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Existence and Ulam stability results for a class of boundary value problem of neutral pantograph equations with complex order

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Abstract
The research reported in this paper deals with the existence of solutions of boundary value problem (BVP) of nonlinear neutral pantograph equations by means of complex fractional derivative in weighted spaces. The results were interpreted with the aid of classical fixed point theorems. The Ulam–Hyers–Rassias stability and Ulam–Hyers stability of differential equations are studied by utilizing the complex fractional derivative through the fixed point method.

Keywords Neutral pantograph equation · Boundary value problem · Stirling asymptotic formula · Fractional derivative · Existence · Ulam stability

Mathematics Subject Classification 26A33 · 34A08 · 34B18

1 Introduction

The purpose of this study confines to the problems in the area of fractional calculus. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. In a note to L'Hospital in 1695 Leibniz posed the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders ?" The story goes that L'Hospital was fairly interested about that question and responded by another question to Leibniz. "What if the order will be $\frac{1}{2}$ Leibniz in a communication dated September 30, 1695 answered: "Itwill lead to a paradox, from which one day useful consequences will be drawn." The query posed by Leibniz for a

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fractional derivative was an ongoing problem in the last 300 years. Several mathematicians contributed to this subject over the years. Those like Liouville, Riemann, and Weyl made major roles to the theory of fractional calculus. The story of the fractional calculus maintained with contributions from Fourier, Leibniz, Abel, Letnikov, and Grnwald. At the present time, the fractional calculus attracts various scientists and engineers, for detailed study, the books [16,24,25,28] and references therein. Emerging trends and widespread applications in various fields like physics, chemistry, engineering, finance and other sciences has made fractional differential equations (FDEs) a special area of interest among scholars and researchers. The reformulations and expressions of enormous models in terms of FDEs, has made their physical meaning inclusive in the mathematical models more reasonably. The interdisciplinary applications which are elegantly modelled using fractional derivatives have laid an attractive platform to many physicists, biologists, engineers and mathematicians. In particular, the existence of solutions to fractional BVP is currently under strong research, one can refer to [3,5,6,13,14,36].

Last few decades have brought a rapid development of the non-integer order differential and integral calculus, from which both the theory and its applications benefit significantly. However, most of the work done in this field so far has been based on the use of real order fractional derivatives and integrals. It is worth to mention that there are several authors who also applied complex order fractional derivative. Fractional operators of complex order are initiated as follows (see [23,32]). In 1977, Ross Bertram [30] considered a use for a derivative of complex order in the fractional calculus. In [11], Carla M.A. Pinto studied a complex order van der Pol oscillator. Later, R. Andriambololona et al. [2] proposed some definitions of complex order integrals and complex order derivatives using operator approach. For instance, some basic theory for fractional differential equation with complex order was investigated by Neamaty et al. [27]. They derived sufficient conditions for existence of solutions of fractional boundary value problems with complex order. Recently, Teodor M. Atanackovi et al. established complex order fractional derivatives in models that describe viscoelastic materials in [4].

Recently, many studies about the delay differential equations (DDEs) have appeared in science literature. The pantograph equation is one of the most important kinds of DDEs that arise in a variety of applications in physics and engineering. Pantograph type always has the delay term fall after the initial value but before the desire approximation being calculated. When the delay term of pantograph type involved with the derivative(s), the equation is named as neutral DDEs of pantograph type. Pantograph type equations have been studied widely due to the several applications in which these equations arise. The pantograph equations also appear in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. In this point, it is usually solved these kinds of DDEs analytically and numerically. Therefore, there are many efficient numerical methods in literature that can be used to approximate solutions to the multi-pantograph DDEs. The abstract fractional neutral pantograph equation was studied in the paper of K. Balachandran et al. [7]. The authors investigated existence and uniqueness by using a classical fixed point theorem. The existence of solutions of this kind of BVP has been studied by Benchohra et al., for example, in [8–10].

In 1940, Ulam [17] raised a problem whether an approximate solution of a functional equation can be approximated by a solution of the corresponding equation. This question was first answered by Hyers 1 year later; we refer the readers to [20,31]. Since then, several results have been published in connection with various generalizations of Ulam's type stability theory or the Ulam's–Hyers (U–H) stability theory. U–H stability definition has applicable significance since it means that if one is studying a U–H stable system then one does not have to reach the exact solution which usually is quite difficult or time consuming. All that is required is to get a function which satisfies the stability definition; see [1,18,26,29,34,35] and references therein. U–H stability guarantees that there is a close exact solution. This is reasonably useful in many applications e.g. numerical analysis, optimization, biology and economics etc., where finding the exact solution is rather difficult.

Let the fractional BVPs for the nonlinear neutral pantograph equations with complex order be given by

$$\begin{cases} D^{\theta}_{0+}x(t) = f\left(t, x(t), x(\lambda t), D^{\theta}_{0+}x(\lambda t)\right), \quad \theta = m + i\alpha, \quad \lambda \in (0,1), \quad t \in J := [0,T], \\ ax(0) + bx(T) = c, \end{cases} \tag{1.1}$$

where D^{θ}_{0+} is the Caputo fractional derivative of order $\theta \in \mathbb{C}$. Let $\alpha \in \mathbb{R}^+$, $0 < \alpha < 1$, $m \in (0, 1]$, and $f : J \times \mathbb{R}^3 \to \mathbb{R}$ is given continuous function. Here a, b, c are real constants with $a + b \neq 0$. The key point of this manuscript is explained step by step as follows:

- Nonlinear fractional neutral pantograph differential system with complex order is considered.
- Common fractional derivative, fixed point theorems and hypotheses on nonlinear terms are utilized to prove the existence of fixed point.
- U–H and U–H–Rassias stability of the problem (1.1) are given.

The rest of this paper is systematized as follows: In Sect. 2, we give some notations; recall some concepts and preparation results. In Sect. 3, by utilizing fixed point techniques and hypotheses on nonlinear terms, some sufficient conditions are established for existence and uniqueness of solution to the system (1.1). In Sect. 4, Ulam–Hyers stability and UlamHyers–Rassias stability of the problem (1.1) is proved.

2 Preliminaries

In the present section, we deal with several basic definitions and properties of fractional calculus which is base for the forthcoming sections. The results in this section are taken from [4,23,32].

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous function J into $\mathbb{R}$ with the norm

$$\|x\|_{\infty} := \sup\{|x(t)| : t \in J\}.$$

By $L^1(J)$ we denote the space of Lebesgue-integrable function $x : J \to \mathbb{R}$ with the norm $\|x\|_{L^1} = \int_0^T |x(t)|\, dt$.

Definition 2.1 ([28]) The Riemann–Liouville fractional integral of order $\theta \in \mathbb{C}$, $(Re(\theta) > 0)$ of a function $f : (0, \infty) \to \mathbb{R}$ is

$$I^{\theta}_{0+} f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s) ds.$$

Definition 2.2 ([28]) For a function f given by on the interval J, the Caputo fractional-order $\theta \in \mathbb{C}$, $(Re(\theta) > 0)$ of f, is defined by

$$(D^{\theta}_{0+} f)(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} f^{(n)}(s) ds,$$

where $n = [Re(\theta)] + 1$ and $[Re(\theta)]$ denotes the integral part of the real number ν.

Definition 2.3 ([21]) The Stirling asymptotic formula of the Gamma function for $z \in \mathbb{C}$ is following

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{\frac{z-1}{2}} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right], \quad (|arg(z)| < \pi; |z| \to \infty), \tag{2.1}$$

and its results for $|\Gamma(u + iv)|$, $(u, v \in \mathbb{R})$ is

$$|\Gamma(u + iv)| = (2\pi)^{\frac{1}{2}} |v|^{u-\frac{1}{2}} e^{-u-\pi|v|/2} \left[1 + O\left(\frac{1}{v}\right)\right], \quad (v \to \infty). \tag{2.2}$$

Lemma 2.4 (see Lemma 7.1.1, [17]) *Let $z, w : [0, T) \to [0, \infty)$ be continuous functions where $T \leq \infty$. If w is nondecreasing and there are constants $k \geq 0$ and $0 < \nu < 1$ such that*

$$z(t) \leq w(t) + k \int_0^t (t - s)^{\nu-1} z(s) ds, \quad t \in [0, T),$$

then

$$z(t) \leq w(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\nu))^n}{\Gamma(n\nu)} (t - s)^{n\nu-1} w(s) \right) ds, \quad t \in [0, T).$$

Remark 2.5 Under the hypothesis of Lemma 2.4, let $w(t)$ be a nondecreasing function on $[0, T)$. Then we have $z(t) \leq w(t) E_{\nu,1}(k\Gamma(\nu)t^{\nu})$.

Now we consider the Ulam stability for the problem

$$D_{0+}^{\theta} x(t) = f\left(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t)\right), \quad \lambda \in (0, 1), \quad t \in [0, T], \tag{2.3}$$

and the following inequations:

$$\left|D_{0+}^{\theta} z(t) - f\left(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t)\right)\right| \leq \epsilon, t \in [0, T], \tag{2.4}$$

$$\left|D_{0+}^{\theta} z(t) - f\left(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t)\right)\right| \leq \epsilon\varphi(t), t \in [0, T], \tag{2.5}$$

$$\left|D_{0+}^{\theta} z(t) - f\left(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t)\right)\right| \leq \varphi(t), t \in [0, T]. \tag{2.6}$$

Definition 2.6 Equation (2.3) is U–H stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality (2.4) there exists a solution $x \in C(J, \mathbb{R})$ of Eq. (2.3) with

$$|z(t) - x(t)| \leq C_f \epsilon, \ t \in J.$$

Definition 2.7 Equation (2.3) is generalized U–H stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that for each solution $z \in C(J, \mathbb{R})$ of the inequality (2.4) there exists a solution $x \in C(J, \mathbb{R})$ of Eq. (2.3) with

$$|z(t) - x(t)| \leq \psi_f \epsilon, \quad t \in J.$$

Definition 2.8 Equation (2.3) is U–H–Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C(J, \mathbb{R})$ of the inequality (2.5) there exists a solution $x \in C(J, \mathbb{R})$ of Eq. (2.3) with

$$|z(t) - x(t)| \leq C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 2.9 Equation (2.3) is generalized U–H–Rassias stable with respect to $\varphi \in C(J, \mathbb{R})$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C(J, \mathbb{R})$ of the inequality (2.6) there exists a solution $x \in C(J, \mathbb{R})$ of Eq. (2.3) with

$$|z(t) - x(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 2.10 A function $z \in C(J, \mathbb{R})$ is a solution of (2.4) if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depend on z) such that

1. $|g(t)| \leq \epsilon, t \in J$;
2. $D_{0+}^{\theta} z(t) = f\left(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t)\right) + g(t), t \in J$.

Remark 2.11 Clearly,

1. Definition 2.6⇒ Definition 2.7.
2. Definition 2.8⇒ Definition 2.9.

Remark 2.12 A solution of the fractional neutral pantograph differential equations with complex order inequality

$$\left|D_{0+}^{\theta} z(t) - f\left(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t)\right)\right| \leq \epsilon, \quad t \in J,$$

is called an fractional ϵ-solution of the problem (2.3).

Theorem 2.13 [15] (Banach's fixed point theorem) *Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.14 [15] (Schaefer's fixed point theorem). *Let $P : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ completely continuous operator. If the set*

$$\zeta = \{x \in C(J, \mathbb{R}) : x = \delta(Px) \quad \textit{for some } \delta \in [0, T]\}$$

is bounded, then P has at least a fixed point.

3 Solution representation and existence results

Let us start by defining what we mean by a solution of the problem (1.1). We adopt some ideas in [10,33].

Definition 3.1 A function $x \in C(J, \mathbb{R})$ is said to be a solution of (1.1) if x satisfies the equation $D_{0+}^{\theta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t))$ on J, and the condition $ax(0)+bx(T) = c$.

For the existence of solutions for the problem (1.1), we need the following auxiliary lemma.

Lemma 3.2 *Let $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}^+$ and $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = x_0 + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), x(\lambda s), D_{0+}^{\theta} x(\lambda s)) ds \tag{3.1}$$

if and only if x is a solution of the initial value problem for the following fractional neutral pantograph differential equation with complex order

$$D_{0+}^{\theta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t)), \quad \lambda \in (0, 1), \quad t \in J := [0, T], \tag{3.2}$$

$$x(0) = x_0. \tag{3.3}$$

As a consequence of Lemma 3.2 we have the following result which is useful in what follows.

Lemma 3.3 *Let $\theta = m + i\alpha$, $0 < m \leq 1$, $\alpha \in \mathbb{R}^+$ and $f : J \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s, x(s), x(\lambda s), D_{0+}^{\theta} x(\lambda s)) ds - \frac{1}{a+b} \left[\frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} f(s, x(s), x(\lambda s), D_{0+}^{\theta} x(\lambda s)) ds - c \right] \tag{3.4}$$

if and only if x is a solution of the BVP for fractional neutral pantograph differential equations with complex order

$$D_{0+}^{\theta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t)), \quad \lambda \in (0, 1), \quad t \in [0, T], \tag{3.5}$$

$$ax(0) + bx(T) = c. \tag{3.6}$$

Our first result is based on the Banach contraction principle.

Theorem 3.4 *Assume that the following hypotheses are fulfilled.*

(H1) $f : J \times \mathbb{R}^3 \to \mathbb{R}$ is continuous function.
(H2) There exist constants $K > 0$ and $L > 0$ such that

$$|f(t, u, v, w) - f(t, \overline{u}, \overline{v}, \overline{w})| \leq K (|u - \overline{u}| + |v - \overline{v}|) + L |w - \overline{w}|,$$

for any $u, v, w, \overline{u}, \overline{v}, \overline{w} \in \mathbb{R}$ and $t \in J$.

If $\Omega_{K,L,m,T,a,b,\theta} < 1$, then the problem (1.1) has a unique solution on J.

Proof Transform the problem (1.1) into a fixed point problem.

On considering the operator $P : C(J, \mathbb{R}) \to C(J, \mathbb{R})$, it is equivalent Volterra integral equation can be represented in the operator form as:

$$(Px)(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_x(s) ds - \frac{1}{a+b} \left[\frac{b}{\Gamma(\theta)} \int_0^T (T-s)^{\theta-1} K_x(s) ds - c \right]. \tag{3.7}$$

For sake of brevity, let us take

$$\begin{aligned} K_x(t) &= D_{0+}^{\theta} x(t) \\ &= f(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t)) \\ &= f(t, x(t), x(\lambda t), K_x(t)). \end{aligned}$$

Clearly, the fixed points of the operator P are solution of the problem (1.1). We shall use the Banach contraction principle to prove that P defined by (3.7) has a fixed point. We shall show that P is a contraction.

Let $x, y \in C(J, \mathbb{R})$. Then, for each $t \in J$ we have

$$|(Px)(t) - (Py)(t)| \leq \frac{1}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| \left|K_x(s) - K_y(s)\right| ds + \frac{|b|}{|\Gamma(\theta)| \, |a+b|} \int_0^T \left|(T-s)^{\theta-1}\right| \left|K_x(s) - K_y(s)\right| ds. \tag{3.8}$$

Here

$$\begin{aligned} \left|K_x(t)-K_y(t)\right| &\leq \left|f(t,x(t),x(\lambda t),K_x(t))-f(t,y(t),y(\lambda t),K_y(t))\right| \\ &\leq K\left(|x(t)-y(t)|+|x(\lambda t)-y(\lambda t)|\right)+L\left|K_x(t)-K_y(t)\right| \\ &\leq 2K\,|x(t)-y(t)|+L\left|K_x(t)-K_y(t)\right| \\ &\leq \left(\frac{2K}{1-L}\right)|x(t)-y(t)|\,. \end{aligned} \tag{3.9}$$

By replacing Eq. (3.9) in the inequality Eq. (3.8), we obtain

$$\begin{aligned} &|(Px)(t)-(Py)(t)| \\ &\leq \left(\frac{2K}{1-L}\right)\frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right||x(s)-y(s)|\,ds \\ &\quad+\left(\frac{2K}{1-L}\right)\frac{|b|}{|a+b|}\frac{1}{|a+b|}\frac{1}{|\Gamma(\theta)|}\int_0^T \left|(T-s)^{\theta-1}\right||x(s)-y(s|\,ds \\ &\leq \left(\frac{2K}{1-L}\right)\frac{1}{|\Gamma(\theta)|}\|x-y\|_\infty\int_0^t (t-s)^{m-1}ds \\ &\quad+\left(\frac{2K}{1-L}\right)\frac{|b|}{|a+b|}\frac{\|x-y\|_\infty}{|\Gamma(\theta)|}\int_0^T (T-s)^{m-1}ds \\ &\leq \left(\frac{2K}{1-L}\frac{T^m}{m\,|\Gamma(\theta)|}\left[1+\frac{|b|}{|a+b|}\right]\right)\|x-y\|_\infty\,. \end{aligned}$$

Thus

$$\|Px-Py\|_\infty \leq \Omega_{K,L,m,T,a,b,\theta}\,\|x-y\|_\infty\,,$$

where

$$\Omega_{K,L,m,T,a,b,\theta}:=\left(\frac{2K}{1-L}\frac{T^m}{m\,|\Gamma(\theta)|}\left[1+\frac{|b|}{|a+b|}\right]\right),$$

depends only on the parameters of the problem. And since $\Omega_{K,L,m,T,a,b,\theta} < 1$, the results follows in view of the contraction mapping principle. □

We now can prove the following existence result.

Theorem 3.5 *Under the hypotheses of Theorem 3.4 and*

(H3) There exist $l, p, q, r \in C(J,\mathbb{R})$ with $l^* = \sup_{t\in J} l(t) < t$ such that

$$|f(t,u,v,w)| \leq l(t)+p(t)\,|u|+q(t)\,|v|+r(t)\,|w|\,,$$

for $t \in J,\ u, v, w \in \mathbb{R}$,

hold. Then the problem (1.1) has at least one solution on J.

Proof We shall use Schaefer's fixed point theorem to prove that P defined by (3.7) has a fixed point. The proof will be given in several steps.

First, we prove that P is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C(J,\mathbb{R})$. Then for each $t \in J$

$$
\begin{aligned}
&|(Px_n)-(Px)(t)| \\
&\le \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right|\left|K_{x_n}(s)-K_x(s)\right|ds \\
&\quad+\frac{|b|}{|\Gamma(\theta)|\,|a+b|}\int_0^T \left|(T-s)^{\theta-1}\right|\left|K_{x_n}(s)-K_x(s)\right|ds \\
&\le \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right|\sup_{s\in J}\left|K_{x_n}(s)-K_x(s)\right|ds \\
&\quad+\frac{|b|}{|\Gamma(\theta)|\,|a+b|}\int_0^T \left|(T-s)^{\theta-1}\right|\sup_{s\in J}\left|K_{x_n}(s)-K_x(s)\right|ds \\
&\le \frac{\left\|K_{x_n}(\cdot)-K_x(\cdot)\right\|_\infty}{|\Gamma(\theta)|}\left[\int_0^t (t-s)^{m-1}ds+\frac{|b|}{|a+b|}\int_0^T (T-s)^{m-1}ds\right] \\
&\le \frac{T^m\left\|K_{x_n}(\cdot)-K_x(\cdot)\right\|_\infty}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right).
\end{aligned}
$$

Since f is a continuous function, we have

$$\|Px_n-Px\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Second, we show to that P maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, it is enough to show that for any $\eta^*>0$, there exists a positive constant χ such that for each $x\in B_{\eta^*}=\{x\in C(J,\mathbb{R}):\|x\|_\infty\le\eta^*\}$, we have $\|Px\|_\infty\le\chi$.

$$
\begin{aligned}
|(Px)(t)| \le{}& \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right||K_x(s)|\,ds \\
&\frac{|b|}{|a+b|}\int_0^T \left|(T-s)^{\theta-1}\right||K_x(s)|\,ds+\frac{|c|}{|a+b|},
\end{aligned}
\tag{3.10}
$$

and by (H3), we have

$$
\begin{aligned}
|K_x(t)| &\le |f(t,x(t),x(\lambda t),K_x(t))| \\
&\le l(t)+p(t)\,|x(t)|+q(t)\,|x(\lambda t)|+r(t)\,|K_x(t)| \\
&\le l^*+p^*\,|x(t)|+q^*\,|x(\lambda t)|+r^*\,|K_x(t)| \\
&\le \frac{l^*+p^*\,|x(t)|+q^*\,|x(\lambda t)|}{1-r^*}.
\end{aligned}
\tag{3.11}
$$

By replacing Eq. (3.11) in the inequality (3.10), we get

$$
\begin{aligned}
|(Px)(t)| \le{}& \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right|\left(\frac{l^*+p^*\,|x(s)|+q^*\,|x(\lambda s)|}{1-r^*}\right)ds \\
&+\frac{|b|}{|a+b|\,|\Gamma(\theta)|}\int_0^T \left|(T-s)^{\theta-1}\right|\left(\frac{l^*+p^*\,|x(s)|+q^*\,|x(\lambda s)|}{1-r^*}\right)ds+\frac{|c|}{|a+b|}. \\
:={}&A_1+A_2.
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right|\left(\frac{l^*+p^*\,|x(s)|+q^*\,|x(\lambda s)|}{1-r^*}\right)ds \\
&= \frac{T^m}{(1-r^*)\,|\Gamma(\theta)|}\left(\frac{l^*}{m}+\frac{(p^*+q^*)\,\|x\|_\infty}{m}\right).
\end{aligned}
$$

$$A_2 = \frac{|a+b|\,|b|}{|\Gamma(\theta)|}\int_0^T \left|(T-s)^{\theta-1}\right| \left(\frac{l^* + p^*\,|x(s)| + q^*\,|x(\lambda s)|}{1-r^*}\right) ds$$
$$= \frac{|b|\,T^m}{(1-r^*)m\,|\Gamma(\theta)|\,|a+b|}\left(l^* + (p^*+q^*)\,\|x\|_\infty\right). \tag{3.12}$$

To substitute A_1, A_2 values into Eq. (3.12),we have

$$|(Px)(t)| \le \frac{T^m l^*}{(1-r^*)}\frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)$$
$$+\frac{T^m(p^*+q^*)}{(1-r^*)}\frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)\|x\|_\infty + \frac{|c|}{|a+b|}$$
$$:= \chi.$$

Next, we prove that P maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $t_1, t_2 \in [0, T]$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in above steps, and let $x \in B_{\eta^*}$. Using the fact f is bounded on the compact set $J \times B_{\eta^*}$(these $\sup_{(t,x)\in J\times B_{\eta^*}} \|K_x(t)\| := C_0 < \infty$). We will get

$$|(Px)(t_2) - (Px)(t_2)|$$
$$\le \left|\frac{1}{\Gamma(\theta)}\int_0^{t_1}\left[(t_2-s)^{\theta-1}-(t_1-s)^{\theta-1}\right]K_x(s)ds + \frac{1}{\Gamma(\theta)}\int_{t_1}^{t_2}(t_2-s)^{\theta-1}K_x(s)ds\right|$$
$$\le \frac{C_0}{|\Gamma(\theta)|}\int_0^{t_1}\left|\left[(t_1-s)^{\theta-1}-(t_2-s)^{\theta-1}\right]\right|ds + \frac{C_0}{|\Gamma(\theta)|}\int_{t_1}^{t_2}\left|(t_2-s)^{\theta-1}\right|ds$$
$$\le \frac{C_0}{|\Gamma(\theta)|}\left|2(t_1-t_2)^{\theta} + t_2^{\theta} - t_1^{\theta}\right|$$

which tends to zero as $t_1 - t_2 \to 0$. Hence along with the Arzela-Ascoli theorem and above discussion, it is concluded that $P : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and completely continuous.

Now it remains to show that the set

$$\zeta = \{x \in C(J,\mathbb{R});\, x = \delta(Px) \quad \text{for some} \quad 0 < \delta < 1\}$$

is bounded. Let $x \in \zeta$, then $x = \delta(Px)$ for some $0 < \delta < 1$. Thus, for each $t \in J$, we have

$$x(t) = \frac{\delta}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}K_x(s)ds - \frac{\delta}{a+b}\left[\frac{b}{\Gamma(\theta)}\int_0^T (T-s)^{\theta-1}K_x(s)ds - c\right].$$

This implies by (H3) that for $t \in J$, we have

$$|(Px)(t)| \le \frac{T^m l^*}{(1-r^*)}\frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)$$
$$+\frac{T^m(p^*+q^*)}{(1-r^*)}\frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)\|x\|_\infty + \frac{|c|}{|a+b|}.$$

Thus for every $t \in J$, we have

$$\begin{aligned}\|Px\|_{\infty} &\leq \frac{T^m l^*}{(1-r^*)} \frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right) \\ &\quad + \frac{T^m(p^*+q^*)}{(1-r^*)} \frac{1}{m\,|\Gamma(\theta)|}\left(1+\frac{|b|}{|a+b|}\right)\|x\|_{\infty} + \frac{|c|}{|a+b|} \\ &:= R.\end{aligned}$$

This shows that the set ζ is bounded. Thus Schaefer's fixed point theorem helps to deduct the solution of the problem which states P has a fixed point which is a solution of problem. □

4 U–H–Rassias stability

We arrive at a situation to state and prove our stability results for the problem (1.1).

Theorem 4.1 *Suppose that the hypotheses (H1),(H2) and $\Omega_{K,L,m,T,a,b,\theta} < 1$ hold. Then, the problem (1.1) is U–H stable.*

Proof Let $\epsilon > 0$ and let $z \in C(J, \mathbb{R})$ be a function which satisfies the inequality

$$\left|D^{\theta}_{0+}z(t) - f(t, z(t), z(\lambda t), D^{\theta}_{0+}z(\lambda t))\right| \leq \epsilon, \quad \text{for some}, \quad t \in J, \tag{4.1}$$

and let $x \in C(J, \mathbb{R})$ be the unique solution of the following problem

$$\begin{aligned}&D^{\theta}_{0+}x(t) = f(t, x(t), x(\lambda t), D^{\theta}_{0+}x(\lambda t)), \quad t \in J, \quad \theta = m + i\alpha, \\ &x(0) = z(0), \quad x(T) = z(T),\end{aligned}$$

where $m \in (0, 1]$, $\alpha \in \mathbb{R}^+$ and $0 < \lambda < 1$.

Using Lemma 3.3, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_x(s) ds$$

with

$$A_x = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\theta)} \int_0^T (t-s)^{\theta-1} K_x(s) ds\right].$$

On the other hand, if $x(T) = z(T)$ and $x(0) = z(0)$, then $A_x = A_z$.
Indeed,

$$\begin{aligned}|A_x - A_z| &\leq \frac{|b|}{|a+b|\,|\Gamma(\theta)|} \int_0^T \left|(T-s)^{\theta-1}\right| |K_x(s) - K_z(s)|\, ds \\ &\leq \left(\frac{2K}{1-L}\right) \frac{|b|}{|a+b|} I^{\theta}_{0+} |x(T) - z(T)| \\ &= 0.\end{aligned}$$

Thus, $A_x = A_z$. We have

$$x(t) = A_z + \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_x(s) ds.$$

By integration of the inequality (4.1) and using Remark 2.10, we obtain

$$\left|z(t) - A_z - \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} K_z(s)ds\right| \le \frac{\epsilon T^m}{m\,|\Gamma(\theta)|}\left(1 + \frac{|b|}{|a+b|}\right).$$

We have for any $t \in J$

$$\begin{aligned}|z(t) - x(t)| &\le \left|z(t) - A_z - \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} K_z(s)ds\right| \\ &\quad + \frac{1}{|\Gamma(\theta)|}\int_0^t \left|(t-s)^{\theta-1}\right| |K_z(s) - K_x(s)|\,ds \\ &\le \left(1 + \frac{|b|}{|a+b|}\right)\frac{\epsilon T^m}{m\,|\Gamma(\theta)|} + \left(\frac{2K}{1-L}\right)\frac{1}{|\Gamma(\theta)|}\int_0^t (t-s)^{m-1}\,|z(s) - x(s)|\,ds.\end{aligned}$$

Using Gronwall inequality, Lemma 2.4 and Remark 2.5, we obtain

$$|z(t) - x(t)| \le \left(1 + \frac{|b|}{|a+b|}\right)\frac{\epsilon T^m}{m\,|\Gamma(\theta)|} E_{m,1}\left(\frac{2K}{1-L}\frac{1}{|\Gamma(\theta)|}\Gamma(m)T^m\right).$$

Thus, the problem (1.1) is U–H stable. □

Let us assume the following assumption for further discussion:

(H4) There exists an increasing function $\varphi \in C(J, \mathbb{R})$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$I_{0+}^{\theta}\varphi(t) \le \lambda_\varphi \varphi(t).$$

Theorem 4.2 *If the assumptions (H1),(H2),(H4) and* $\Omega_{K,L,m,T,a,b,\theta} < 1$ *hold. Then, the problem* (1.1) *is generalized U–H–Rassias stable.*

Proof Let $z \in C(J, \mathbb{R})$ be solution of the following inequality

$$\left|D_{0+}^{\theta} z(t) - f(t, z(t), z(\lambda t), D_{0+}^{\theta} z(\lambda t))\right| \le \epsilon\varphi(t), \quad t \in J, \quad \epsilon > 0, \tag{4.2}$$

and let $x \in C(J, \mathbb{R})$ be the unique solution of the following problem

$$\begin{aligned}&D_{0+}^{\theta} x(t) = f(t, x(t), x(\lambda t), D_{0+}^{\theta} x(\lambda t)), \quad t \in [0, T], \quad \theta = m + i\alpha, \\ &x(0) = z(0), \quad x(T) = z(T),\end{aligned}$$

where $m \in (0, 1]$, $\alpha \in \mathbb{R}^+$ and $0 < \lambda < 1$.
By Lemma 3.3, we get

$$x(t) = A_z + \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} K_x(s)ds$$

with

$$A_z = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\theta)}\int_0^T (T-s)^{\theta-1} K_z(s)ds\right].$$

By integration of the inequality (4.2) and using (H4), we obtain

$$\left|z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_z(s) ds\right| \leq \left(1 + \frac{|b|}{|a+b|}\right) \epsilon \lambda_\varphi \varphi(t).$$

We have for any $t \in J$

$$\begin{aligned} &|z(t) - x(t)| \\ &\leq \left|z(t) - A_z - \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} K_z ds\right| + \frac{1}{|\Gamma(\theta)|} \int_0^t \left|(t-s)^{\theta-1}\right| |K_z(s) - K_x(s)| ds \\ &\leq \left(1 + \frac{|b|}{|a+b|}\right) \epsilon \lambda_\varphi \varphi(t) + \left(\frac{2K}{1-L}\right) \frac{1}{|\Gamma(\theta)|} \int_0^t (t-s)^{m-1} |z(s) - x(s)| ds. \end{aligned}$$

Using Gronwall inequality,

$$|z(t) - x(t)| \leq \left(1 + \frac{|b|}{|a+b|}\right) \epsilon \lambda_\varphi \varphi(t) E_{m,1} \left(\frac{2K}{1-L} \frac{1}{|\Gamma(\theta)|} \Gamma(m) T^m\right), \quad t \in J.$$

Thus, the problem (1.1) is generalized U–H–Rassias stable. □

5 An example

Consider the following fractional neutral-pantograph equation with complex order

$$D_{0^+}^\theta x(t) = \frac{\left(6 + |x(t)| + \left|x\left(\frac{t}{2}\right)\right| + \left|D_{0^+}^\theta x\left(\frac{t}{2}\right)\right|\right)}{10e^{t+1}\left(1 + |x(t)| + \left|x\left(\frac{t}{2}\right)\right| + \left|D_{0^+}^\theta x\left(\frac{t}{2}\right)\right|\right)}, \quad \text{for each} t \in J := [0, 1], \quad (5.1)$$

$$x(0) = 0, \quad x(1) = 1, \quad (5.2)$$

where $\theta = m + i\alpha$, $\alpha = \frac{1}{2}$ and $m = 1$.
Set

$$f(t, u, v, w) = \frac{6 + |u| + |v| + |w|}{10e^{t+1}(1 + |u| + |v| + |w|)}, \quad t \in J, \quad u, v, w \in \mathbb{R}.$$

For every $u, v, w, \overline{u}, \overline{v}, \overline{w} \in \mathbb{R}$ and $t \in J$,

$$|f(t, u, v, w) - f(t, \overline{u}, \overline{v}, \overline{w})| \leq \frac{1}{10e}\left(|u - \overline{u}| + |v - \overline{v}| + |w - \overline{w}|\right).$$

Hence, the condition (H2) holds with $K = L = \frac{1}{10e}$.
Also, we have, for each $u, v, w \in \mathbb{R}$ and $t \in J$,

$$|f(t, u, v, w)| \leq \frac{1}{10e^{t+1}}\left(6 + |u| + |v| + |w|\right).$$

Thus condition (H3) holds with $l(t) = \frac{3}{5e^{t+1}}$, $p(t) = q(t) = r(t) = \frac{1}{10e^{t+1}}$.

Clearly, $l^* = \frac{3}{5e}$, $p^* = q^* = \frac{1}{10e}$ and $r^* = \frac{1}{10e} < 1$.
The condition of Theorem 3.4 is satisfied for the suitable values of $\alpha = \frac{1}{2}$, $m = 1$ with $a = b = T = 1$. Indeed, $\Omega_{K,L,m,T,a,b,\theta} < 1$.

It follows from Theorem 3.4 that problem (5.1–5.2) has a unique solution on J. In addition, Theorem 4.1 implies that the problem (5.1–5.2) is Ulam–Hyers stable.

6 Conclusion

The main idea of this paper is to give some significant and general results on stability analysis of neural pantograh equations with fractional complex order. Existence and uniqueness of fixed point, U–H stability and U–H–Rassias stability for proposed problem have been proved. With the intention of achieve this plan, fixed point theorems fractional calculus, nonlinear neutral pantograph theory and appropriate hypotheses on nonlinear terms have been employed.

References

1. Andras, S., Kolumban, J.J.: On the Ulam–Hyers stability of first order differential systems with nonlocal initial conditions. Nonlinear Anal. **82**, 1–11 (2013)
2. Andriambololona, R., Tokiniaina, R., Rakotoson, H.: Definitions of complex order integrals and complex order derivatives using operator approach. Int. J. Latest Res. Sci. Technol. **1**(4), 317–323 (2012)
3. Arara, A., Benchohra, M., Hamidi, N., Nieto, J.J.: Fractional order differential equations on an unbounded domain. Nonlinear Anal. **72**(2), 580–586 (2010)
4. Atanackovi, T.M., Konjik, S., Pilipovic, S., Zorica, D.: Complex order fractional derivatives in viscoelasticity. Mech. Time-Depend. Mater. **1**, 1–21 (2016)
5. Bai, Z.: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**(2), 916–924 (2010)
6. Bai, Z., Lu, H.: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**(2), 495–505 (2005)
7. Balachandran, K., Kiruthika, S., Trujillo, J.J.: Existence of solutions of nonlinear fractional pantograph equations. Acta Math. Sci. **33B**(3), 1–9 (2013)
8. Benchohra, M., Bouriah, S.: Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order. Morccan J. Pure Appl. Anal. **1**(1), 22–37 (2015)
9. Benchohra, M., Lazreg, J.E.: Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. Romanian J. Math. Comput. Sci. **4**, 60–72 (2014)
10. Benchohra, M., Hamani, S., Ntouyas, S.K.: Boundary value problems for differential equations with fractional order. Surv. Math. Appl. **3**, 1–12 (2008)
11. Carla, M.A., Pinto, J.A.: Tenreiromachado. Complex order van der Pol oscillator. Nonlinear Dyn. **65**(3), 247–254 (2010)
12. Derfel, G.A., Iserles, A.: The pantograph equation in the complex plane. J. Math. Anal. Appl. **213**, 117–132 (1997)
13. El-Shahed, M.: Positive solutions for boundary value problem of nonlinear fractional differential equation. Abstract and Applied Analysis , 2007 Article ID 10368, 8 pages (2007)
14. Goodrich, C.S.: Existence of a positive solution to a class of fractional differential equations. Appl. Math. Lett. **23**, 1050–1055 (2010)
15. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
16. Hilfer, R.: Application of Fractional Calculus in Physics. World Scientific, Singapore (1999)
17. Hyers, D.H., Isac, G., Rassias, T.M.: Stability of functional equation in several variables, Vol. 34. Progress in nonlinea differential equations their applications. Boston: Birkhauser (1998)
18. Ibrahim, R.W.: Generalized Ulam–Hyers stability for fractional differential equations. Int. J. Math. (2012). https://doi.org/10.1142/S0129167X12500565
19. Iserles, A.: On the generalized pantograph functional-differential equations. Eur. J. Appl. Math. **4**, 1–38 (1993)
20. Jung, S.M.: Hyers–Ulam stability of linear differential equations of first order. Appl. Math. Lett. **17**, 1135–1140 (2004)
21. Kilbas, A.A., Srivasta, H.M., Trujillo, J.J.: Theory and Application of Fractional Differential Equations. Elsevier B. V, Netherlands (2016)
22. Liu, M.Z., Li, D.: Runge–Kutta methods for the multi-pantograph delay equation. Appl. Math. Comput. **155**, 853–871 (2004)
23. Love, E.R.: Fractional derivatives of imaginary order. J. Lond. Math. Soc. **2**(2–3), 241–259 (1971)
24. Magin, R.L.: Fractional Calculus in Bioengineering. Begell House, Danbury (2006)

25. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
26. Muniyappan, P., Rajan, S.: Hyers–Ulam–Rassias stability of fractional differential equation. Int. J. Pure Appl. Math. **102**, 631–642 (2015)
27. Neamaty, A., Yadollahzadeh, M., Darzi, R.: On fractional differential equation with complex order. Prog. Fract. Differ. Equ. Appl. **1**(3), 223–227 (2015)
28. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
29. Rabha, W.: Ibrahim, Ulam stability of boundary value problem. Kragujev J. Math. **37**(2), 287–297 (2013)
30. Ross, B., Northover, F.H.: A use for a derivative of complex order in the fractional calculus. IJPAM **9**(4), 400–406 (1977)
31. Rus, I.A.: Ualm stabilities of ordinary differential equations in a Banach space. Carpath. J. Math. **26**, 103–107 (2010)
32. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives-Theory and Applications. Gordon and Breach Science Publishers, Amsterdam (1993)
33. Vivek, D., Kanagarajan, K., Harikrishnan, S.: Dynamics and stability results for integro-differential equations with complex order. Discontin. Nonlinearity Complex. **7**(2), 179–187 (2018)
34. Wang, J., Zhou, Y.: New concepts and results in stability of fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 2530–2538 (2012)
35. Wang, J., Lv, L., Zhou, Y.: Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electron. J. Qual. Theory Differ. Eq. **63**, 1–10 (2011)
36. Zhang, S.: Existence of solution for a boundary value problem of fractional order. Acta Math Sci Ser B Engl. Ed. **26**(2), 220–228 (2006)