Studies over the existence of a certain impulse-based fuzzy integrodifferential equations of the Sobolev type

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Abstract

The present investigation employs impulses and a non-local constraint to prove the existence are some various types of abstract differential and integrodifferential equations related to the Sobolev type. Semigroup theory, specifically variants of constant formula, is utilized to get the analytical results for those equations. Furthermore, findings using the Banach fixed point approach were examined using fuzzy numbers with values spanning the $\mathscr{E}_n$ range, which includes the normal, convex, upper semi-continuous, and compactly supported interval. A description is given for each situation to illustrate the principle.

Keywords Existence and uniqueness, Fuzzy solution, Neutral integrodifferential equation, Impulsive differential equation, Fixed point theorem

Mathematics Subject Classification 34A07, 34A12, 12H20, 47H20, 34K45

Introduction

There are numerous applications for the intriguing theory of differential equations throughout abstract spaces in the fields of analysis and other mathematics. Ordinary differential equations (ODEs), functional differential equations, partial differential equations (PDEs), and sometimes a combination of interacting systems of ordinary and partial differential equations would be used, depending on the nature of the problems. In the fields of applied mathematics, engineering, biology, and the physical sciences, nonlinear differential and integral equations in abstract spaces have been utilized to deal with several problems. With significant applications in many areas of analysis and other disciplines, the theory of nonlinear differential and integral equations in abstract spaces is growing quickly.

The fuzzy semigroups of linear operators to solve fuzzy differential equations were originally proposed in the fuzzy literature by the authors of Gomes et al. [1]. Next, fuzzy Cauchy problems were studied by Kaleva [2] using nonlinear iteration semigroups (with exponential formula). Ding and Kandel [3] examined how differential equations and fuzzy sets may be used to create fuzzy logic systems, also known as fuzzy dynamical systems, which are similar to fuzzy neutral functional differential equations. With its wide range of applications, semigroup theory has recently been the subject of much study in the classic literature. For the interested reader, we recommend the fascinating work by Pazy [4], in which the

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Radhakrishnan *et al. BMC Research Notes* (2024) 17:39

Page 2 of 18

author uses semigroups to solve partial and ordinary differential equations.

PDEs are frequently employed to simulate a broad variety of scientific and engineering issues. They are typically expressed in several forms of differential or integrodifferential equations in abstract spaces. Several writers have studied integrodifferential equations in abstract spaces, including [5–8]. In general, an integrodifferential equation is an abstract representation of a large number of partial integrodifferential equations that emerge in issues involving wave propagation and other physical phenomena. By addressing Sobolev type nonlinear interodifferential equations, Radhakrishnan et al. [9] investigated the existence of Sobolev type nonlinear neutral integrodifferential equations. Many writers have analyzed differential equations of the Sobolev type [10–12].

Nonlocal Cauchy problem, namely, the differential equation with a non-local initial condition $z(\tau_0) + e(\tau_1, \ldots, \tau_n, z) = z_0$ $(0 \le \tau_0 < \tau_1 < \ldots < \tau_n \le \tau_0 + a$ and e is a given function) is one of the important topics in the study of analysis. The primary motivation behind interest in the area revolves around the non-local initial condition's high efficacy over the standard one while treating physical issues. In fact, the conventional beginning condition $z(0) = z_0$ could not be incorporated into a number of fascinating empirical events that the non-local initial condition represents. For instance, the function $e(\tau_1, \ldots, \tau_n, z)$ may be given by

$$(\tau_1, \ldots, \tau_n, z) = \sum_{k=1}^{n} c_k z(\tau_k)$$

(c_k, $k = 1, \ldots, n$ are constants). In this case, we are permitted to have the measurements at $\tau = 0, \tau_1, \ldots, \tau_n$, rather than just at $\tau = 0$. More specially, letting $e(\tau_1, \ldots, \tau_n, z) = -z(\tau_k)$ and $z_0 = 0$ yields a periodic problem and letting $e(\tau_1, \ldots, \tau_n, z) = -z(\tau_0) + z(\tau_n)$ gives a backward problem. Byszewski [13] was the first to delve at the existence of solutions to evolution equations in Banach spaces with non-local constraints.

In reality, experts agree that discretely emerging discontinuities enrich its continuous. The latter are also known as jumps or, from an energy standpoint, impulses. Many evolution processes are accompanied by abrupt shifts in condition at specific points in time. These processes are subject to short-term perturbations that are insignificant in contrast to the process's lifespan. As a result, it's reasonable to presume that these disturbances occur instantly, in the form of impulses. As a result, differential equations with impulsive effects serve as a natural description of observable evolution events in a variety of real-world issues, such as [14–21]. In some situations, such as the so-called neutral differential difference equations, the delayed argument occurs in both the derivative of the state variable and the independent variable. A neutral functional differential equation involves the derivatives concerning past events history or derivatives of functionals of the prior history, and the present state of the system. The book on neutral functional differential equations by Hale and Verduyn Lunel [22] and its references are a useful resource.

In accordance with the premise that "all things occurring in the real world are unstable and unexpected," Zadeh[23] developed fuzzy set theory in 1965. In a number of research areas, the notion was proposed and implemented successfully. This hypothesis has lately been investigated further, with a variety of applications being proposed. To explain fuzzy conceptions, Diamand et al. [24] established the metric space of fuzzy sets theory. Kaleva [25, 26] looked at fuzzy differential equations in broad sense. For further discussion on the several types of fuzzy differential equations, see [27–32].

Motivated by the literature, we are using the fixed point approach to investigate several types of Sobolev type Fuzzy neutral integro-differential equations with impulses in a fuzzy environment.

Problem formulation

The authors of this paper have to investigate if fuzzy neutral impulsive nonlinear integrodifferential equations of the Sobolev type are subject to non-local conditions

$$[\mathbb{B}z(\nu) + \mathscr{P}(\tau, z(\nu))]' + \mathbb{A}z(\nu) = \mathbb{F}(\nu, z(\nu)) + \int_0^\nu \mathscr{K}\left(\nu, \mu, z(\mu)\right) d\mu$$
$$\nu \in [0, a], \nu \ne \nu_k$$

(1.1)

$$z(0) + \sum_{i=1}^{n} c_i z(v_i) = z_0 \qquad (1.2)$$

$$\Delta z(v_k) = \mathcal{I}_k(z_{v_k}), \quad k = 1, 2, ..., m, \qquad (1.3)$$

where $\mathbb{A}, \mathbb{B} : \mathbb{J} \to \mathbb{E}_n$ denotes a fuzzy coefficient and $\mathbb{J} = [0, a]$. The set of all upper semi continuous convex regular numbers that are uncertain, with limited $\lambda-$level intervals is designated as $\mathbb{E}_n$. The function $\mathbb{F}, \mathscr{P}, \mathscr{K} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ is nonlinear fuzzy function and z_0 is a initial value and $\mathcal{I}_k z(v_k) = \Delta z(v_k) = z(v_k^+) - z(v_k^-)$, for all $k = 1, 2, ..., m$; $0 = v_0 < v_1 < v_2 < ... < v_m < v_{m+1} = a$;

Preliminaries

By giving each $z \in \mathbb{R}^n$ a membership grade, a fuzzy subset of $\mathbb{R}^n$ is constructed using a membership function. In this way, the purpose of membership is addressed.

$u : \mathbb{R}^n$ to the closed interval $[0, 1]$.

Across the analysis, the subsequent requirements were introduced: w maps $\mathbb{R}^n$ onto $[0, 1]$, $[u]^0$ constitutes a finite subset of $\mathbb{R}^n$, u is fuzzy convex, and u is upper semi-continuous. In this instance, consider $\mathbb{E}_n$ represent the space of all fuzzy subsets u of $\mathbb{R}_n$, encompassing upper semi-continuous, normal, and fuzzy convex sets along with bounded supports. The space of all fuzzy subsets u of $\mathbb{R}$ is precisely expressed by $\mathbb{E}^1$.

A fuzzy amount $\mathscr{A}$ in $\mathbb{R}$ comprises a set that is fuzzy defined by the membership function χ_A spanning $\mathbb{R}$ to $[0, 1]$. A number that is fuzzy $\mathscr{A}$ also transforms into

$$\mathscr{A} = \int_{z \in \mathbb{R}} \frac{\chi_{\mathscr{A}}}{z}$$

with $\chi_{\mathscr{A}}(\cdot)$ in the closed interval 0 and 1

Let z in $\mathbb{R}^n$ and $\mathcal{D}$ be a nonempty subsets of $\mathbb{R}^n$. Now the Hausdroff separation of $\mathcal{B}$ from $\mathcal{D}$ is defined by

$$d(z, \mathcal{D}) = \inf\{\|z - b\| : b \in \mathcal{D}\}.$$

Let $\mathcal{D}$ and $\mathcal{B}$ be nonempty subsets of $\mathbb{R}^n$. The Hausdroff separation of $\mathcal{B}$ from $\mathcal{D}$ is defined by

$$\mathscr{H}_*^d(\mathcal{B}, \mathcal{D}) = \sup\{d(b, \mathcal{D}) : b \in \mathcal{B}\}.$$

In general,

$$\mathscr{H}_*^d(\mathcal{B}, \mathcal{D}) \neq \mathscr{H}_*^d(\mathcal{D}, \mathcal{B}).$$

With regard to two mathematical functions $\mathcal{A}$ and $\mathcal{B}$, the Hausdroff gap between them can be expressed as $\mathscr{H}^d(\mathcal{A}, \mathcal{B})$. The greatest measure d^∞ on $\mathbb{E}_n$ is characterized as

$$d^\infty(x, y) = \sup\{\mathscr{H}^d\left([x]^\lambda, [y]^\lambda\right) : \lambda \in (0, 1]\}, \text{ for all } x, y \in \mathbb{E}_n,$$

and is obviously metric on $\mathbb{E}_n$.

The writers of this paper make an inference that there is an operator $\mathcal{E}$ on $\mathbb{E}_n$, which is provided by formula

$$\mathcal{E} = \left[\mathcal{I} + \sum_{i=n}^{n} c_i \mathbb{B}^{-1} \mathscr{S}(\tau_i) \mathbb{B}\right]^{-1}.$$

We primary investigate the following fuzzy functional differential equation (FFDE) along with non-local initial condition of Sobolev type

$$(\mathbb{B}z(v))' = \mathbb{A}z(v) + \mathbb{F}(v, z(v)), \quad v \in \mathbb{J} = [0, a] \quad (2.1)$$

$$z(0) + \sum_{i=1}^{n} c_i z(v_i) = z_0, \qquad (2.2)$$

where $\mathbb{A}, \mathbb{B} : \mathbb{J} \to \mathbb{E}_n$ denotes a fuzzy coefficient and $\mathbb{J} = [0, a]$, $\mathbb{E}_n$ is the collection of all upper semi continuous convex normal fuzzy numbers with bounded $\lambda-$level intervals. The function $\mathbb{F} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ is nonlinear fuzzy function and z_0 is a initial value.

Definition 2.1 A continuous function $z(v)$ of the integral equation

$$z(v) = \mathbb{B}^{-1} \mathscr{S}(\tau) \mathbb{B} \mathcal{E} z_0 - \sum_{i=1}^{n} c_i \mathbb{B}^{-1} \mathscr{S}(v) \mathcal{E} \times \left\{ \int_0^{v_i} \mathbb{B}^{-1} \mathscr{S}(v_i - \mu) \mathbb{F}(\mu, z(\mu)) d\mu \right\}$$

$$+ \int_0^v \mathbb{B}^{-1} \mathscr{S}(v - \mu) \mathbb{F}(\mu, z(\mu)) d\mu + \mathbb{B}^{-1} \mathbb{A} \mathscr{S}(v) \mathbb{B} \mathcal{E} z_0$$

Radhakrishnan *et al. BMC Research Notes*   (2024) 17:39

Page 4 of 18

is said be a solution, of problem (2.1, 2.2) on $\mathbb{J}$.

Remark 2.1 A solution of (2.1, 2.2) satisfies the condition (2.2). Then

$$z(0) = \mathcal{E}z_0 + \mathbb{A}\mathcal{E}z_0 - \sum_{i=1}^{n} c_i \mathcal{E}\mathbb{B}^{-1}$$

$$\left\{ \int_0^{v_i} \mathscr{S}(v_i - \mu)\mathbb{B}^{-1}\mathbb{F}(\mu, z(\mu))d\mu \right\}$$

and

$$z(v_j) = \mathbb{B}^{-1}\mathscr{S}(v_j)\mathbb{B}\mathcal{E}z_0 - \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v_j)\mathbb{B}\mathcal{E}$$

$$\left\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v_i - \mu)\mathbb{F}(\mu, z(\mu))d\mu \right\}$$

$$+ \int_0^{v} \mathbb{B}^{-1}\mathscr{S}(v_j - \mu)\mathbb{F}(\mu, z_\mu(\mu))d\mu$$

$$+ \mathbb{B}^{-1}\mathbb{A}\mathscr{S}(v_j)\mathbb{B}\mathcal{E}z_0.$$

Therefore

$$z(0) + \sum_{j=1}^{n} c_j z(v_j) = \left[\mathscr{I} + \sum_{i=1}^{n} c_i z(v_i)\mathbb{B}^{-1}\mathscr{S}(\tau_j)\mathbb{B} \right]\mathcal{E}z_0 - \left[\mathscr{I} + \sum_{i=1}^{n} c_i z(v_i)\mathbb{B}^{-1}\mathscr{S}(\tau_j)\mathbb{B} \right]$$

$$\times \sum_{i=1}^{n} c_i \mathcal{E}\left\{ \int_0^{v_i} \mathscr{S}(v_i - \mu)\mathbb{B}^{-1}\mathbb{F}(\mu, z(\mu))d\mu \right\}$$

$$+ \int_0^{v_j} \mathscr{S}(v_j - \mu)\mathbb{B}^{-1}\mathbb{F}(\mu, x_\mu(\mu))d\mu = z_0.$$

The subsequent assumptions are required in order to demonstrate the existence result:

(A_1) $\mathbb{A} : \mathbb{J} \to \mathbb{E}_n$ is a fuzzy coefficient and $\{\mathscr{C}(v), v \in \mathbb{E}_n\}$ of bounded linear operator in Banach space. There exist constants $\mathbb{M}_s \geq 0, \mathbb{M}_c \geq 0$ such that $|\mathscr{S}(v)| \leq \mathbb{M}_s$, for every

$v \in [0, a]$. Furthermore, take $\mathbb{M}_a = \sup_{0 \leq t \leq a} |\mathbb{A}\mathscr{S}(v)|$ and $\mathbb{M}_B = |\mathbb{B}^{-1}|$. Let $\mathbb{M}_c = \sum |c_i|$.

(A_2) The function $\mathbb{F} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ satisfies the following conditions: $\mathscr{H}^d\left([\mathbb{F}(v, z(v)]^\lambda, \mathbb{F}(v, y(v)]^\lambda\right)$
$\leq \mathbb{L}_f \mathscr{H}^d(z_v(v), z_v(v))$, for $v \in \mathbb{J}$ and $z_v, y_v \in \mathbb{E}_n$.

(A_3) For our convenience, we choose

$$\mathbb{M}_b \mathbb{M}_S |\mathbb{B}\mathcal{E}x_0| + b\mathbb{M}_b \mathbb{M}_s \mathbb{L}_f r$$
$$+ \mathbb{L}_c \mathbb{M}_b^2 |\mathscr{B} \mathcal{E}|\mathbb{M}_c b\mathbb{M}_s \mathbb{L}_f \leq \mathcal{L}; \quad \text{and}$$
$$\delta = b\mathbb{M}_b \mathbb{M}_s \mathbb{L}_f + b\mathbb{M}_c \mathbb{M}_b^2 |\mathscr{B} \mathscr{E}| |\mathbb{M}_c \mathbb{M}_s \mathbb{L}_f.$$

Existence and uniqueness

Theorem 3.1 *Assuming* $(A_1) - (A_3)$ *retain, there exists a solution for* (2.1)-(2.2) *on* $\mathbb{J}$.

Proof Consider the subset $\mathcal{X}$ of $\mathbb{C}(\mathbb{J}, \mathbb{E}_n)$. specified through

$$\mathcal{X} = \{z : z(v) \in \mathbb{C}(\mathbb{J}, \mathbb{E}_n, |z(v)| \leq r, v \in \mathbb{J}\}.$$

Again, we define a mapping $\mathscr{F} : \mathcal{X} \to \mathcal{X}$ by

$$\mathscr{F}(z(v)) = \mathbb{B}^{-1}\mathscr{C}(v)\mathbb{B}\mathcal{E}z_0 - \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{C}(v)\mathcal{E}$$

$$\left\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v_i - \mu)\mathbb{F}(\mu, z_\mu(\mu))d\mu \right\}$$

$$+ \int_0^{v} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\mathbb{F}(\mu, z_\mu(\mu))d\mu$$

$$+ \mathbb{B}^{-1}\mathbb{A}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0.$$

First to show that the operator $\mathscr{F}$ maps $\mathcal{X}$ into itself. Now

$$\begin{aligned}
|\mathscr{F}(z(\nu))| \leq & |\mathbb{B}^{-1}\mathscr{C}(\nu)\mathbb{B}\mathcal{E}z_0| + |\int_0^\nu \mathscr{S}(\nu-\mu)\mathbb{B}^{-1}\mathbb{F}(\mu,z(\mu))d\mu| \\
& + |\sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{C}(\nu)\mathbb{B}\mathcal{E}\int_0^{\nu_i}\mathbb{B}^{-1}\mathscr{S}(\nu_i-\mu)\mathbb{F}(\mu,z(\mu))d\mu| \\
\leq & \mathbb{M}_b\mathbb{M}_c|\mathscr{B}\mathscr{E} x_0| + b\mathbb{M}_b\mathbb{M}_s\mathbb{L}_f r + \mathbb{L}_c\mathbb{M}_b^2|\mathscr{B}\mathscr{E}|\mathbb{M}_c b\mathbb{M}_s\mathbb{L}_f.
\end{aligned}$$

From the assumption (A_3), $|\mathscr{F}(z(\nu))| \leq \mathcal{L}$. Therefore the $\mathscr{F}$ maps $\mathcal{X}$ into itself. Also, if $x, y \in \mathcal{X}$.

$$\begin{aligned}
& \mathcal{H}^d\Big(\mathscr{F}(z(\nu)), \mathscr{F}(y(\nu))\Big) \\
= & \mathscr{H}^d\Big(\Big[\mathbb{B}^{-1}\mathscr{S}(\nu)\mathbb{B}\mathcal{E}z_0 - \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(\nu)\mathcal{E} \times \Big\{\int_0^{\nu_i}\mathbb{B}^{-1}\mathscr{S}(\nu_i-\mu)\mathbb{F}(\mu,z(\mu))d\mu\Big\} \\
& + \int_0^\nu \mathbb{B}^{-1}\mathscr{S}(\nu-\mu)\mathbb{F}(\mu,z_\mu(\mu))d\mu + \mathbb{B}^{-1}\mathbb{A}\mathscr{S}(\nu)\mathbb{B}\mathcal{E}z_0\Big]^\lambda, \Big[\mathbb{B}^{-1}\mathscr{S}(\nu)\mathbb{B}\mathcal{E}y_0 \\
& - \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(\nu)\mathcal{E} \times \Big\{\int_0^{\nu_i}\mathbb{B}^{-1}\mathscr{S}(\nu_i-\mu)\mathbb{F}(\mu,y(\mu))d\mu\Big\} \\
& + \int_0^\tau \mathbb{B}^{-1}\mathscr{S}(\nu-\mu)\mathbb{F}(\mu,y(\mu))d\mu + \mathbb{B}^{-1}\mathbb{A}\mathscr{S}(\nu)\mathbb{B}\mathcal{E}y_0\Big]^\lambda\Big) \\
\leq & \mathscr{H}^d\Big(\Big[\sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(\nu)\mathcal{E} \times \Big\{\int_0^{\nu_i}\mathbb{B}^{-1}\mathscr{S}(\nu_i-\mu)\mathbb{F}(\mu,z_\mu(\mu))d\mu\Big\} \\
& + \int_0^\nu \mathbb{B}^{-1}\mathscr{S}(\nu-\mu)\mathbb{F}(\mu,z(\mu))d\mu + \mathbb{B}^{-1}\mathbb{A}\mathscr{S}(\nu)\mathbb{B}\mathcal{E}z_0\Big]^\lambda, \Big[\sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{C}(\nu)\mathcal{E} \\
& \times \Big\{\int_0^{\nu_i}\mathbb{B}^{-1}\mathscr{S}(\nu_i-\mu)\mathbb{F}(\mu,y(\mu))d\mu\Big\} + \int_0^\nu \mathbb{B}^{-1}\mathscr{S}(\nu-\mu)\mathbb{F}(\mu,y(\mu))d\mu\Big]^\lambda\Big) \\
\leq & [b\mathbb{M}_b\mathbb{M}_s\mathbb{L}_f + b\mathbb{L}_c\mathbb{M}_b^2|\mathbb{B}\mathcal{E}\mathbb{M}_c\mathbb{M}_s\mathbb{L}_f|\mathscr{H}^d\Big(z(\nu),y(\nu)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d^\infty(\mathscr{F}(z(\nu)), \mathscr{F}y(\nu)) = & \sup_{\nu\in\mathbb{J}}\mathscr{H}^d\Big(\mathscr{F}(z(\nu)), \mathscr{F}(y(\nu))\Big) \\
\leq & \sup_{\nu\in\mathbb{J}}[a\mathbb{M}_b\mathbb{M}_s\mathbb{L}_f + a\mathbb{L}_c\mathbb{M}_b^2|\mathscr{B}\mathscr{E}\mathbb{M}_c\mathbb{M}_s\mathbb{L}_f|\mathscr{H}^d\Big(x(\tau),y(\tau)\Big) \\
\leq & [b\mathbb{M}_b\mathbb{M}_s\mathbb{L}_f + b\mathbb{L}_c\mathbb{M}_b^2|\mathbb{B}\mathcal{E}\mathbb{M}_c\mathbb{M}_s\mathbb{L}_f|d^\infty(z(\nu),y(\nu)).
\end{aligned}$$

Hence,

$$\mathscr{H}^1\left(\mathscr{F}\,(z(v)),\mathscr{F}\,y(v)\right) = \sup_{v\in\mathscr{J}} d^\infty(\mathscr{F}\,(z(v)),\mathscr{F}\,(y(v))$$

$$\leq \sup_{t\in\mathbb{J}}[a\mathbb{M}_b\mathbb{M}_s\mathbb{L}_f + a\mathbb{L}_c\mathbb{M}_b^2|\mathbb{B}\mathcal{E}|\mathbb{M}_c\mathbb{M}_s\mathbb{L}_f d^\infty(z(v),y(v))$$

$$\leq \Delta\mathscr{H}^1\left(z(v),y(v)\right).$$

Since $\Delta < 1$, this show that the operator ζ is contraction on $\mathbb{E}_n$ and so, by Banach fixed point theorem, there exists a unique fixed point $z \in \mathscr{F}$ such that $\mathscr{F}\,(z(v)) = z(v)$. This fixed point is the solution of (2.1)-(2.2). Thus Theorem 3.1 is proved.

Case study on Sobolev type of FFDEs

Consider the fuzzy differential equation of the form with a non-local condition

$$(z(v))' = \tilde{\mathbf{3}}z(v) + \tilde{\mathbf{3}}vz(v)^2 \tag{3.1}$$

$$z(0) = \sum_{i=1}^n c_i(v_i). \tag{3.2}$$

The λ level set of fuzzy number $\tilde{\mathbf{3}}$: $[3]^\lambda = [\lambda + 2, 4 - \lambda]$. Now λ level set of $\mathbb{F}(v, z(v)) = \tilde{\mathbf{3}}vz(v)^2$ is

$$[\mathbb{F}(v, z(v)]^\lambda = [\tilde{\mathbf{3}}vz(v)^2]^\lambda$$
$$= v\left[(\lambda + 2)z_l^\lambda(v + k)^2, (4 - \lambda)z_r^\lambda(v + k)^2\right]$$

The λ- level set of $\sum_{i=1}^n c_i z(v_i)$: $\left[\sum_{i=1}^n c_i z(v_i)\right]^\lambda = \left[\sum_{i=1}^n c_i z_l^\lambda(v_i),\right.$ $\left.\sum_{i=1}^n c_i x_r^\lambda(v_i)\right]$

$$\mathscr{H}^d\left([\mathbb{F}(v, z(v))]^\lambda, [\mathbb{F}(v, y(v))]^\lambda\right)$$
$$= \mathscr{H}^d\left(v[(\lambda + 2)(z_l^\lambda(v))^2, (4 - \lambda)(z_r^\lambda(v))^2],\right.$$
$$\left.t[(\lambda + 2)(y_l^\lambda(v))^2, (4 - \lambda)(y_r^\lambda(v))^2]\right)$$
$$= t\max\{(\lambda + 2)|(z_l^\lambda(v))^2 - (y_l^\lambda(v))^2|, (4 - \lambda)|(z_r^\lambda(v))^2 - (y_r^\lambda(v))^2|\}$$
$$= v\max\{(\lambda + 2)|z_l^\lambda(v) + y_l^\lambda(v)||z_l^\lambda(v) - y_l^\lambda(v)|,$$
$$(4 - \lambda)|z_r^\lambda(v)) + (y_r^\lambda(v))||z_r^\lambda(v) - y_r^\lambda(v)|\}$$
$$\leq (4 - \lambda)t|z_r^\lambda(v)) + y_r^\lambda(v)|\max\{|z_l^\lambda(v) - y_l^\alpha(v)|, |z_r^\alpha(v) - y_r^\alpha(v)|\}$$
$$\leq (4 - \lambda)b|z_r^\lambda(v)) + y_r^\lambda(v)|\max\{|z_l^\lambda(v) - y_l^\lambda(v)|, |z_r^\lambda(v) - y_r^\lambda(v)|\}$$
$$\leq 4b|z_r^\lambda(v)) + y_r^\lambda(v)|\max\{|z_l^\lambda(v) - y_l^\lambda(v)|, |z_r^\lambda(v) - y_r^\lambda(v)|\}$$
$$= \mathbb{L}_f\mathcal{H}^d\left([z(\tau)]^\lambda, [y(\tau)]^\lambda\right),$$

where $\mathbb{L}_f = 4b|x_r^\lambda(v)) + y_r^\lambda(v)|$ meets the inequality stated within the circumstance (A_2).

$$\mathscr{H}^d\left([\sum_{i=1}^n c_i z(v_i)]^\lambda, [\sum_{i=1}^n c_i y(v_i)]^\lambda\right) = \mathscr{H}^d\left(\left[\sum_{i=1}^n c_i z_l^\lambda(v_i), \sum_{i=1}^n c_i z_r^\lambda(v_i)\right], \left[\sum_{i=1}^n c_i y_l^\lambda(v_i), \sum_{i=1}^n c_i y_r^\lambda(v_i)\right]\right)$$
$$\leq \mathbb{M}_c \mathcal{H}^d\left([z(v_i)]^\lambda, y(\tau_i)]^\lambda\right),$$

where $\mathbb{M}_c = |\sum_{i=1}^n c_i|$ satisfies the inequality which is given in condition (A_2).

Thus, all conditions of Theorem 3.1 are satisfied. Hence the system (3.1)-(3.2) has a unique fuzzy solution.

Fuzzy neutral integrodifferential equation

We shall explore at the non-linear neutral fuzzy integrodifferential equation of the form in this section.

$$\left[\mathbb{B}z(v) + \mathscr{P}(v, z(v))\right]' + \mathbb{A}z(v) = \mathbb{F}(v, z(v)) + \int_0^v \mathscr{K}(v, \mu, z(\mu))d\mu$$
$$v \in (0, a], \; v \neq v_k \qquad (4.1)$$

$$z(0) + \sum_{i=1}^n c_i z(v_i) = z_0, \qquad (4.2)$$

where $\mathbb{A}, \mathbb{B} : \mathbb{J} \to \mathbb{E}_n$ denotes a fuzzy coefficient and $\mathbb{J} = [0, a]$. The function $\mathbb{F}, \mathscr{P}, \mathscr{K} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ is non-linear fuzzy function and z_0 is a initial value.

To prove the existence of (4.1)-(4.2), the following requirements must be fulfilled:

(A_4) The function $\mathscr{P} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ satisfy the condition

$$\mathscr{H}^d\left([\mathscr{P}(v, z(v))]^\lambda, [\mathscr{P}(v, y(v))]^\lambda\right) \leq \mathbb{L}_p \mathscr{H}^d\left(z(v), y(v)\right).$$

There exists $\mathbb{L}_p \geq 0$.

(A_5) The function $\mathscr{K} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ satisfy the condition.

$$\mathscr{H}^d\left([\mathscr{K}(v, z(v))]^\lambda, [\mathscr{K}(v, y(v))]^\lambda\right) \leq \mathbb{L}_k \mathscr{H}^d\left(z(v), y(v)\right).$$

There exists $\mathbb{L}_k \geq 0$.

(A_6) For convenience

$$\mathbb{M}_b|\mathbb{B}\mathcal{E}z_0|\mathbb{M}_s + \mathbb{M}_c\mathbb{M}_b^2|\mathbb{B}\mathcal{E}|\mathbb{L}_p + \mathscr{M}_s[\mathscr{L}_p + a\mathscr{M}_s(\mathscr{L}_p + \mathscr{L}_f + a\mathscr{L}_k]$$
$$+ \mathscr{M}_b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + a\mathscr{L}_k + b(\mathscr{L}_p] \leq \nabla.$$

In this section, we'll suppose that there is an operator $\mathbb{E}$ on $\mathbb{E}_n$, which can be found using the formula

$$\mathcal{E} = \left[\mathbb{I} + \sum_{i=1}^n c_i \mathbb{B}^{-1} \mathscr{S}(v_i)\mathbb{B}\right]^{-1}$$

with

$$\mathcal{E}\left\{\mathbb{B}^{-1}\mathscr{P}(v, z(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) + \int_0^{v_i} \mathbb{A}\mathscr{S}(v_i - \mu)\mathbb{B}^{-1}\mathscr{P}(\mu, z(\mu))d\mu\right.$$
$$\left. + \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v_i - \mu)\left[\mathbb{F}(\mu, z(\mu)) + \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\right]d\mu\right\} \in \mathbb{E}_n.$$

Radhakrishnan *et al. BMC Research Notes*   (2024) 17:39

Page 8 of 18

Definition 4.1 An expression for the integral equation $z(v)$

Remark 4.1 The fuzzy functional neutral integrodifferential equation (4.1, 4.2) has a fuzzy solution that satisfies (4.3).

$$
\begin{aligned}
z(v) =\;& \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathbb{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) \Big\} \\
& + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v-\mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu)) \\
& + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v)) \\
& + \int_0^{v} \mathscr{S}(v-\mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu.
\end{aligned}
\tag{4.3}
$$

is said to be a fuzzy solution of (4.1, 4.2) on $\mathbb{E}_n$.

$$
\begin{aligned}
z(0) =\;& \mathbb{E}z_0 + \sum_{i=1}^{n} c_i \mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathcal{E}(0, z(0))\} \\
& \times \Big[\mathbb{A}\mathscr{P}(\mu, z_\mu(v)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\} \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
z(v_j) =\;& \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathcal{E}(0, z(0))\} \\
& + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}\Big\{ \int_0^{v_i} \mathscr{S}(v_i-\mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z_\mu(v)) + \mathbb{F}(\mu, z_\mu(\mu)) \\
& + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\} + \mathbb{B}^{-1}\mathscr{S}(v_j)\mathscr{P}(0, z(0)) \\
& - \mathbb{B}^{-1}\mathscr{P}(v_j, z_v(v_j)) \int_0^{v_j} \mathscr{S}(v_j-\mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z_\mu(\mu)) \\
& + \mathbb{F}(\mu, z_\mu(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& z(0) + \sum_{j=1}^{n} c_j z(v_j) \\
&= \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathcal{E}(0, z(0))\} \\
&+ \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v_j)\mathcal{E}\Big\{ \int_0^{v_i} \mathscr{S}(v_i-\mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z_\mu(v)) + \mathbb{F}(\mu, z_\mu(\mu)) \\
&+ \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\} + \mathbb{B}^{-1}\mathscr{S}(v_j)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v_j, z_v(v_j)) \\
&\times \int_0^{v_j} \mathscr{S}(v_j-\mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z_\mu(\mu)) + \mathbb{F}(\mu, z_\mu(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu.
\end{aligned}
$$

Theorem 4.2   *If the assumptions (A_1)-(A_6) are true, then (4.1)-(4.2) offers an ambiguous conclusion on $\mathbb{J}$.*

Proof   Let $\mathcal{F}_1$ be the subset of $\mathbb{C}(\mathbb{J}, \mathbb{E}_n)$ defined by

$$\mathcal{F}_1 = \{z : z(v) \in \mathbb{E}_n, |z(v)| \le r, \text{ for } t \in \mathbb{J}\}.$$

We define a mapping $\tilde{\tilde{\mathscr{F}}} : \mathcal{F}_1 \to \mathcal{F}_1$ by

$$(\tilde{\tilde{\mathscr{F}}} z)(v) = \mathbb{B}^{-1}\mathscr{S}(v)[\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 - \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}$$
$$\times \Big\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v_i - \mu)\mathbb{F}(\mu, z(\mu))d\mu + \int_0^t \mathscr{S}(v - \mu)\mathbb{B}^{-1}\mathbb{F}(\mu, z(\mu))d\mu.$$

First to show that the operator $\tilde{\tilde{\mathscr{F}}}$ maps $\mathcal{F}_1$ into itself.
Now

From the assumption (A_6), $|(\tilde{\tilde{\mathscr{F}}} z)(v)| \le \eta$. Therefore $\tilde{\tilde{\mathscr{F}}}$ maps $\mathcal{F}_1$ into itself. Moreover, if $z, y \in \mathcal{F}_1$, then

$$|(\tilde{\tilde{\mathscr{F}}} z)(v)| = |\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0) \Big\}$$
$$+ \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu))$$
$$+ \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v))$$
$$+ \int_0^{v} \mathscr{S}(v - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu|$$
$$\le |\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0| + |\sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \mathcal{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0) \Big\}|$$
$$+ |\sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)$$
$$+ \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu \Big\}| + |\mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v))$$
$$+ \int_0^{v} \mathscr{S}(v - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv \Big]d\mu|$$
$$\le \mathbb{M}_b|\mathscr{B}\,\mathscr{E}\,x_0|\mathbb{M}_s + \mathbb{M}_c\mathbb{M}_b^2|\mathscr{B}\,\mathscr{E}\,|\mathbb{L}_p\mathscr{M}_s[\mathscr{L}_p + b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + a\mathscr{L}_k]$$
$$+ \mathscr{M}_b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + b\mathscr{L}_k + a\mathscr{L}_p].$$

Radhakrishnan *et al. BMC Research Notes*   (2024) 17:39

Page 10 of 18

$$\mathscr{H}^d\left([\tilde{\mathscr{F}}z)(v)]^\lambda, [(\tilde{\mathscr{F}}z)(v)]^\lambda\right)$$

$$=\mathscr{H}^d\Bigg(\Big[\mathscr{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathscr{B}\;\mathscr{E}\;\Big\{\mathscr{B}^{-1}\mathscr{P}(\tau, x(\tau) - \mathscr{B}^{-1}\mathscr{S}(\tau_i)\mathscr{P}(0, x(0)\Big\}$$

$$+ \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{\int_0^{v_i}\mathbb{B}^{-1}\mathscr{S}(v-\mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu))$$

$$+ \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\Big]d\mu\Big\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathbb{P}(v, z(v)) + \int_0^v \mathscr{S}(v-\mu)$$

$$\times \mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\Big]d\mu\Big]^\lambda, \Big[\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}y_0$$

$$+ \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{\mathbb{B}^{-1}\mathscr{P}(v, y(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, y(0)\Big\}$$

$$+ \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\Big\{\int_0^{v_i}\mathbb{B}^{-1}\mathscr{S}(v-\mu)\Big[\mathbb{A}\mathscr{P}(\mu, y(\mu) + \mathbb{F}(\mu, y(\mu))$$

$$+ \int_0^\mu \mathscr{K}(\mu, v, y(v))dv\Big]d\mu\Big\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, y(0)) - \mathbb{B}^{-1}\mathscr{P}(v, y(v)) + \int_0^v \mathscr{S}(v-\mu)$$

$$\times \mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, y(\mu)) + \mathbb{F}(\mu, y(\mu)) + \int_0^\mu \mathscr{K}(\mu, v, y(v))d\tau\Big]d\mu\Big]^\lambda\Bigg)$$

$$\leq \mathbb{M}_b|\mathbb{B}\mathcal{E}x_0|\mathbb{M}_s + \mathbb{M}_c\mathbb{M}_b^2|\mathscr{B}\;\mathscr{E}|\mathbb{L}_p\mathscr{M}_s[\mathscr{L}_p + b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + a\mathscr{L}_k]$$

$$+ \quad \mathscr{M}_b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + a\mathscr{L}_k + a\mathscr{L}_p] \leq \nabla\mathscr{H}^d\Big([z(v)]^\lambda, [y(v)]^\lambda\Big),$$

where $\S = \mathbb{M}_b|\mathscr{B}\;\mathscr{E}\;z_0|\mathbb{M}_s + \mathbb{M}_c\mathbb{M}_b^2|\mathscr{B}\;\mathscr{E}\;|\mathbb{L}_p\mathscr{M}_s[\mathscr{L}_p + b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + b\mathscr{L}_k]\mathscr{M}_b\mathscr{M}_s[\mathscr{L}_p + \mathscr{L}_f + b\mathscr{L}_k + b\mathscr{L}_p]$.

Now,

$$d^\infty\Big([\tilde{\mathscr{F}}z(v)]^\lambda, [\tilde{\mathscr{F}}y(v)]^\lambda\Big) \leq \sup_{\lambda\in(0,1)}\mathcal{H}^d([\mathscr{F}z(v)]^\lambda, [\mathscr{F}y(v)]^\lambda)$$

$$\leq \S d^\infty\Big(z(v), y(v)\Big).$$

Therefore,

$$\mathscr{H}^1\Big([\tilde{\mathscr{F}}z(v)]^\lambda, [\tilde{\mathscr{F}}y(v)]^\lambda\Big) \leq \sup_{\lambda\in(0,1)} d^\infty([\mathscr{F}z(v)]^\lambda, [\mathscr{F}y(v)]^\lambda)$$

$$\leq \S\mathcal{H}^1\Big(z(v), y(v)\Big).$$

Since $\S < 1$, the above equation, demonstrate the contraction associated with the operator $\tilde{\mathscr{F}}$ over $\mathbb{E}_n$ and hence by Banach fixed point theorem there exists a unique fixed point $z \in \mathcal{F}_1$ such that $(\tilde{\mathscr{F}}z)(v) = z(v)$. This fixed point is then the solution of the problem (4.1, 4.2).

Case study: Consider the non-local condition on the fuzzy neutral indegrodifferential equation

$$(z(v) - \mathbf{4}z(v))' = \tilde{\mathbf{3}}x(\tau) + \tilde{\mathbf{3}}\tau x^2(\tau) + \tilde{\mathbf{3}}\tau x^2(\tau); \quad (4.4)$$

$$x(0) = \sum_{i=1}^n c_i(\tau_i). \quad (4.5)$$

The λ level set of fuzzy number $\tilde{\mathbf{3}}$: $[3]^\lambda = [\lambda + 2, 4 - \lambda]$. Now λ level set of $\mathscr{F}(\tau, x(\tau)) = \tilde{\mathbf{3}}\tau x(\tau)^2$ is

$$[\mathscr{F}(v, z(v))]^\lambda = [\tilde{\mathbf{3}}vz(v)^2]^\lambda$$

$$= v\Big[(\lambda + 2)z_l^\lambda(v)^2, (4 - \lambda)z_r^\lambda(v)^2\Big].$$

The λ- level set of $\sum_{i=1}^{n} c_i z(v_i) : \left[\sum_{i=1}^{n} c_i z(v_i) \right]^{\lambda}$ where, $\mathbb{L}_k = 4b|z_r^{\lambda}(v)) + y_r^{\lambda}(v)|$ satisfies the inequality

$= \left[\sum_{i=1}^{n} c_i z_l^{\lambda}(v_i), \sum_{i=1}^{n} c_i x_r^{\lambda}(v_i) \right]$

$$\mathscr{H}^d \left([\mathbb{F}(v, z(v))]^{\lambda}, [\mathbb{F}(v, y(v))]^{\lambda} \right)$$

$$= \mathscr{H}^d \left(v[(\lambda+2)(z_l^{\lambda}(v))^2, (4-\lambda)(z_r^{\lambda}(v))^2], \right.$$

$$\left. t[(\lambda+2)(y_l^{\lambda}(v))^2, (4-\lambda)(y_r^{\lambda}(v))^2] \right)$$

$$= t \max\{(\lambda+2)|(z_l^{\lambda}(v))^2 - (y_l^{\lambda}(v))^2|, (4-\lambda)|(z_r^{\lambda}(v))^2 - (y_r^{\lambda}(v))^2|\}$$

$$= v \max\{(\lambda+2)|z_l^{\lambda}(v) + y_l^{\lambda}(v)||z_l^{\lambda}(v) - y_l^{\lambda}(v)|,$$

$$(4-\lambda)|z_r^{\lambda}(v)) + (y_r^{\lambda}(v))||z_r^{\lambda}(v) - y_r^{\lambda}(v)|\}$$

$$\leq (4-\lambda)t|z_r^{\lambda}(v)) + y_r^{\lambda}(v)| \max\{|z_l^{\lambda}(v) - y_l^{\lambda}(v)|, |z_r^{\lambda}(v) - y_r^{\lambda}(v)|\}$$

$$\leq (4-\lambda)b|z_r^{\lambda}(v)) + y_r^{\lambda}(v)| \max\{|z_l^{\lambda}(v) - y_l^{\lambda}(v)|, |z_r^{\lambda}(v) - y_r^{\lambda}(v)|\}$$

$$\leq 4b|z_r^{\lambda}(v)) + y_r^{\lambda}(v)| \max\{|z_l^{\lambda}(v) - y_l^{\lambda}(v)|, |z_r^{\lambda}(v) - y_r^{\lambda}(v)|\}$$

$$= \mathbb{L}_f \mathscr{H}^d \left([z(v)]^{\lambda}, [y(v)]^{\lambda} \right),$$

where $\mathbb{L}_f = 4b|z_r^{\lambda}(v)) + y_r^{\lambda}(v)|$ satisfies the inequality which is given in condition (A_2).

which is given in condition (A_5).

Thus, all conditions of Theorem 4.1 are satisfied. Hence

$$\mathscr{H}^d \left([\sum_{i=1}^{n} c_i z(v_i)]^{\lambda}, [\sum_{i=1}^{n} c_i y(v_i)]^{\lambda} \right) = \mathcal{H}^d \left(\left[\sum_{i=1}^{n} c_i x_l^{\lambda}(v_i), \sum_{i=1}^{n} c_i x_r^{\lambda}(v_i) \right], \left[\sum_{i=1}^{n} c_i y_l^{\lambda}(v_i), \sum_{i=1}^{n} c_i y_r^{\lambda}(v_i) \right] \right)$$

$$\leq \mathbb{M}_c \mathscr{H}^d \left([z(v_i)]^{\lambda}, y(v_i)]^{\lambda} \right),$$

where $\mathbb{M}_c = |\sum_{i=1}^{n} c_i|$ satisfies the inequality which is given in condition (A_2).

the system (4.4, 4.5) has a unique fuzzy solution.

$$\mathscr{H}^d \left([\mathscr{P}(v, z(v))]^{\lambda}, [\mathscr{P}(v, y(v))]^{\lambda} \right) = \mathscr{H}^d \left(v[(\lambda+3)(z_l^{\lambda}(v)^2), (5-\lambda)(z_r^{\lambda}(v)^2)], \right.$$

$$\left. \tau[(\lambda+1)(y_l^{\lambda}(v)^2), (3-\lambda)(y_r^{\lambda}(v))^2] \right)$$

$$= v \max\{(\lambda+1)|(z_l^{\lambda}(v)) - (y_l^{\lambda}(v))|, (3-\lambda)|(z_r^{\lambda}(v)) - (y_r^{\lambda}(v))|\}$$

$$\leq 5b \mathscr{H}^d \left([z(v_i]^{\lambda}, y(v_i]^{\lambda} \right) = \mathbb{L}_p \mathscr{H}^d \left([z(v_i]^{\lambda}, y(v_i)]^{\lambda} \right),$$

where, $\mathbb{L}_p = 5b|x_r^{\lambda}(\tau)) + y_r^{\lambda}(\tau)|$ satisfies the inequality which is given in condition (A_4).

$$\mathscr{H}^d \left([\int_0^v \mathscr{K}(v, \mu, z(\mu))]^{\lambda}, [\int_0^v \mathscr{K}(v, \mu, z(\mu))]^{\lambda} \right) = \mathscr{H}^d \left(v[(\lambda+2)(z_l^{\lambda}(v)^2), (4-\lambda)(z_r^{\lambda}(v)^2)], \right.$$

$$\left. \tau[(\lambda+1)(y_l^{\lambda}(v)^2), (3-\lambda)(y_r^{\lambda}(v))^2] \right)$$

$$= \tau \max\{(\lambda+2)|(z_l^{\lambda}(v)) - (y_l^{\lambda}(v))|,$$

$$(4-\lambda)|(z_r^{\lambda}(v)) - (y_r^{\lambda}(v))|\}$$

$$\leq 4b \mathscr{H}^d([z(v_i]^{\lambda}, y(v_i]^{\lambda}) = \mathbb{L}_k \mathscr{H}^d \left([z(v_i)]^{\lambda}, y(v_i)]^{\lambda} \right),$$

Fuzzy impulsive neutral integrodifferential equation

Take into account the following: Neutral integrodifferential equation with fuzziness

$$[\mathbb{B}z(v) + \mathscr{P}(v, z(v))]' + \mathbb{A}z(v) = \mathbb{F}(v, z(v)) + \int_0^v \mathscr{K}(v, \mu, z(\mu))d\mu$$
$$v \in (0, z], v \neq v_k \tag{5.1}$$

$$z(0) + \sum_{i=1}^n c_i z(v_i) = z_0 \tag{5.2}$$

$$\Delta z(v_k) = \mathcal{I}_k(z_{v_k}); k = 1, 2, ..., m, \tag{5.3}$$

where $\mathbb{A}, \mathbb{B} : \mathbb{J} \to \mathbb{E}_n$ denotes a fuzzy coefficient and $\mathbb{J} = [0, a]$, $\mathbb{E}_n$ is the collection of all upper semi continuous convex normal fuzzy numbers with bounded $\lambda-$ level intervals. The function $\mathbb{F}, \mathscr{P}, \mathscr{K} : \mathbb{J} \times \mathbb{E}_n \to \mathbb{E}_n$ is nonlinear fuzzy function and z_0 is a initial value and $\mathcal{I}_k z(v_k) = \Delta z(v_k) = z(v_k^+) - z(v_k^-) \in \mathbb{E}_n$. Denote $\mathbb{J}_0 = [0, v_1]$, $v_k = (v_k, v_{k+1}]$, $k = 1, 2, \ldots, m$ and define the following space:

Let $\mathscr{P}\mathscr{C}([0, a], \mathbb{X}) = \{z : z$ is a function from $[0, a]$ into $\mathbb{X}$ such that $z(v)$ is continuous at $v \neq v_k$ and left continuous at $v = v_i$ and the right limit $z(v_k^+)$ exists for $k = 1, 2, \ldots, m\}$. Similarly as in ([33]), we see that $\mathscr{P}\mathscr{C}([0, a], \mathbb{X})$ is a Banach space with norm

$$\|z\|_{\mathscr{P}\mathscr{C}} = \sup_{v \in [0,a]} \|z(v)\|.$$

To demonstrate the system's existence (5.1, 5.2, 5.3). The following requirements must be fulfilled in order for it to work:

(A_7)

$$\mathcal{H}^d\left([\mathcal{I}_k(z(v_k^-))]^\lambda, [\mathcal{I}_k(y(v_k^-))]^\lambda\right) \leq l_i \mathscr{H}^d\left([z(v)]^\lambda, [y(v)]^\lambda\right),$$

$$\sum_{i=1}^k l_i = \mathcal{L}_i.$$

In this part, we presume that an operator $\mathcal{E}$ exists on $\mathbb{E}_n$, which is described by the formula.

$$\mathcal{E} = \left[\mathbb{I} + \sum_{i=1}^n c_i \mathbb{B}^{-1}\mathscr{S}(v_i)\mathbb{B}\right]^{-1}$$

with

$$\mathcal{E}\left\{\mathbb{B}^{-1}\mathscr{P}(v, z(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) + \int_0^{v_i} \mathbb{A}\mathscr{S}(v_i - \mu)\mathbb{B}^{-1}\mathscr{P}(\mu, z(\mu))d\mu\right.$$
$$\left. + \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v_i - \mu)\left[\mathbb{F}(\mu, z(\mu)) + \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\right]d\mu\right\} \in \mathbb{E}_n.$$

Definition 5.1 The integral equation's z fuzzy solution

$$z(v) = \Psi\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}$$
$$\times \left\{\mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) - \sum_{0 < v_k < v}\mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k))\right\}$$
$$+ \sum_{i=1}^n c_i\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\left\{\int_0^{v_i}\mathbb{B}^{-1}\mathscr{S}(v - \mu)\left[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu)))\right.\right.$$
$$\left.\left. + \int_0^\mu\mathscr{K}(\mu, v, z(v))dv\right]d\mu\right\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v))$$
$$+ \int_0^v\mathscr{S}(v - \mu)\mathbb{B}^{-1}\left[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^\mu\mathscr{K}(\mu, v, z(v))dv\right]d\mu$$
$$- \sum_{0 < v_k < v}\mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k)) \tag{5.4}$$

is said to be a solution of (5.1, 5.2, 5.3) on $\mathbb{J}$.

Remark 5.1 The fuzzy neutral integrodifferential equation (5.1, 5.2, 5.3) of the Sobolev type satisfies (5.3).

Theorem 5.2 *If assumptions* (A_1)-(A_7) *hold, then* (5.1, 5.2, 5.3) *has a fuzzy solution on* $\mathbb{J}$.

Proof Let $\mathcal{F}_2$ be the subset of $\mathscr{P}\,\mathscr{C}\,(\mathbb{J},\mathbb{E}_n)$ defined by

$$
\begin{aligned}
z(0) &= \mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}\,(v,z(v)) - \mathbb{B}^{-1}\mathscr{S}\,(v_i)\mathscr{P}\,(0,z(0)) - \sum_{0<v_k<v}\mathbb{B}^{-1}\mathscr{S}\,(v_k - v_i)\mathcal{I}_k(z(v_k))\} \\
&\quad + \sum_{i=1}^{n} c_i \mathcal{E}\Big\{ \int_0^{v_i} \mathscr{S}\,(v_i - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}\,(\mu,z_\mu(v)) + f(\mu,z(\mu)) + \int_0^{\mu}\mathscr{K}\,(\mu,v,z(v))dv\Big]d\mu\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
z(v_j) &= \mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathcal{E}z_0 + \sum_{i=1}^{n} c_i\mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathcal{E}\Big\{ \mathbb{B}^{-1}\mathscr{P}\,(v,z(v)) - \mathbb{B}^{-1}\mathscr{S}\,(v_i)\mathcal{E}(0,z(0))\} \\
&\quad - \sum_{0<v_k<v}\mathbb{B}^{-1}\mathscr{S}\,(v_j - v_i)\mathcal{I}_k(z(v_j)) \\
&\quad + \sum_{i=1}^{n} c_i\mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathcal{E}\Big\{ \int_0^{v_i}\mathscr{S}\,(v_i - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}\,(\mu,z_\mu(v)) + \mathbb{F}(\mu,z_\mu(\mu)) \\
&\quad + \int_0^{\mu}\mathscr{K}\,(\mu,v,z(v))dv\Big]d\mu\Big\} + \mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathscr{P}\,(0,z(0)) - \mathbb{B}^{-1}\mathscr{P}\,(v_j,z_v(v_j))\int_0^{v_j}\mathscr{S}\,(v_j - \mu) \\
&\quad \times \mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}\,(\mu,z_\mu(\mu)) + \mathbb{F}(\mu,z_\mu(\mu)) + \int_0^{\mu}\mathscr{K}\,(\mu,v,z(v))dv\Big]d\mu \\
&\quad - \sum_{0<v_k<v}\mathbb{B}^{-1}\mathscr{S}\,(v_j - v_i)\mathcal{I}_k(z(v_j)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&z(0) + \sum_{j=1}^{n} c_j z(v_j) \\
&= \mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathcal{E}z_0 + \sum_{i=1}^{n} c_i\mathbb{B}^{-1}\mathscr{S}(v_j)\mathscr{E}\ \Big\{ \mathbb{B}^{-1}\mathscr{P}\,(v,z(v)) - \mathbb{B}^{-1}\mathscr{S}\,(v_i)\mathcal{E}(0,z(0))\} \\
&\quad - \sum_{0<v_k<v}\mathbb{B}^{-1}\mathscr{S}\,(v_j - v_i)\mathcal{I}_k(z(v_k)) \\
&\quad + \sum_{i=1}^{n} c_i\mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathcal{E}\Big\{ \int_0^{\tau_i}\mathscr{S}\,(v_i - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}\,(\mu,z_\mu(v)) + \mathbb{F}(\mu,z_\mu(\mu)) \\
&\quad + \int_0^{\mu}\mathscr{K}\,(\mu,v,z(v))dv\Big]d\mu\Big\} + \mathbb{B}^{-1}\mathscr{S}\,(v_j)\mathscr{P}\,(0,z(0)) - \mathbb{B}^{-1}\mathscr{P}\,(v_j,z_v(v_j)) \\
&\quad \times \int_0^{v_j}\mathscr{S}\,(v_j - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}\,(\mu,z_\mu(\mu)) + \mathbb{F}(\mu,z_\mu(\mu)) + \int_0^{\mu}\mathscr{K}\,(\mu,v,z(v))dv\Big]d\mu \\
&\quad - \sum_{0<v_k<v}\mathbb{B}^{-1}\mathscr{S}\,(v_j - v_i)\mathcal{I}_k(z(v_k))\ =\ z_0.
\end{aligned}
$$

$$\mathcal{F}_2 = \{z : z(v) \in \mathbb{E}_n, |z(v)| \le r, \text{ for } t \in \mathbb{J}\}.$$

We define a mapping $\tilde{\tilde{\mathscr{F}}}_1 : \mathcal{F}_2 \to \mathcal{F}_2$ by

Let $\quad \mathbb{M}_b |\mathbb{B}\mathcal{E}z_0|\mathbb{M}_s + \mathbb{M}_b\mathbb{M}_s\mathbb{L}_p + a\mathbb{M}_b\mathbb{M}_s\mathbb{L}_p r + a\mathbb{M}_b\mathbb{M}_s$
$[\mathbb{L}_p r + a\mathbb{L}_k r] + \mathbb{M}_c\mathbb{M}_b^2 |\mathscr{B} \mathcal{E}|\mathbb{M}_s\Big[\mathbb{M}_s\mathbb{L}_p r + \mathbb{M}_s\mathbb{L}_p + a\mathbb{M}_s$
$[\mathscr{L}_f r + a\mathbb{L}_k] + \mathbb{M}_s\mathbb{L}_i\Big] + \mathbb{M}_s\mathbb{L}_i = \mathfrak{L}_B.$ Hence,

$$
\begin{aligned}
(\tilde{\tilde{\mathscr{F}}}_1 z)(v) = \quad & \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E} \\
\times \quad & \left\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) - \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(\tau_k)) \right\} \\
+ \quad & \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\left\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu))) \right. \\
+ \quad & \left. \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv\Big]d\mu \right\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v)) \\
+ \quad & \int_0^v \mathscr{S}(v - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))dv\Big]d\mu \\
- \quad & \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k)).
\end{aligned}
$$

To begin, we illustrate that the operator $\tilde{\tilde{\mathscr{F}}}_1$ transfers $\mathcal{F}_2$ to itself. Now

$$
\begin{aligned}
|(\tilde{\tilde{\mathscr{F}}}_1 z)(v)| = & |\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E} \\
\times \quad & \left\{ \mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) - \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k)) \right\} \\
+ \quad & \sum_{i=1}^{n} c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\left\{ \int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\Big[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu))) \right. \\
+ \quad & \left. \int_0^{\mu} \mathscr{K}(\mu, v, x(\tau))d\tau\Big]d\mu \right\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v)) \\
+ \quad & \int_0^v \mathscr{S}(v - \mu)\mathbb{B}^{-1}\Big[\mathbb{A}\mathscr{P}(\mu, z\mu)) + \mathbb{F}(\mu, z(\mu)) + \int_0^{\mu} \mathscr{K}(\mu, v, z(v))d\tau\Big]d\mu \\
- \quad & \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k))| \\
\le \quad & \mathbb{M}_b |\mathbb{B}\mathcal{E}z_0|\mathbb{M}_s + \mathbb{M}_b\mathbb{M}_s\mathbb{L}_p + b\mathbb{M}_b\mathbb{M}_s\mathbb{L}_p r + b\mathbb{M}_b\mathbb{M}_s[\mathbb{L}_p r + b\mathbb{L}_k r] \\
+ \quad & \mathbb{M}_c\mathbb{M}_b^2 |\mathscr{B} \mathcal{E} |\mathbb{M}_s\Big[\mathbb{M}_s\mathbb{L}_p r + \mathbb{M}_s\mathbb{L}_p + b\mathbb{M}_s[\mathscr{L}_f r + a\mathbb{L}_k] + \mathbb{M}_s\mathbb{L}_i\Big] + \mathbb{M}_s\mathbb{L}_i.
\end{aligned}
$$

$|(\tilde{\tilde{\mathscr{F}}}_1 z)(v)| \leq \mathfrak{L}_B.$

Therefore $\tilde{\tilde{\mathscr{F}}}_1$ maps $\mathcal{F}_2$ into itself. Moreover, if $z, y \in \mathcal{F}_2$, then

Now,

$$d^\infty\left([\tilde{\tilde{\mathscr{F}}}_1 z(v)]^\lambda, [\tilde{\tilde{\mathscr{F}}}_1 y(v)]^\lambda\right) \leq \sup_{\lambda \in (0,1)} \mathscr{H}^d\left([\tilde{\tilde{\mathscr{F}}} z(v)]^\lambda, [\tilde{\tilde{\mathscr{F}}} y(v)]^\lambda\right)$$
$$\leq \mathfrak{L}_B d^\infty\left(z(v), y(v)\right).$$

Therefore

$$\mathscr{H}^d\left([\tilde{\tilde{\mathscr{F}}}_1 z)(v)]^\lambda, [(\tilde{\tilde{\mathscr{F}}}_1 y)(v)]^\lambda\right)$$

$$=\mathscr{H}^d\left(\left[\mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}z_0 + \sum_{i=1}^n c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\right.\right.$$

$$\times \left\{\mathbb{B}^{-1}\mathscr{P}(v, z(v) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, z(0)) - \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k))\right\}$$

$$+ \sum_{i=1}^n c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\left\{\int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\left[\mathbb{A}\mathscr{P}(\mu, z(\mu) + \mathbb{F}(\mu, z(\mu)))\right.\right.$$

$$+ \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\bigg]d\mu\bigg\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(v, z(v))$$

$$+ \int_0^v \mathscr{S}(v - \mu) \quad \mathbb{B}^{-1}\left[\mathbb{A}\mathscr{P}(\mu, z(\mu)) + \mathscr{F}(\mu, z(\mu)) + \int_0^\mu \mathscr{K}(\mu, v, z(v))dv\right]d\mu$$

$$- \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(z(v_k))\bigg]^\lambda, \left[\mathbb{B}^{-1}\mathscr{S}(\tau)\mathbb{B}\mathbb{E}y_0 + \sum_{i=1}^n c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\right.$$

$$\times \left\{\mathbb{B}^{-1}\mathscr{P}(v, y(v)) - \mathbb{B}^{-1}\mathscr{S}(v_i)\mathscr{P}(0, y(0)) - \sum_{0 < v_k < v} \mathbb{B}^{-1}\mathscr{S}(v_k - v_i)\mathcal{I}_k(y(v_k))\right\}$$

$$+ \sum_{i=1}^n c_i \mathbb{B}^{-1}\mathscr{S}(v)\mathbb{B}\mathcal{E}\left\{\int_0^{v_i} \mathbb{B}^{-1}\mathscr{S}(v - \mu)\left[\mathbb{A}\mathscr{P}(\mu, y(\mu) + \mathbb{F}(\mu, y(\mu)))\right.\right.$$

$$+ \int_0^\mu \mathscr{K}(\mu, v, y(v))dv\bigg]d\mu\bigg\} + \mathbb{B}^{-1}\mathscr{S}(v)\mathscr{P}(0, z(0)) - \mathbb{B}^{-1}\mathscr{P}(\tau, y(\tau))$$

$$+ \int_0^v \mathscr{S}(v - \mu)\mathscr{B}^{-1}\left[\mathscr{A}\mathscr{P}(\mu, y(\mu)) + \mathscr{F}(\mu, y(\mu)) + \int_0^\mu \mathscr{K}(\mu, \tau, y(\tau))d\tau\right]d\mu$$

$$- \sum_{0 < v_k < v} \mathscr{B}^{-1}\mathscr{S}(\tau_k - \tau_i)\mathcal{I}_k(y(\tau_k))\bigg]^\lambda\bigg)$$

$$\leq \left[\mathbb{M}_b|\mathscr{B}\,\mathscr{E}\,z_0|\mathbb{M}_s + \mathbb{M}_b\mathbb{M}_s\mathbb{L}_p + b\mathbb{M}_b\mathbb{M}_s\mathbb{L}_p r + a\mathbb{M}_b\mathbb{M}_s[\mathbb{L}_p r + b\mathbb{L}_k r]\right.$$

$$+ \mathbb{M}_c\mathbb{M}_b^2|\mathscr{B}\,\mathscr{E}\,|\mathbb{M}_s\left[\mathbb{M}_s\mathbb{L}_p r + \mathbb{M}_s\mathbb{L}_p + a\mathbb{M}_s[\mathscr{L}_f r + a\mathbb{L}_k] + \mathbb{M}_s\mathbb{L}_i\right] + \mathbb{M}_s\mathbb{L}_i\right]\mathscr{H}^d\left([x(\tau)]^\lambda, [y(\tau)]^\lambda\right)$$

$$\leq \mathfrak{L}_B\mathcal{H}^d\left([z(v)]^\lambda, [y(v)]^\lambda\right).$$

$$\mathscr{H}^1\left([\tilde{\mathscr{F}}_1 z(\nu)]^\lambda, [\tilde{\mathscr{F}}_1 y(\nu)]^\lambda\right) \leq \sup_{\lambda \in (0,1)} d^\infty\left([\mathscr{F} z(\nu)]^\lambda, [\mathscr{F} y(\nu)]^\lambda\right)$$

$$\leq \mathfrak{L}_B \mathscr{H}^1(z(\nu), y(\nu)).$$

Since $\mathfrak{L}_B < 1$, the above equation, the operator $\tilde{\mathscr{F}}_1$ is a contraction on $\mathbb{E}_n$, the Banach fixed point theorem indicates that there exists a unique fixed point z such that $(\tilde{\mathscr{F}}_1 z)(\nu) = z(\nu)$. At this fixedpoint, the problems (5.1)–(5.3) arrive at their conclusion.

Case study

Consider the fuzzy neutral indegrodifferential equation with non-local condition

$$(x(\tau) - \tilde{4}\tau x^2(\tau))' = \tilde{2}[x(\tau) + \tau x^2(\tau)] + \tilde{3}\tau x(\tau)^2 \tag{6.1}$$

$$x(0) = \sum_{i=1}^{n} c_i x((\tau_k)) \tag{6.2}$$

$$\Delta x(\tau_k) = \mathcal{I}_k(x(\tau_k)) = 2e^{2\tau} x(\tau^-). \tag{6.3}$$

Let $\mathscr{P}(\tau, x(\tau)) = 4\tau x^2(\tau), \mathscr{F}(\tau, x(\tau) = \tilde{2}x^2(\tau), \int_0^\tau \mathscr{K}(\tau, \mu, x(\mu))d\mu = \tilde{3}\tau x^2(\tau), \mathcal{I}_k(x(\tau_k)) = 2e^{2\tau} x(\tau_k^-).$

The λ level set of fuzzy numbers

$$\tilde{0} : [0]^\lambda = [\lambda - 1, 1 - \lambda];$$
$$\tilde{2} : [2]^\lambda = [\lambda + 1, 3 - \lambda];$$
$$\tilde{3} : [3]^\lambda = [\lambda + 2, 4 - \lambda];$$
$$\tilde{4} : [4]^\lambda = [\lambda + 3, 5 - \lambda];$$

Now α level set of functions are

$$[\mathscr{F}(\tau, x(\tau)]^\lambda = [\tilde{2}\tau x^2(\tau)]^\lambda$$
$$= \tau\left[(\lambda + 1)x_l^\lambda(\tau)^2, (2 - \lambda)x_r^\lambda(\tau^2)\right]$$
$$[\mathscr{P}(\tau, x(\tau))]^\lambda = [\tilde{4}\tau x(\tau)]^\lambda$$
$$= \tau\left[(\lambda + 3)x_l^\lambda(\tau), (5 - \lambda)x_r^\lambda(\tau)\right]$$
$$\left[\int_0^\tau \mathscr{K}(\tau, \mu, x(\mu))d\mu\right]^\lambda = [\tilde{3}\tau x(\tau)^2]^\lambda$$
$$= \tau\left[(\lambda + 2)x_l^\lambda(\tau)^2, (4 - \lambda)x_r^\lambda(\tau)^2\right].$$

The λ- level set of $\sum_{i=1}^{n} c_i x(\tau_i) : \left[\sum_{i=1}^{n} c_i x(\tau_i)\right]^\lambda$
$= \left[\sum_{i=1}^{n} c_i x_l^\lambda(\tau_i), \sum_{i=1}^{n} c_i x_r^\lambda(\tau_i)\right]$

$$\mathcal{H}^d\left([\mathscr{F}(\tau, x(\tau))]^\lambda, [\mathscr{F}(\tau, y(\tau))]^\lambda\right) = \mathcal{H}^d\Big(\tau[(\lambda + 1)(x_l^\lambda(\tau)), (3 - \lambda)(x_r^\lambda(\tau))],$$
$$\tau[(\lambda + 1)(y_l^\lambda(\tau)), (3 - \lambda)(y_r^\lambda(\tau))]\Big)$$
$$= \tau \max\{(\lambda + 1)|(x_l^\lambda(\tau)) - (y_l^\lambda(\tau))|, (3 - \lambda)|(x_r^\lambda(\tau)) - (y_r^\lambda(\tau))|\}$$
$$= \tau \max\{(\lambda + 1)|x_l^\lambda(\tau) + y_l^\lambda(\tau)||x_l^\lambda(\tau) - y_l^\lambda(\tau)|,$$
$$(3 - \lambda)|x_r^\lambda(\tau)) + (y_r^\lambda(\tau))||x_r^\lambda(\tau) - y_r^\lambda(\tau)|\}$$
$$\leq (3 - \lambda)t|x_r^\lambda(\tau)) + y_r^\lambda(\tau)| \max\{|x_l^\lambda(\tau) - y_l^\lambda(\tau)|, |x_r^\lambda(\tau) - y_r^\lambda(\tau)|\}$$
$$\leq (3 - \lambda)b|x_r^\lambda(\tau)) + y_r^\lambda(\tau)| \max\{|x_l^\lambda(\tau) - y_l^\lambda(\tau)|, |x_r^\lambda(\tau) - y_r^\lambda(\tau)|\}$$
$$\leq 3b|x_r^\lambda(\tau)) + y_r^\lambda(\tau)| \max\{|x_l^\lambda(\tau) - y_l^\lambda(\tau)|, |x_r^\lambda(\tau) - y_r^\lambda(\tau)|\}$$
$$= \mathbb{L}_f \mathcal{H}^d\left([x(\tau)]^\lambda, [y(\tau)]^\lambda\right),$$

where $\mathbb{L}_f = 3b|x_r^\lambda(\tau)) + y_r^\lambda(\tau)|$ complies with the inequality indicated within the case of (A_2).

$$\mathcal{H}^d\left([\sum_{i=1}^{n} c_i x(\tau_i)]^\lambda, [\sum_{i=1}^{n} c_i y(\tau_i)]^\lambda\right) = \mathcal{H}^d\left(\left[\sum_{i=1}^{n} c_i x_l^\lambda(\tau_i), \sum_{i=1}^{n} c_i x_r^\lambda(\tau_i)\right], \left[\sum_{i=1}^{n} c_i y_l^\lambda(\tau_i), \sum_{i=1}^{n} c_i y_r^\lambda(\tau_i)\right]\right)$$
$$\leq \mathbb{M}_c \mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i]^\lambda\right),$$

where $\mathbb{M}_c = |\sum_{i=1}^{n} c_i|$ complies with the inequality indicated within the case of (A_2).

$$
\begin{aligned}
\mathcal{H}^d\left([\mathscr{P}\,(\tau, x(\tau))]^\lambda, [\mathscr{P}\,(\tau, y(\tau))]^\lambda\right) =& \mathcal{H}^d\Big(\tau[(\lambda+3)(x_l^\lambda(\tau)^2), (5-\lambda)(x_r^\lambda(\tau)^2)], \\
& \tau[(\lambda+1)(y_l^\alpha(\tau)^2), (3-\lambda)(y_r^\lambda(\tau))^2]\Big) \\
=& \tau \max\{(\lambda+1)|(x_l^\lambda(\tau)) - (y_l^\lambda(\tau))|, (3-\lambda)|(x_r^\lambda(\tau)) - (y_r^\lambda(\tau))|\} \\
\leq& 5b\mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda\right) = \mathbb{L}_p \mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda\right),
\end{aligned}
$$

where, $\mathbb{L}_p = 5b|x_r^\lambda(\tau)) + y_r^\lambda(\tau)|$ fulfills the inequality stipulated in premise (A_4).

$$
\begin{aligned}
\mathcal{H}^d\left(\left[\int_0^\tau \mathscr{K}\,(\tau, \mu, x(\mu))\right]^\lambda, \left[\int_0^\tau \mathscr{K}\,(\tau, \mu, x(\mu))\right]^\lambda\right) =&\ \mathcal{H}^d\Big(\tau[(\lambda+2)(x_l^\lambda(\tau)^2), (4-\lambda)(x_r^\lambda(\tau)^2)], \\
& \tau[(\lambda+1)(y_l^\lambda(\tau)^2), (3-\lambda)(y_r^\lambda(\tau))^2]\Big) \\
=&\ \tau \max\{(\lambda+2)|(x_l^\lambda(\tau)) - (y_l^\lambda(\tau))|, \\
& (4-\lambda)|(x_r^\lambda(\tau)) - (y_r^\lambda(\tau))|\} \\
\leq&\ 4b\mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda\right) = \mathbb{L}_k \mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda\right),
\end{aligned}
$$

where, $\mathbb{L}_k = 4b|x_r^\lambda(\tau)) + y_r^\lambda(\tau)|$ meets the inequality stated in the scenario (A_5).

$$
\begin{aligned}
\mathcal{H}^d\left([\mathcal{I}_k(x_k(\tau^-))]^\lambda, [\mathcal{I}_k(y_k(\tau^-))]^\lambda\right) =&\ \mathcal{H}^d\left(\left[2e^{-2\tau}x(\tau_i)\right]^\lambda, [2e^{-2\tau}y(\tau^-)]^\lambda\right) \\
\leq&\ (3-\lambda)2e^{-2\tau}\mathcal{H}^d([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda) \\
\leq&\ \mathbb{L}_i \mathcal{H}^d\left([x(\tau_i)]^\lambda, y(\tau_i)]^\lambda\right),
\end{aligned}
$$

where, $\mathbb{L}_i = 3e^{-2b}$ satisfies the inequality which is given in condition (A_7). As a result, all of Theorem 5.1's requirements are met. As an outcome, the fuzzy solution for the system (6.1, 6.2, 6.3) appears unique.

Conclusion

The outcomes of this work demonstrated the existence of specific types of impulsive neutral integrodifferential equations with Sobolev-type non-local conditions in a fuzzy environment. Fuzzy intervals that are normal, convex, upper semi-continuous, and compactly supported were used in conjunction with the fixed point strategy to examine the findings. An example is given in order to illustrate the concept for each case. Future research will expand this type of study to include control theory and fractional calculus.

Data Availability
Not Applicable.

Declarations

Ethical approval and consent to participate
Not applicable.

Consent for publication
Not Applicable.

Competing interests
Authors don't have a competing interests

Radhakrishnan *et al. BMC Research Notes*   (2024) 17:39

Page 18 of 18

References

1. Gomes LT, de Barros LC, Bede B. Fuzzy differential equations in various approached. New York: Springer; 2015.
2. Kaleva O. Nonlinear iteration semigroups of fuzzy Cauchy problems. Fuzzy Sets Syst. 2012;209:104–10.
3. Ding Z, Kandel A. On the controllability of fuzzy dynamical systems (I). J Fuzzy Math. 2000;8:203–14.
4. Pazy A. Semigroups of linear operators and applications to partial differential equations. New York: Springer Verlag; 1983.
5. Balachandran K, Park JY. Sobolev type integrodifferential equation with non-local condition in Banach spaces. Taiwan J Math. 2003;7:155–63.
6. Ezzinbi K, Toure H, Zabsonre I. Existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces. Nonlinear Anal Theor Methods Appl. 2009;70:2761–71.
7. Hernandez E, Dos Santos JPC. Existence results for partial neutral integro-differential equations with unbounded delay. Appl Anal. 2007;86:223–37.
8. Radhakrishnan B. Existence of solution for semilinear neutral impulse mixed integrodifferential inclusion Sobolev type in Banach space. Dyn Contin Discrete Impuls Syst Ser A Math Anal. 2017;24:317–32.
9. Radhakrishnan B. Existence results for second order neutral integrodifferential equations of Sobolev type in Banach Spaces. Thai J Math. 2018;16:25–47.
10. Agarwal S, Bahuguna D. Existence of solutions to Sobolev type partial neutral differential equations. J Appl Math Stoch Anal. 2006;163:1–10.
11. Balachandran K, Park JY, Chandrasekaran M. Nonlocal Cauchy problem for delay integrodifferential equations of Soboleve type in Banach spaces. Appl Math Lett. 2002;15:845–54.
12. Lagnese JE. Exponential stability of solutions of differential equation of Sobolev type. SIAM J Math Anal. 1972;3:625–36.
13. Byszewski L. Theorems about the existence and uniqueness of solutions of a semi-linear evolution integral Cauchy problem. J Math Anal Appl. 1991;162:494–505.
14. Bedi P, Kumar A, Khan A. Controllability of neutral impulsive fractional differential equations with Atangana-Baleanu-Caputo derivatives. Chaos Solitons Fract. 2021;150:111–53.
15. Bedi P, Kumar A, Abdeljawad T, Khan A. Existence of mild solutions for impulsive neutral Hilfer fractional evolution equations. Adv Differ Equ. 2020;155:1–16.
16. Radhakrishnan B, Tamilarasi M. Existence results for quasilinear random impulsive abstract differential inclusions in Hilbert space. J Anal. 2018. https://doi.org/10.1007/s41478-018-0132-3.
17. Rogovchenko Y. Nonlinear impulsive evolution systems and applications to population models. J Math Anal Appl. 1997;207:300–15.
18. Shu Z. On impulsive processes. Int Non-linear Mech. 2002;37:213–24.
19. Wu SJ, Guo XL, Lin ZS. Existence and uniqueness of solutions to random impulsive differential systems. Acta Math Appl Sin. 2006;4:627–32.
20. Yong Z, Wu SJ. Existence and Uniqueness of solutions to stochastic differential equations with random impulsive under lipschitz conditions. Chin J Appl Probab Statist. 2010;28:347–56.
21. Zavalishchin ST, Sesekin AN. Dynamic impulse systems: theory and applications. Alphen aan den Rijn: Kluwer Academic Publishers; 1997.
22. Hale JK, Verduyn Lunel SM. Introduction to functional-differential equations. New York: Springer-Verlag; 1993.
23. Zadeh LA. Fuzzy Sets. Inf Control. 1965;81:338–53.
24. Diamond P, Koleden PE. Metric Space of Fuzzy Sets. Singapore: World Scientific; 1994.
25. Kaleva O. On the convergence of fuzzy sets. Fuzzy Sets Syst. 1985;17:53–65.
26. Kaleva O. Fuzzy differential equations. Fuzzy Sets Syst. 1987;24:301–17.
27. Arshad S. On existence and uniqueness of solution of fuzzy fractional differential equations. Iran J Fuzzy Syst. 2013;10:137–51.
28. Gal CG, Gal SG. Semigroups of operators on spaces of fuzzy-number-valued functions with applications to fuzzy differential equations. J Fuzzy Math. 2005;3:647–82.
29. Jeong JU. Existence results for fractional order fuzzy differential equations with infinite delay. Int Math Forum. 2010;5:3221–30.
30. Ngo HV, Lupulescu V, O'Regan D. A note on initial value problems for fractional fuzzy differential equations. Fuzzy Set Syst. 2018;347:54–69.
31. Nieto JJ, López RR. Bounded solutions for fuzzy differential and integral equations. Chaos Solitons Fract. 2006;27:1376–86.
32. Zimmermann HJ. Fuzzy set theory and applications. Berlin: Springer Science and Business; 2011.
33. Liang J, Liu JH, Xiao TJ. Non-local impulsive problems for nonlinear differential equations in Banach spaces. Math Comp Model. 2009;49:798–804.

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