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Qualitative analysis for random differential equations with proportional fractional derivative

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Abstract. This manuscript is devoted to investigate the existence, uniqueness and stability of random differential equations with Hilfer-generalized proportional fractional derivative. The concerned investigation of existence and uniqueness is obtained using Schauder fixed point theorem and Banach contraction principle respectively. At the end, the ulam stability is studied along with application.

1 Introduction

Fractional order differential equations have been applied in Science and Engineering, see [7, 9, 14]. There has been a significant development in ordinary differential equations involving fractional order derivatives [2, 4, 6, 12, 16, 19]. In the present paper, we analyze random differential equation(RDE) involving fractional order of the form

$$\begin{cases} D^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t,\omega) = \mathfrak{g}(t,\omega,\mathfrak{h}(t,\omega)), & t \in J := [0,T], T > 0, \\ I^{1-\nu,\vartheta;\psi}\mathfrak{h}(t,\omega)| = \mu(\omega), \end{cases} \tag{1.1}$$

where $D^{\alpha,\beta,\vartheta;\psi}$ is ψ-HFD of orders $\alpha \in (0,1)$, type $\beta \in [0,1]$ and $\vartheta \in (0,1]$. Further, $\mathfrak{h}$ is a random function, ω is the random variable and $I^{1-\nu,\vartheta;\psi}$ is ψ-fractional integral of orders $1-\nu(\nu = \alpha+\beta-\alpha\beta)$. Let Ω be the probability space with a continuous function $\mathfrak{g} : J \times \Omega \times R \to R$ such that $\omega \in \Omega$.

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Evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system is of statistical nature, that is, the information is probabilistic; the common approach in mathematical modeling of such systems is the use of RDE or stochastic differential equations. The analyses of fractional differential equations with random parameters have been studied in, [8, 10, 15, 18].

The paper is constructed as follows: In Section 2, we present the basic definition. In Section 3, existence and stability results are established. In section 4, we present an application to verify the theory.

2 Preliminaries

Basic definitions and results introduced in this section. Let $J = [a,b]$ $(0 \leq a \leq b)$ be a finite interval. The space of continuous function $\mathfrak{h}$, defined by $C[a,b]$ associated with the norm

$$\|\mathfrak{h}\|_{C_{\nu,\psi}} = \sup\{|\mathfrak{h}(t,\omega)| : t \in J\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\nu,\psi}[a,b] = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(a))^{\nu}\,\mathfrak{g}(t,\omega) \in C\right\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{\nu,\psi}} = \sup_{t\in J}\left|(\psi(t)-\psi(a))^{\nu}\,\mathfrak{g}(t,\omega)\right|.$$

The weighted space $C^{n}_{\nu,\psi}$ of functions $\mathfrak{g}$ on J is defined by

$$C^{n}_{\nu,\psi} = \left\{f : J \to R : \mathfrak{g}(t) \in C^{n-1}; \mathfrak{g}(t) \in C_{\nu,\psi}\right\}, 0 \leq \nu < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C^{n}_{\nu,\psi}} = \sum_{k=0}^{n-1}\|\mathfrak{g}^{k}\|_{C} + \|\mathfrak{g}^{n}\|_{C_{\nu,\psi}}.$$

For $n = 0$, we have, $C^{0}_{\nu} = C_{\nu}$.
Here we present the following weighted space for our problem as follows

$$C^{\alpha,\beta}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, D^{\alpha,\beta,\vartheta;\psi}\mathfrak{g} \in C_{\nu;\psi}\right\}$$

and

$$C^{\nu}_{1-\nu;\psi} = \left\{\mathfrak{g} \in C_{1-\nu;\psi}, D^{\nu,\vartheta;\psi}\mathfrak{g} \in C_{1-\nu;\psi}\right\}.$$

It is obvious that

$$C^{\nu}_{1-\nu;\psi} \subset C^{\alpha,\beta}_{1-\nu;\psi}.$$

Definition 2.1. [5]If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$. then the fractional integral

$$\left(I^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \int_{0}^{t}\psi'(s)e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\frac{(\psi(t)-\psi(s))^{\alpha-1}}{\vartheta^{\alpha}\Gamma(\alpha)}\mathfrak{h}(s)ds. \tag{2.1}$$

Definition 2.2. [5] If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the generalized left proportional fractional derivative of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi'(t) \neq 0$ is describe as

$$\left(D^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \mathfrak{h}(s)ds. \qquad (2.2)$$

Definition 2.3. [5]If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the generalized Caputo proportional fractional derivative of order α of the function $\mathfrak{h}$ with respect to another function isdefined by with $\psi'(t) \neq 0$ is describe as

$$\left(D^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = I^{n-\alpha,\vartheta;\psi}\left(D^{n,\vartheta;\psi}\mathfrak{h}\right)(t). \qquad (2.3)$$

Definition 2.4. [5] The ψ-Hilfer generalized proportional fractional derivative of order α and type β over $\mathfrak{h}$ with respect to another function is defined by

$$\left(D^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}\right)(t) = I^{\beta(1-\alpha),\vartheta;\psi}\left(D^{1,\vartheta;\psi}\right) I^{(1-\beta)(1-\alpha),\vartheta;\psi}\mathfrak{h}(t). \qquad (2.4)$$

Next, we shall give the definitions and the criteria of generalized Ulam-Hyers-Rassias (UHR) stability. Let $\varepsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$\left|D^{\alpha,\nu,\vartheta;\psi}\mathfrak{h}(t,\omega) - \mathfrak{g}(t,\omega,\mathfrak{h}(t,\omega))\right| \leq \varphi(t). \qquad (2.5)$$

Definition 2.5. [17] The Eq. (1.1) is generalized UHR stable with respect to $\varphi \in C_{1-\nu}$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{v} \in C_{1-\nu}$ of the inequality (2.5) there exists a solution $\mathfrak{h} \in C_{1-\nu}$ of Eq. (1.1) with

$$|\mathfrak{v}(t,\omega) - \mathfrak{h}(t,\omega)| \leq C_{f,\varphi}\varphi(t,\omega).$$

Lemma 2.1. Let $\alpha, \beta > 0$, Then we have the following semigroup property

$$(I^{\alpha,\vartheta;\psi} I^{\beta,\vartheta;\psi}\mathfrak{g})(t) = (I^{\alpha+\beta,\vartheta;\psi}\mathfrak{g})(t),$$

and

$$(D^{\alpha,\vartheta;\psi} I^{\alpha,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.2. Let $n-1 < \alpha < n$ where $n \in N$, $\vartheta \in (0,1]$, $0 \leq \beta \leq 1$, with $\nu = \alpha + \beta(n-\alpha)$, such that $n-1 < \nu < n$. If $\mathfrak{g} \in C_\nu$ and $\mathfrak{I}^{n-\nu,\vartheta;\psi}\mathfrak{g} \in C^n_\nu$, then

$$(I^{\alpha,\vartheta;\psi}\alpha, \beta^{\beta,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t) - \sum_{k=1}^{n} \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\nu-k}}{\vartheta^{\nu-k}\Gamma\nu - k + 1} I^{k-\nu,\vartheta;\psi}(a).$$

Lemma 2.3. [17](Grönwall's Lemma) Let $\alpha > 0$, $a(t,\omega) > 0$ is locally integrable function on $J \times \Omega$ and if $\mathfrak{g}(t,\omega)$ be a increasing and nonnegative continuous function on $J \times \Omega$, such that $|\mathfrak{g}(t,\omega)| \leq K$ for some constant K. Moreover if $\mathfrak{h}(t,\omega)$ be a nonnegative locally integrable function on $J \times \Omega$ with

$$\mathfrak{h}(t,\omega) \leq a(t,\omega) + g(t,\omega)\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{h}(s,\omega)ds, \quad (t,\omega) \in J \times \Omega,$$

with some $\alpha > 0$. Then

$$\mathfrak{h}(t,\omega) \leq a(t,\omega) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\mathfrak{g}(t,\omega)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s)(\psi(t)-\psi(s))^{n\alpha-1}\right] a(s,\omega)ds, \quad (t,\omega) \in J \times \Omega.$$

Theorem 2.1. [21](Schauder fixed point theorem) Let B be closed, convex and nonempty subset of a Banach space C. Let $T : B \to B$ be a continuous mapping such that $T(B)$ is a relatively compact subset of C. Then T has at least one fixed point in B.

3 Main results

Existence theory of solutions to proposed problem (1.1) is presented in this section. First we declare the hypothesis:

(H1) There exists a Carathèodory function $\ell : [0,T] \times \Omega \to R$, such that

$$|\mathfrak{g}_1(s,\omega,\mathfrak{h}(s,\omega)) - \mathfrak{g}_2(s,\omega,\eta(s,\omega))| \leq \ell(t,\omega)\,|\mathfrak{h}-\eta|,$$

(H2) for every $t \in [0,T]$ and $\omega \in \Omega$.

Lemma 3.1. A function $\mathfrak{h}$ is the solution of RFDE (1.1), if and only if $\mathfrak{h}$ satisfies the random integral equation

$$\begin{aligned}\mathfrak{h}(t,\omega) =& \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\ &+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds. \end{aligned} \tag{3.1}$$

Theorem 3.1. Assume that hypothesis $(H1)$ is satisfied. Then, Eq.(1.1) has at least one solution.

Proof. Consider the operator $T(\omega) : \Omega \times C_{1-\nu,\psi} \to C_{1-\nu,\psi}$. Hence $\mathfrak{h}$ is a solution for the problem (1.1) if and only if $\mathfrak{h}(t,\omega) = (T\mathfrak{h})(t,\omega)$, where the equivalent integral Eq. (3.1) which can be written in the operator form

$$\begin{aligned}(T\mathfrak{h})(t,\omega) =& \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} \\ &+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds. \end{aligned} \tag{3.2}$$

Clearly, the fixed points of the operator T is solution of the problem (1.1). Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s,\omega,0)$. For any $\mathfrak{h} \in J \times \Omega$, we have

$$
\begin{aligned}
&\left|(T\mathfrak{h})(t,\omega)(\psi(t)-\psi(a))^{1-\nu}\right| \\
&\leq \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))|\,ds \\
&\leq \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)} \\
&\quad+\frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))-\mathfrak{g}(s,\omega,0)+\mathfrak{g}(s,\omega,0)|\,ds \\
&\leq \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\ell(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\|\mathfrak{h}\|_{C_{1-\nu,\psi}} \\
&\quad+\frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\|\tilde{\mathfrak{g}}\|_{C_{1-\nu,\psi}} \\
&\leq \frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha}\|\mathfrak{h}\|_{C_{1-\nu,\psi}} \\
&\quad+\frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha}\|\tilde{\mathfrak{g}}\|_{C_{1-\nu,\psi}}
\end{aligned}
$$

This proves that T transforms the ball $B_r=\left\{\mathfrak{h}\in C_{1-\nu,\psi}:\|\mathfrak{h}\|_{C_{1-\nu,\psi}}\leq r\right\}$, into itself. We shall show that the operator $T:B_r\to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in the following steps.

Step 1: T is continuous.

Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n\to\mathfrak{h}$ in $C_{1-\nu,\psi}$. Then for each $t\in J,\ \omega\in\Omega$,

$$
\begin{aligned}
&\left|((T\mathfrak{h}_n)(t,\omega)-(T\mathfrak{h})(t,\omega))(\psi(t)-\psi(a))^{1-\nu}\right| \\
&\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\omega,\mathfrak{h}_n(s,\omega))-\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))|\,ds \\
&\leq \frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(s))^{\alpha+\nu-1}\|\mathfrak{g}(\cdot,\omega,\mathfrak{h}_n(\cdot,\omega))-\mathfrak{g}(\cdot,\omega,\mathfrak{h}(\cdot,\omega))\| \\
&\leq \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}\|\mathfrak{g}(\cdot,\omega,\mathfrak{h}_n(\cdot,\omega))-\mathfrak{g}(\cdot,\omega,\mathfrak{h}(\cdot,\omega))\|_{C_{1-\nu,\psi}}.
\end{aligned}
$$

Due to continuity of $\mathfrak{g}$, we have

$$\|T\mathfrak{h}_n-T\mathfrak{h}\|_{C_{1-\nu,\psi}}\to 0 \text{ as } n\to\infty.$$

Step 2: $T(B_r)$ is uniformly bounded.

This is clear since $T(B_r)\subset B_r$ is bounded.

Step 3: We show that $T(B_r)$ is equi-continuous.

Let $t_1>t_2\in J$ with B_r be a bounded set of $C_{1-\nu,\psi}$ as in Step 2, and $\mathfrak{h}\in B_r$. Then

$$\left|(\psi(t_1)-\psi(a))^{1-\nu}(T\mathfrak{h})(t_1,\omega)-(\psi(t_2)-\psi(a))^{1-\nu}(T\mathfrak{h})(t_2,\omega)\right|$$
$$\leq \left|\frac{(\psi(t_1)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_1} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_1)-\psi(a))}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds\right.$$
$$\left.-\frac{(\psi(t_2)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_2} e^{\frac{\vartheta-1}{\vartheta}(\psi(t_2)-\psi(a))}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds\right|$$
$$\leq \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^{\tau_1}\left[(\psi(\tau_1)-\psi(a))^{1-\nu}(\psi(\tau_1)-\psi(s))^{\alpha-1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\right]$$
$$\times\psi'(s)\left|\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))\right|ds$$
$$+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_{\tau_2}^{\tau_1}(\psi(\tau_2)-\psi(a))^{1-\nu}(\psi(\tau_2)-\psi(s))^{\alpha-1}\psi'(s)\left|\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))\right|ds$$

right hand side of the inequality approaches to zero, as $t_1 \to t_2$ Therefore by Step 1-3 together with the Arzela-Ascoli theorem, we say that T is continuous and compact. Hence by the Theorem 2.1, the operator T has a fixed point which is a solution of the problem (1.1).

Lemma 3.2. Assume that the hypothesis $(H1)$ is satisfied. If

$$\frac{\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}<1.$$

Then, Eq. (1.1) has unique fixed point.

Proof. Consider the operator $T(\omega):\Omega\times C_{1-\nu,\psi}\to C_{1-\nu,\psi}$ defined by

$$(T\mathfrak{h})(t,\omega)=\frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}$$
$$+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds.$$

Clearly the operator T is well defined. Now for any $\mathfrak{h}_1,\mathfrak{h}_2\in C_{1-\nu}$, we obtain

$$\left|((T\mathfrak{h}_1)(t,\omega)-(T\mathfrak{h}_2)(t,\omega))(\psi(t)-\psi(a))^{1-\nu}\right|$$
$$\leq\frac{(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\omega,\mathfrak{h}_1(s,\omega))-\mathfrak{g}(s,\omega,\mathfrak{h}_2(s,\omega))\right|ds$$
$$\leq\frac{\ell(\psi(t)-\psi(a))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(t)-\psi(a))^{\alpha+\nu-1}\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}}$$
$$\leq\frac{\ell}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)(\psi(b)-\psi(a))^{\alpha}\left\|\mathfrak{h}_1-\mathfrak{h}_2\right\|_{C_{1-\nu,\psi}}.$$

it follows that T ia a contraction map, there exists a unique solution of problem (1.1).

Theorem 3.2. The hypotheses $(H1)$ and $(H2)$ are satisfied. Then Eq. (1.1) is g-UHR stable.

Proof. Let $\mathfrak{v}$ be solution of inequality (2.5) and by Lemma 3.2, $\mathfrak{h}$ is a unique solution of Eq. (1.1) is as follows

$$\mathfrak{h}(t,\omega) =\frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds.$$

By inequality (2.5), we obtain

$$\left|\mathfrak{v}(t,\omega)-\frac{\phi(0,\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1} - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{v}(s,\omega))ds\right| \leq \lambda_{\varphi}\varphi(t).$$

Hence for every $t\in J$, we have

$$\begin{aligned}
&|\mathfrak{v}(t,\omega)-\mathfrak{h}(t,\omega)|\\
&\leq \left|\mathfrak{v}(t,\omega)-\frac{\mu(\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\right.\\
&\qquad\left. - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))ds\right|\\
&\leq \left|\mathfrak{v}(t,\omega)-\frac{\phi(0,\omega)}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\nu-1}\right.\\
&\qquad\left. - \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\omega,\mathfrak{v}(s,\omega))ds\right|\\
&\qquad + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\omega,\mathfrak{v}(s,\omega))-\mathfrak{g}(s,\omega,\mathfrak{h}(s,\omega))\right|ds\\
&\leq \lambda_{\varphi}\varphi(t,\omega)+\frac{\ell}{\vartheta^{\alpha}\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{v}(s,\omega)-\mathfrak{h}(s,\omega)\right|ds.
\end{aligned}$$

By Lemma 2.3, there exists a constant $C_{f,\varphi}>0$ such that

$$|\mathfrak{v}(t,\omega)-\mathfrak{h}(t,\omega)| \leq C_{f,\varphi}\lambda_{\varphi}\varphi(t,\omega)$$

Hence the proof.

4 Example

For $\psi(t)=t$, we obtain the particular case of Eq. (1.1) is as follows:

$$\mathfrak{D}^{\alpha,\nu,\vartheta;t}\mathfrak{h}(t,\omega) = \mathfrak{g}(t,\omega,\mathfrak{h}_t), \quad t\in J:=[0,1], \tag{4.1}$$

$$\mathfrak{I}^{1-\nu,\vartheta;t}\mathfrak{h}(t,\omega)| = \phi(t,\omega). \tag{4.2}$$

We choose $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, $\vartheta=\frac{4}{5}$ and $\nu=\frac{3}{4}$. Here

$$\mathfrak{g}(t,\omega,\mathfrak{h}(t,\omega)) = \frac{1}{9e^t}\left(\frac{\mathfrak{h}^2(t,\omega)}{1+\mathfrak{h}^2(t,\omega)}+cos2\right).$$

For $\mathfrak{h}$, $\mathfrak{y}\in R$, we have

$$|\mathfrak{g}(t,\omega,\mathfrak{h}(t,\omega)) - \mathfrak{g}(t,\omega,\mathfrak{y}(\omega))| \leq \frac{1}{9}|\mathfrak{h}-\mathfrak{y}|.$$

Now all the assumptions in Theorem 3.2 are satisfied, the problem has a unique solution. Next, set $\varphi(t,\omega) = e^{t+\omega}$,

$$I^{\frac{1}{2};t}\varphi(t,\omega) \leq \frac{1}{\Gamma(\frac{3}{2})}\varphi(t,\omega) = \lambda_{\varphi}\varphi(t,\omega).$$

5 Application

In this section, we analyze a economical model described by a differential equation through the fractional derivative. In economy, the standard measure of the value added created by the production of goods and services in a country during a certain period is the gross domestic product (GDP). As a result, it also accounts for the income collected by that production, as well as the overall amount spent on final goods and services (less imports). While GDP is the most significant indicator for capturing economic activity, it falls short of ensuring an appropriate estimate of people's material well-being, for which other metrics may be more appropriate. The differential equation is utilised to express the rate of change in the current state as a function of the current state in this case. The change in the domestic product GDP, $\mathfrak{h}(t)$, over time is a simple illustration of this type of dependence. The current GDP is proportional to the rate of change in GDP, given by

$$\mathfrak{h}'(t) = \lambda\mathfrak{h}(t),$$

where λ is the growth rate, so if $\mathfrak{h}(0) = \mathfrak{h}_0$ denotes the initial population then the solution is given by, $\mathfrak{h}(t) = \mathfrak{h}_0 e^{\lambda t}$. This solution tells that the GDP increases exponentially when the growth rate is positive. Now by applying the fractional derivative to the above equation due to the non uniformity we declare the equation as

$$\mathfrak{D}^{\alpha,\nu,\vartheta;\psi(t)}\mathfrak{h}(t) = \lambda\mathfrak{h}(t),$$

the solution of the described fractional differential equation is given by $\mathfrak{h}(t) = \mathfrak{h}_0(E_\alpha(g(\psi(t) - \psi(0))^\alpha))$.

Here, while predicting the economical growth there exists a six essential components in the economy which are responsible for economic growth. Improving or growing their size will result in economic growth. Land, water, forests, and natural gas are examples of other resources.

Natural in a country, in fact. It's difficult, if not impossible, to increase the number of hand resources. To minimise depletion of finite natural resources, countries must carefully balance supply and demand. Land management improvements will increase land quality while also contributing to economic prosperity.

Increasing physical capital investment, such as factories, machines, and roads, will lower the cost of economic activity. More is produced by better manufacturing and machinery than by physical labour. This large production has the power to enhance productivity. A strong highway system, for example, can minimise inefficiency in moving raw materials or goods throughout the country, hence increasing GDP. As the population grows, so does the availability of workers or employees, resulting in more employees. One disadvantage of a large population is that it can result in high unemployment rates.

The quality of labour can be improved by increasing investment in human capital. This rise in quality will result in improved talents, abilities, and training. The development of a skilled workforce is influenced significantly by the productivity of skilled workers. Investing in STEM students or funding coding academies, for example, will increase the number of people available for higher-paying jobs than investing in blue-collar industries.

The growth of technology is another important aspect. Technology can boost production while reducing labour costs, resulting in faster growth and development. As a result of the rise, industries will be able to create more at a reduced cost. Long-term growth can be aided by technology.

Rules and laws are part of an organisational framework that controls economic activity. There are no companies that specifically foster growth.

Here all the mentioned factors are not a fixed quantity whereas it is random, the impact of these random elements will reflect in the behaviour of the system vastly. So it is important to add the random factor which plays a major role in the system. Thus we consider the

$$\mathfrak{D}^{\alpha,\nu,\vartheta;\psi(t)}\mathfrak{h}(t,\omega) = \lambda(\omega)\mathfrak{h}(t,\omega),$$

where the growth rate coefficient $r(\omega)$ as well as the dependent variable $\mathfrak{h}(t,\omega)$ for a given t are supposed to be random variables of the outcome of an experiment ω taking values in the set of all outcomes Ω. Thus, modeling population growth is many times a difficult task due to the scarcity and scattering of the data, and to errors and uncertainty in it. Incorporating randomness in the population model is a natural alternativeand its much important.

6 Conclusion

The main purpose of this article is to develope the theory of RDE involving ψ-Hilfer generalized proportional fractional derivative. By the use of classical fixed point theory, we obtain the existence and uniqueness result. The uniqueness of the solution is obtained by Banach Contraction Principal. The stability of the solutions is studied by the concept proposed by Ulam. Finally, an model is been provided to show the the importance of the random coefficient.

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