalities (3.4)-(3.6), we obtain

$$\mathbb{E}\left\|(\Phi w_1)(t) - (\Phi w_2)(t)\right\|^2 \leq \chi\|w_1 - w_2\|^2_{\mathcal{PC}} \tag{3.7}$$

where, $\chi = \max_{1\leq l\leq\mathcal{K}}[\chi_1, \mathbb{N}_{\mathcal{D}_l}, \widehat{\chi_l}]$. From (3.7), Φ is a contraction mapping on $\varpi_{\mathfrak{R}}$. □

4 Stability

This segment is devoted to establish the asymptotic behaviour of mild solutions of stochastic system (1.2). The following hypotheses are imposed to prove the results.

(H8)A semigroup $\{P(t)\}_{t\geq 0}$ is said to be exponential stable if there exists $\varphi > 0$ and $\mathbb{N} > 0$ such that

$$\left\|P(t)\right\| \leq \mathbb{N}e^{-\varphi t}, \quad t \geq 0$$

(H9)There exists continuous functions $\xi_i : \mathfrak{R}^+ \to \mathfrak{R}^+$, $i = 1, 2, 3, 5$ and $\xi_{4,l} : \mathfrak{R}^+ \to \mathfrak{R}^+$, $l = 1, 2, \cdots, \mathcal{K}$ and non-negative real numbers $T_1, T_3, T_5 \geq 0$ such that

$(i)\mathbb{E}\left\|A^p Q_1(t, w)\right\|^2 \leq \xi_1(t)$

$(ii)\mathbb{E}\left\|Q_2(t, w)\right\|^2 \leq T_1\mathbb{E}\|w\|^2 + \xi_2(t)$

$(iii)\mathbb{E}\left\|Q_3(t, w)\right\|^2_{\mathcal{L}_2^{\frac{1}{2}}} \leq T_3\mathbb{E}\|w\|^2 + \xi_3(t)$

$(iv)\mathbb{E}\left\|\mathcal{D}_l(t, w)\right\|^2 \leq \xi_{4,l}(t), \quad l = 1, 2, \cdots, \mathcal{K}$

$(v)\int_{\mathcal{U}}\mathbb{E}\left\|\sigma(t, w, u)\right\|^2 \leq T_5\mathbb{E}\|w\|^2 + \xi_5(t)$

Moreover, there exists non-negative real numbers $q_1, q_2, q_3, q_{4,l}, q_5 \geq 0$, $l = 1, 2, \cdots, \mathcal{K}$ and $\delta > \varphi > 0$ such that $\xi_i(t) \leq q_i e^{-\delta t}$ $i = 1, 2, 3, 5$ and $\xi_{4,l}(t) \leq q_{4,l}e^{-\delta t}$, $l = 1, 2, \cdots, \mathcal{K}$.

(H10)The function $\mathsf{G} : \mathcal{J}_3 \to \mathcal{L}_2^2(\mathsf{S}_2, W)$ satisfies

$$\int_0^t e^{\varphi s} \left\|\mathsf{G}(s)\right\|^2_{\mathcal{L}_2^2} ds < \infty.$$

Lemma 4.1. *[17] Suppose* $-\mathsf{A}$ *be the generator of an analytic semigroup* $\{\mathsf{P}(t)\}_{t\geq 0}$ *on* W *and* $0 \in \rho(-\mathsf{A})$*, then*
(i)For any $0 < \delta \leq p$ *implies* $\mathcal{D}(\mathsf{A}^p) \subset \mathcal{D}(\mathsf{A}^\delta)$*, then embedding* $\mathsf{Z_P} \hookrightarrow \mathsf{Z}_\delta$ *is continuous.*
(ii)For $t > 0$, $\mathsf{A}^p\mathsf{P}(t)$ *is a bounded operator and* $\|\mathsf{A}^p\mathsf{P}(t)\| \leq \mathbb{N}_p t^{-p} e^{-\varphi t}$ *for any* $\varphi > 0$.

Theorem 4.1. *Assume the hypotheses (H1)-(H10) and* $\left\{\frac{7\mathbb{N}^2\mathsf{T}_1}{\varphi} + 7\mathbb{N}^2(\mathsf{T}_3 + \mathsf{T}_5)\right\} < \varphi$ *gets satisfied. In that case, the unique mild solution of the stochastic system 1.2 exponentially decays to zero in the mean square (i.e.) there exists* $\tilde{\mathfrak{K}}$ *and* $\tilde{\Theta}$ *being positive constants,*

$$\mathbb{E}\left\|\mathsf{w}(t)\right\|^2 \leq \tilde{\mathfrak{K}} e^{-\tilde{\Theta} t}, \quad t \geq 0$$

where, $\tilde{\mathfrak{K}} = \max\{\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3\}$*;* $\tilde{\Theta} = \min\{\Theta_1, \Theta_2, \Theta_3\}$, $\Theta_1 = \varphi - \gamma_1$*;* $\Theta_2 = \varphi - \gamma_2$*;* $\Theta_3 = \varphi$, $\mathfrak{K}_1 = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_6 + \mathsf{S}_2$, $\mathfrak{K}_2 = \mathcal{L}_7 + \mathcal{L}_{12} + \mathsf{S}_2$ *and* $\mathfrak{K}_3 = \mathsf{P}_{4,l}$.

Proof. For any $t \in [0, t_1]$, we have

$$\begin{aligned} \mathsf{w}(t) &= \mathsf{P}(t)[\mathsf{w}_0 + \mathsf{Q}_1(0, \mathsf{w}(0))] - \mathsf{Q}_1(t, \mathsf{w}(t)) + \int_0^t \mathsf{AP}(t-s)\mathsf{Q}_1(s, \mathsf{w}(s))ds + \int_0^t \mathsf{P}(t-s) \\ &\times \mathsf{Q}_2(s, \mathsf{w}(s))ds + \int_0^t \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s) + \int_0^t \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^\mathsf{H}(s) \\ &+ \int_0^t \int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathrm{u})\tilde{\mathbb{N}}(ds, d\mathrm{u}) \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\|\mathsf{w}(t)\|^2 &\leq 7\mathbb{E}\left\|\mathsf{P}(t)[\mathsf{w}_0 + \mathsf{Q}_1(0, \mathsf{w}(0))]\right\|^2 + 7\mathbb{E}\left\|\mathsf{Q}_1(t, \mathsf{w}(t))\right\|^2 + 7\mathbb{E}\left\|\int_0^t \mathsf{AP}(t-s)\mathsf{Q}_1(s, \mathsf{w}(s))ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t \mathsf{P}(t-s)\mathsf{Q}_2(s, \mathsf{w}(s))ds\right\|^2 + 7\mathbb{E}\left\|\int_0^t \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s)\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^t \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^\mathsf{H}(s)\right\|^2 + 7\mathbb{E}\left\|\int_0^t \int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathrm{u})\tilde{\mathbb{N}}(ds, du)\right\|^2 \\ &= \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 \end{aligned}$$

Through (H8),(H9),

$$\begin{aligned} \mu_1 &= 7\mathbb{E}\left\|\mathsf{P}(t)[\mathsf{w}_0 + \mathsf{Q}_1(0, \mathsf{w}(0))]\right\|^2 \\ &\leq 14\mathbb{N}^2 e^{-2\varphi t}\mathbb{E}\|\mathsf{w}_0\|^2 + 14\mathbb{N}^2\sigma e^{-2\varphi t}\mathsf{q}_1 \\ &\leq \mathcal{L}_1 e^{-\varphi t} \end{aligned} \tag{4.1}$$

where $\mathcal{L}_1 = 14\mathbb{N}^2\mathbb{E}\|\mathsf{w}_0\|^2 + 14\sigma\mathbb{N}^2\mathsf{q}_1$
Consider,

$$\begin{aligned} \mu_2 &= 7\mathbb{E}\left\|\mathsf{Q}_1(t, \mathsf{w}(t))\right\|^2 \leq 7\sigma\mathsf{q}_1 e^{-2\varphi t} \\ \mu_2 &\leq \mathcal{L}_2 e^{-\varphi t} \end{aligned} \tag{4.2}$$

thereby, $\mathcal{L}_2 = 7\sigma q_1$
Along with (H8) and Holder's inequality

$$\begin{aligned}
\mu_3 &= 7\mathbb{E}\left\|\int_0^t AP(t-s)Q_1(s,w(s))ds\right\|^2 \\
&= 7\mathbb{E}\left\|\int_0^t A^{1-p}P(t-s/2)P(t-s/2)A^pQ_1(s,w(s))ds\right\|^2 \\
&\leq 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\left(\int_0^t (t-s)^{-2(1-p)}e^{-\varphi(t-s)}ds\right)\left(\int_0^t e^{-\varphi(t-s)}\mathbb{E}\left\|A^pQ_1(s,w(s))\right\|^2 ds\right) \\
&\leq 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}}\int_0^t e^{-\varphi(t-s)}\mathbb{E}\left\|A^pQ_1(s,w(s))\right\|^2 ds
\end{aligned} \tag{4.3}$$

Also,

$$\mu_4 = 7\mathbb{E}\left\|\int_0^t P(t-s)Q_2(s,w(s))ds\right\|^2 \leq \frac{7\mathbb{N}^2}{\varphi}\int_0^t e^{-\varphi(t-s)}\mathbb{E}\left\|Q_2(s,w(s))\right\|^2 ds \tag{4.4}$$

By (H8),

$$\begin{aligned}
\mu_5 &= 7\mathbb{E}\left\|\int_0^t P(t-s)Q_3(s,w(s))dZ(s)\right\|^2 \leq 7\mathbb{N}^2\int_0^t e^{-2\varphi(t-s)}\mathbb{E}\|Q_3(s,w(s))\|_{\mathcal{L}_2^1}^2 ds \\
&\leq 7\mathbb{N}^2\int_0^t e^{-\varphi(t-s)}\mathbb{E}\|Q_3(s,w(s))\|_{\mathcal{L}_2^1}^2 ds
\end{aligned} \tag{4.5}$$

By (H10), a constant $\mathcal{L}_6 > 0$ is obtained such that

$$12H\mathbb{N}^2 t_1^{2H-1}\int_0^t e^{\varphi s}\|G(s)\|_{\mathcal{L}_2^2}^2 ds \leq \mathcal{L}_6 \quad \text{for } t \geq 0.$$

Hence,

$$\mu_6 \leq \mathcal{L}_6 e^{-\varphi t} \tag{4.6}$$

Consider,

$$\begin{aligned}
\mu_7 &= 7\mathbb{E}\left\|\int_0^t\int_{\mathcal{U}} P(t-s)\sigma(s,w(s),u)\tilde{\mathbb{N}}(ds,du)\right\|^2 \leq 7\mathbb{N}^2\int_0^t\int_{\mathcal{U}} e^{-2\varphi(t-s)}\mathbb{E}\left\|\sigma(s,w(s),u)\tilde{\mathbb{N}}(ds,du)\right\|^2 \\
&\leq 7\mathbb{N}^2\int_0^t\int_{\mathcal{U}} e^{-\varphi(t-s)}\mathbb{E}\left\|\sigma(s,w(s),u)\tilde{\mathbb{N}}(ds,du)\right\|^2
\end{aligned} \tag{4.7}$$

Through equations (4.1)-(4.7),

$$\begin{aligned} e^{\varphi t}\mathbb{E}\left\|\mathsf{w}(t)\right\|^2 \leq\ & \mathcal{L}_1+\mathcal{L}_2+\frac{7\mathbb{N}^2}{\varphi}\int_0^t e^{\varphi s}\mathbb{E}\left\|\mathsf{Q}_2(s,\mathsf{w}(s))\right\|^2 ds + 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}} \\ \times & \int_0^t e^{\varphi s}\mathbb{E}\left\|\mathsf{A}^p\mathsf{Q}_1(s,\mathsf{w}(s))\right\|^2 ds + \mathcal{L}_6 + 7\mathbb{N}^2\int_0^t e^{\varphi s}\mathbb{E}\|\mathsf{Q}_3(s,\mathsf{w}(s))\|_{\mathcal{L}_2^1}^2 ds \\ + & 7\mathbb{N}^2\int_0^t\int_{\mathcal{U}} e^{\varphi s}\mathbb{E}\left\|\sigma(s,\mathsf{w}(s),\mathfrak{u})\tilde{\mathbb{N}}(ds,d\mathfrak{u})\right\|^2 \end{aligned}$$

By (H9),

$$\begin{aligned} e^{\varphi t}\mathbb{E}\|\mathsf{w}(t)\|^2 \leq\ & \mathcal{L}_1+\mathcal{L}_2+\frac{7\mathbb{N}^2}{\varphi}\int_0^t e^{\varphi s}\left(\mathsf{T}_1\mathbb{E}\|\mathsf{w}\|^2+\xi_2(s)\right)ds + 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}}\int_0^t e^{\varphi s}\xi_1(s)ds \\ + & \mathcal{L}_6+7\mathbb{N}^2\int_0^t e^{\varphi s}\left(\mathsf{T}_3\mathbb{E}\|\mathsf{w}\|^2+\xi_3(s)\right)ds + 7\mathbb{N}^2\int_0^t e^{\varphi s}\left(\mathsf{T}_5\mathbb{E}\|\mathsf{w}\|^2+\xi_5(t)\right)ds \\ \leq\ & \mathcal{L}_1+\mathcal{L}_2+\frac{7\mathbb{N}^2}{\varphi}\int_0^t e^{\varphi s}\left(\mathsf{T}_1\mathbb{E}\|\mathsf{w}\|^2+\mathfrak{q}_2e^{-\delta s}\right)ds + 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}}\int_0^t e^{\varphi s}\mathfrak{q}_1e^{-\delta s}ds \\ + & \mathcal{L}_6+7\mathbb{N}^2\int_0^t e^{\varphi s}\left(\mathsf{T}_3\mathbb{E}\|\mathsf{w}\|^2+\mathfrak{q}_3e^{-\delta s}\right)ds + 7\mathbb{N}^2\int_0^t e^{\varphi s}\left(\mathsf{T}_5\mathbb{E}\|\mathsf{w}\|^2+\mathfrak{q}_5e^{-\delta s}\right)ds \\ \leq\ & \mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_6+\frac{7\mathbb{N}^2\mathfrak{q}_2}{\varphi(\delta-\varphi)}+7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)\mathfrak{q}_1}{\varphi^{2p-1}(\delta-\varphi)}+\frac{7\mathbb{N}^2\mathfrak{q}_3}{(\delta-\varphi)}+\frac{7\mathbb{N}^2\mathfrak{q}_5}{(\delta-\varphi)} \\ + & \left[\frac{7\mathbb{N}^2\mathsf{T}_1}{\varphi}+7\mathbb{N}^2(\mathsf{T}_3+\mathsf{T}_5)\right]\int_0^t e^{\varphi s}\mathbb{E}\left\|\mathsf{w}(s)\right\|^2 ds \\ \leq\ & \mathcal{K}_1+\gamma_1\int_0^t e^{\varphi s}\mathbb{E}\left\|\mathsf{w}(s)\right\|^2 ds \end{aligned}$$

where,

$$\mathcal{K}_1 = \mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_6+\frac{7\mathbb{N}^2\mathfrak{q}_2}{\varphi(\delta-\varphi)}+7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)\mathfrak{q}_1}{\varphi^{2p-1}(\delta-\varphi)}+\frac{7\mathbb{N}^2}{(\delta-\varphi)}[\mathfrak{q}_3+\mathfrak{q}_5]$$

$$\gamma_1 = \left[\frac{7\mathbb{N}^2\mathsf{T}_1}{\varphi}+7\mathbb{N}^2(\mathsf{T}_3+\mathsf{T}_5)\right]$$

Thus we have,

$$\mathbb{E}\|\mathsf{w}(t)\| \leq \mathcal{K}_1 e^{-\Theta_1 t} \tag{4.8}$$

where $\Theta = \varphi - \gamma_1$ Similarly, for any $t \in (s_l, t_{l+1}]$, $l = 1, 2, \cdots \mathcal{K}$,

$$\begin{aligned}
\mathsf{w}(t) &= \mathsf{P}(t-s_l)[\mathcal{D}_l(s_l, \mathsf{w}(t_l^-)) + \mathsf{Q}_1(s_l, \mathsf{w}(s_l))] - \mathsf{Q}_1(t, \mathsf{w}(t)) + \int_{s_l}^{t} \mathsf{A}\mathsf{P}(t-s)\mathsf{Q}_1(s, \mathsf{w}(s))ds \\
&+ \int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{Q}_2(s, \mathsf{w}(s))ds + \int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s) + \int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^{\mathsf{H}}(s) \\
&+ \int_{s_l}^{t}\int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathfrak{u})\tilde{\mathbb{N}}(ds, d\mathfrak{u})
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left\|\mathsf{w}(t)\right\|^2 &\leq 7\mathbb{E}\left\|\mathsf{P}(t-s_l)[\mathcal{D}_l(s_l, \mathsf{w}(t_l^-)) + \mathsf{Q}_1(s_l, \mathsf{w}(s_l))]\right\|^2 + 7\mathbb{E}\left\|\mathsf{Q}_1(t, \mathsf{w}(t))\right\|^2 + 7\mathbb{E}\Bigg\|\int_{s_l}^{t} \mathsf{A}\mathsf{P}(t-s) \\
&\times \mathsf{Q}_1(s, \mathsf{w}(s))ds\Bigg\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{Q}_2(s, \mathsf{w}(s))ds\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{Q}_3(s, \mathsf{w}(s))d\mathsf{Z}(s)\right\|^2 \\
&+ 7\mathbb{E}\left\|\int_{s_l}^{t} \mathsf{P}(t-s)\mathsf{G}(s)d\mathsf{Z}^{\mathsf{H}}(s)\right\|^2 + 7\mathbb{E}\left\|\int_{s_l}^{t}\int_{\mathcal{U}} \mathsf{P}(t-s)\sigma(s, \mathsf{w}_s, \mathfrak{u})\tilde{\mathbb{N}}(ds, d\mathfrak{u})\right\|^2
\end{aligned}$$

Using (H8)-(H10) and Holder's inequality

$$\begin{aligned}
e^{\varphi t}\mathbb{E}\left\|\mathsf{w}(t)\right\|^2 &\leq \mathcal{L}_7 + \mathcal{L}_{12} + \frac{7\mathbb{N}^2}{\varphi}\int_{s_l}^{t} e^{\varphi s}\mathbb{E}\left\|\mathsf{Q}_2(s, \mathsf{w}(s))\right\|^2 ds + 7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}} \\
&\times \int_{s_l}^{t} e^{\varphi s}\mathbb{E}\left\|\mathsf{A}^p\mathsf{Q}_1(s, \mathsf{w}(s))\right\|^2 ds + 7\mathbb{N}^2\int_{s_l}^{t} e^{\varphi s} + 7\mathbb{N}^2\int_{s_l}^{t} e^{\varphi s}\mathbb{E}\left\|\mathsf{Q}_3(s, \mathsf{w}(s))\right\|_{\mathcal{L}_2^1}^2 ds \\
&+ 7\mathbb{N}^2\int_{s_l}^{t}\int_{\mathcal{U}} e^{\varphi s}\mathbb{E}\left\|\sigma(s, \mathsf{w}_s, \mathfrak{u})\right\|^2 \tilde{\mathbb{N}}(ds, d\mathfrak{u})
\end{aligned}$$

where

$$\mathcal{L}_7 = 14\mathbb{N}^2[\mathsf{q}_{4,l} + \sigma\mathsf{q}_1] + 7\sigma\mathsf{q}_1$$

$$14\mathbb{N}^2\mathsf{H}t_{l+1}^{2\mathsf{H}-1}\int_{s_l}^{t} e^{\varphi s}\|\mathsf{G}(s)\|_{\mathcal{L}_2^2}^2 \leq \mathcal{L}_{12}, \forall\, t \geq 0.$$

Through (H9), we have

$$
\begin{aligned}
e^{\varphi t}\mathbb{E}\left\|\mathrm{w}(t)\right\|^2 &\le \mathcal{L}_7+\mathcal{L}_{12}+\frac{7\mathbb{N}^2}{\varphi}\int_{s_l}^{t} e^{\varphi s}\left[\mathsf{T}_1\mathbb{E}\left\|\mathrm{w}(s)\right\|^2+\xi_2(s)\right]ds+7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}}\\
&\times\int_{s_l}^{t} e^{\varphi s}\xi_1(s)ds+7\mathbb{N}^2\int_{s_l}^{t} e^{\varphi s}+7\mathbb{N}^2\int_{s_l}^{t} e^{\varphi s}\left[\mathsf{T}_3\mathbb{E}\left\|\mathrm{w}(s)\right\|^2+\xi_3(s)\right]ds\\
&+ 7\mathbb{N}^2\int_{s_l}^{t} e^{\varphi s}\left[\mathsf{T}_5\mathbb{E}\left\|\mathrm{w}(s)\right\|^2+\xi_5(s)\right]ds\\
&\le \mathcal{L}_7+\mathcal{L}_{12}+\frac{7\mathbb{N}^2\mathrm{q}_2}{\varphi(\delta-\varphi)}+7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}(\delta-\varphi)}+\frac{7\mathbb{N}^2}{\delta-\varphi}[\mathrm{q}_3+\mathrm{q}_5]\\
&+\left[\frac{7\mathbb{N}^2\mathsf{T}_1}{\varphi}+7\mathbb{N}^2\mathsf{T}_3+7\mathbb{N}^2\mathsf{T}_5\right]\int_{s_l}^{t} e^{\varphi s}\mathbb{E}\left\|\mathrm{w}(s)\right\|^2 ds\\
&\le \mathcal{K}_2+\gamma_2\int_0^t e^{\varphi s}\mathbb{E}\left\|\mathrm{w}(s)\right\|^2 ds
\end{aligned}
\tag{4.9}
$$

where,

$$
\mathcal{K}_2=\mathcal{L}_7+\mathcal{L}_{12}+\frac{7\mathbb{N}^2\mathrm{q}_2}{\varphi(\delta-\varphi)}+7\mathbb{N}_{1-p}^2\mathbb{N}^2 4^{1-p}\frac{\Gamma(2p-1)}{\varphi^{2p-1}(\delta-\varphi)}+\frac{7\mathbb{N}^2}{\delta-\varphi}[\mathrm{q}_3+\mathrm{q}_5]
$$

$$
\frac{7\mathbb{N}^2\mathsf{T}_1}{\varphi}+7\mathbb{N}^2\mathsf{T}_3+7\mathbb{N}^2\mathsf{T}_5
$$

Thus we acquire

$$
\mathbb{E}\left\|\mathrm{w}(t)\right\|^2 \le \mathcal{K}_2 e^{-\Theta_2 t} \tag{4.10}
$$

thereby, $\Theta_2=\varphi-\gamma_2$.
Also for $t\in(t_l,s_l]$, $l=1,2,\cdots,\mathcal{K}$,

$$
\mathbb{E}\left\|\mathrm{w}(t)\right\|^2 \le \mathcal{K}_3 e^{-\Theta t} \tag{4.11}
$$

where $\mathcal{K}_3=p_{4,l}$, $\Theta=\varphi$ Therefore by (4.8), (4.10) and (4.11) we may generalize

$$
\mathbb{E}\left\|\mathrm{w}(t)\right\|^2 \le \widehat{\mathcal{K}}e^{-\widehat{\Theta}t},\ t\ge 0\ \ and\ \ \widehat{\mathcal{K}},\widehat{\Theta}>0
$$

which is our desired inequality. □

5 Illustration

We may consider the following non-instantaneous impulsive SDEs with Poisson jump.

$$
\begin{aligned}
\partial\left[\mathrm{w}(t,\zeta)+\frac{1}{10}\int_0^1\int_0^1\zeta\sin(\mathrm{w}(s,\vartheta))d\vartheta ds\right]&=\left[\frac{\partial^2}{\partial\zeta^2}\mathrm{w}(t,\zeta)+\frac{\sqrt{2}e^{-t}}{10(1+t^2)}sin(\mathrm{w}(t,\zeta))\right]\partial t\\
&+\frac{te^{-t}}{9}sin(\mathrm{w}(t,\zeta))d\mathrm{Z}(t)+e^{-t}d\mathrm{Z}^{\mathrm{H}}(t)+\int_{\mathcal{U}}\frac{e^{-t}\mathrm{w}(t,\zeta)\mu}{7}\tilde{\mathcal{N}}(dt,d\mu),
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\zeta\in[0,1],\ t\in(0.30^-,\zeta]\cup(0.70,1],\mathrm{w}(t,\zeta)&=\frac{1}{5}sin(\mathrm{w}(0.30^-,\zeta)),\quad t\in(0.30,0.70],\\
\mathrm{w}(t,0)=0=\mathrm{w}(t,1),\mathrm{w}(0,\zeta)&=\mathrm{w}_0,
\end{aligned}
$$

where Z^H denotes the fBm with Hurst parameter $H \in (1/2, 1)$ and $Z(t)$ is a Weiner process. $0 = s_0 = t_0 < t_1 < s_1 < t_2 = \mathfrak{b} < \infty$ with $s_0 = 0, t_1 = 0.30, s_1 = 0.70, t_2 = \mathfrak{b} = 1$. Let $S_1 = S_2 = \mathfrak{R}$, $\beta_1^k = 1$, $\beta_j^k = 0, j > 1, k = 1, 2$. Let $W = \mathcal{U} = \mathcal{L}^2[0, 1]$. We define an operator A by $Ar = -r''$ with

$$\mathcal{D}(A) = \left\{ r \in H^2(0,1) \cap H_0^1(0,1) : r'' \in W \right\} \tag{5.2}$$

Clearly the operator A is compact resolvent and is self-adjoint thereby being a generator of an analytic semigroup $\{P(t)\}_{t\geq 0}$. Consider $p = 1/2$ and $W_p = \mathcal{D}(A^{1/2})$ be a banach space with the norm

$$\|\zeta\|_{1/2} = \left\|A^{1/2}\zeta\right\|, \quad \zeta \in \mathcal{D}(A^{1/2}).$$

$\forall\, r \in \mathcal{D}(A)$ and $\varrho \in \mathfrak{R}$, with $Ar = -r'' = \varrho r$, we obtain $\langle Ar, r\rangle = \langle \varrho r, r\rangle$,i.e.

$$|r'|_{\mathcal{L}^2} = \varrho |r|_{\mathcal{L}^2}.$$

So $\varrho > 0$. A solution r of $Ar = \varrho r$ is of the form

$$r(\zeta) = \mathcal{C}_1 cos(\sqrt{\varrho}) + \mathcal{C}_2 sin(\sqrt{\varrho}\zeta),$$

and the conditions $r(0) = r(1) = 0$ imply that $\mathcal{C}_1 = 0$ and $\varrho = \varrho_k = k^2\pi^2$, $k \in \mathbb{N}$, the corresponding solution is given by

$$r_k(\zeta) = \mathcal{C}_2 sin(\sqrt{\varrho_k}\zeta).$$

We have $\langle r_k, r_m\rangle = 0 \;\forall\, k \neq \mathfrak{m}$ and $\langle r_k, r_k\rangle = 1$ and hence $\mathcal{C}_2 = \sqrt{2}$ and

$$r_k(\zeta) = \sqrt{2} sin(k\pi\zeta), \quad k \in \mathbb{N},$$

is the orthogonal set of eigenvectors of A. Thus we obtain that for $r \in \mathcal{D}(A)$, there exists a sequence of real numbers ς_k such that

$$r(\zeta) = \sum_{k\in\mathbb{N}} \varsigma_k r_k(\zeta) \quad \text{with} \quad \sum_{k\in\mathbb{N}} (\varsigma_k)^2 < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} (\varsigma_k)^2 (\beta_k)^2 < +\infty.$$

We have,

$$A^{1/2} r(\zeta) = \sum_{k\in\mathbb{N}} \sqrt{\beta_k}\varsigma_k r_k(\zeta), \quad r \in \mathcal{D}(A^{1/2}), \quad i.e. \ \sum_{k\in\mathbb{N}} \varsigma_k^2 \beta_k < +\infty.$$

It is well known that $\|A^{-(1/2)}\| = 1$, and it follows that $\|A^{-1}\| \leq 1$.
Let $w(t)(\zeta) = w(t, \zeta)$ and the functions being defined as follows

$$\begin{aligned}
Q_1(t, w)(\zeta) &= \frac{1}{10}\int_0^1\int_0^1 \zeta sin(w(s, \vartheta)) d\vartheta ds,\\
Q_2(t, w)(\zeta) &= \frac{\sqrt{2}e^{-t}}{10(1 + t^2} sin(w(t, \zeta)),\\
Q_3(t, w)(\zeta) &= \frac{te^{-t}}{9} sin(w(t, \zeta)),\\
\sigma(t, w)(\zeta) &= \int_{\mathcal{U}} \frac{e^{-t}w(t, \zeta)}{7}\mu,\\
\mathcal{D}_l(t, w(t_1^-))(\zeta) &= \frac{1}{5} sin(w(0.30^-, \zeta)), \quad G(t) = e^{-t}.
\end{aligned}$$

We get $S_{\mathcal{D}_1} = \mathbb{N}_{\mathcal{D}_1} = 1/25$, $S_{Q_1} = \mathbb{N}_{Q_1} = 1/100$, $S_{Q_2} = \mathbb{N}_{Q_2} = 1/50$, $S_{Q_3} = \mathbb{N}_{Q_3} = 1/81$, $S_\sigma = \mathbb{N}_\sigma = 1/49$. Set $\mathbb{N} = \mathbb{N}_{1/2} = 1$. Now,

$$\begin{aligned}
(i)\chi_1 &= 5\left[\lambda\mathbb{N}_{Q_1} + \mathbb{N}_{1-p}^2\mathbb{N}_{Q_1}\frac{t_1^{2p}}{2p-1} + \mathbb{N}^2 t_1[\mathbb{N}_{Q_2}t_1 + \mathbb{N}_{Q_3} + \mathbb{N}_\sigma t_1]\right] = 0.101,\\
(ii)\widehat{\chi_1} &= 12\mathbb{N}^2[\mathbb{N}_{\mathcal{D}_l} + \lambda\mathbb{N}_{Q_1}] + 6\left[\mathbb{N}_{Q_1}\left(\lambda + \mathbb{N}_{1-p}^2\frac{t_2^{2p}}{2p-1}\right) + \mathbb{N}^2 t_2\left(\mathbb{N}_{Q_2}t_2 + \mathbb{N}_{Q_3} + \mathbb{N}_\sigma t_2\right)\right] = 0.723,\\
(iii)\chi &= \max[\chi_1, \mathbb{N}_{\mathcal{D}_l}, \widehat{\chi_l}] = \max[0.101, 1/25, 0.723] = 0.723 < 1.
\end{aligned}$$

Thus the hypotheses of Theorem 3.1 gets satisfied thereby concluding that the stochastic system (5.1) has a unique mild solution on $\mathcal{J}_3$.

6 Conclusion

In this paper, the existence and stability Results of Stochastic Differential Equations with Non-instantaneous Impulse and Poisson jumps are investigated. For the further study, the system (1.2) can be improved substantially by considering stochastic processes driven by Rosenblatt process or G-Brownian motion for some complicated situations. On the other hand, we hope that some of the results in this paper can be generalized for some non-local problems such as the Non-instantaneous Impulsive Stochastic Differential Equations with non-local conditions involving time varying delays.

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