

EXISTENCE AND UNIQUENESS RESULTS FOR ψ -FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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ABSTRACT. We study boundary value problems (BVPs for short) for the integro-differential equations via ψ -fractional derivative. The results are obtained by using the contraction mapping principle and Schaefer's fixed point theorem. In addition, we discuss the Ulam–Hyers stability.

1. Introduction

In this paper, we investigate the existence and Ulam–Hyers stability results for ψ -fractional integro-differential equation with boundary condition of the form

$$(1.1) \quad {}^c\mathfrak{D}^{\alpha;\psi}u(t) = \mathfrak{F}\left(t, u(t), \int_0^t \mathfrak{h}(t, s, u(s))ds\right), \text{ for each } t \in \mathfrak{I} := [0, T], \alpha \in (0, 1),$$

$$(1.2) \quad au(0) + bu(T) = c,$$

where ${}^c\mathfrak{D}^{\alpha;\psi}$ is the ψ -Caputo fractional derivative of order α . Let $\mathfrak{F}, \psi: \mathfrak{I} \times \mathfrak{R} \rightarrow \mathfrak{R}$, $\mathfrak{h}: \Delta \times \mathfrak{R} \rightarrow \mathfrak{R}$ are continuous functions and a, b, c are real constants with $a + b \neq 0$. Here $\Delta: \{(t, s) : 0 \leq s \leq t \leq T\}$. For sake of brevity, we take

$$\mathfrak{H}u(t) = \int_0^t \mathfrak{h}(t, s, u(s))ds.$$

Fractional differential equations (FDEs) have recently confirmed to be important tools in the modelling of many phenomena in different fields of science and engineering. There are various applications to problems in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [4, 11] and references therein). There is a significant growth in ordinary and partial differential equations involving both Riemann–Liouville and Caputo fractional derivatives in modern years; see the monographs of Hilfer [13], Podlubny [20] and Samko et al. [21]. The theoretical study of these kinds of differential equations is significant for the

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applicability on the reality. For that reason, as a part of theoretical study, the pre-knowledge of the existence of a solution to FDEs is the first action for finding the analytic solution. Many natural phenomena can be formulated by BVPs of FDEs. We mention here some works on FDEs with boundary conditions (see [5, 7–10] and references therein). In [1], Aghajani et al. studied the solvability of a large class of nonlinear fractional integro-differential equations by establishing some fractional integral inequalities and using the nonlinear alternative Leray–Schauder type. Balachandran and Kiruthika analysed the existence of solutions of nonlinear fractional integrodifferential equations of Sobolev type with nonlocal condition in Banach spaces [6]. Very recently, Almeida [2] introduced the so-called ψ -fractional derivative with respect to another function. For more informations on ψ -type derivatives, see [12, 22].

The rest of the paper is arranged as follows. In Section 2, we recall some useful preliminaries. In Section 3, we give some sufficient conditions of the existence of the solutions and Ulam–Hyers stability is considered in Section 4.

2. Fundamental concepts

By $C(\mathcal{I}, \mathfrak{R})$ we denote the Banach space of all continuous functions from \mathcal{I} into \mathfrak{R} with the norm $\|u\|_\infty := \sup\{|u(t)| : t \in \mathcal{I}\}$.

For a detailed study on ψ -fractional derivative, we refer to [2, 23].

DEFINITION 2.1. Let $\alpha > 0$, $\mathcal{I} = [0, T]$ be a finite or infinite interval, \mathfrak{F} an integrable function defined on \mathcal{I} and $\psi \in C^1(\mathcal{I}, \mathfrak{R})$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in \mathcal{I}$. Fractional integrals and fractional derivatives of a function \mathfrak{F} with respect to another function ψ are defined as follows:

$$\begin{aligned} I^{\alpha; \psi} \mathfrak{F}(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s) ds, \\ \mathfrak{D}^{\alpha; \psi} \mathfrak{F}(t) &:= \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I^{n-\alpha; \psi} \mathfrak{F}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} \mathfrak{F}(s) ds, \end{aligned}$$

respectively, where $n = [\alpha] + 1$.

We declare the following generalization of Gronwall's lemma for ψ -fractional derivative. It plays a vital role in the proof of Ulam–Hyers stability.

LEMMA 2.1. [23, Theorem 3] *Let $\mathfrak{Z}, \mathfrak{W}: [0, T] \rightarrow [0, \infty)$ be continuous functions where $T \leq \infty$. If \mathfrak{w} is nondecreasing and there are constants $k \geq 0$ and $0 < \alpha < 1$ such that*

$$\mathfrak{Z}(t) \leq \mathfrak{W}(t) + k \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \mathfrak{Z}(s) ds, \quad t \in [0, T],$$

then

$$\mathfrak{Z}(t) \leq \mathfrak{W}(t) + \int_0^t \left(\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\psi(t) - \psi(s))^{n\alpha-1} \mathfrak{W}(s) \right) ds, \quad t \in [0, T].$$

REMARK 2.1. [23] Under the hypothesis of Lemma 2.1, let $\mathfrak{W}(t)$ be a nondecreasing function on $[0, T]$. Then we have $\mathfrak{Z}(t) \leq \mathfrak{W}(t)E_{\alpha, \psi}(k\Gamma(\alpha)(\psi(t))^\alpha)$.

THEOREM 2.1 (Banach's fixed point theorem). *Let C be a non-empty closed subset of a Banach space X , then any contraction mapping \mathfrak{P} of C into itself has a unique fixed point.*

THEOREM 2.2 (Schaefer's fixed point theorem). *Let $\mathfrak{P}: C(\mathfrak{I}, \mathfrak{R}) \rightarrow C(\mathfrak{I}, \mathfrak{R})$ completely continuous operator. If the set*

$$\kappa = \{u \in C(\mathfrak{I}, \mathfrak{R}) : u = \lambda \mathfrak{P}(u) \text{ for some } \lambda \in (0, T)\}$$

is bounded, then \mathfrak{P} has at least a fixed point.

3. Existence results

Let us begin by defining what we point out by a solution of the problem (1.1)–(1.2).

DEFINITION 3.1. A function $u \in C^1(\mathfrak{I}, \mathfrak{R})$ is said to be a solution of (1.1)–(1.2) if u satisfied the equation ${}^c\mathfrak{D}^{\alpha; \psi}u(t) = \mathfrak{F}(t, u(t))$ on \mathfrak{I} , and the condition $au(0) + bu(T) = c$.

We need the following lemma to derive the existence of solutions for the problem (1.1)–(1.2).

LEMMA 3.1. [22] *Let $\alpha \in (0, 1)$ and let $\mathfrak{F}, \psi: \mathfrak{I} \rightarrow \mathfrak{R}$, $\mathfrak{h}: \Delta \times \mathfrak{R} \rightarrow \mathfrak{R}$ be continuous. A function u is a solution of the ψ -fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, u(s), \mathfrak{H}u(s)) ds$$

if and only if u is a solution of the initial value problem for the ψ -fractional differential equation

$${}^c\mathfrak{D}^{\alpha; \psi}u(t) = \mathfrak{F}(t, u(t), \mathfrak{H}u(t)), \text{ for each } t \in \mathfrak{I} := [0, T], \alpha \in (0, 1), u(0) = u_0.$$

Due to a result of Lemma 3.1 we have the following result which is helpful in what follows.

LEMMA 3.2. *Let $\alpha \in (0, 1)$ and let $\mathfrak{F}, \psi: I \rightarrow \mathfrak{R}$, $\mathfrak{h}: \Delta \times \mathfrak{R} \rightarrow \mathfrak{R}$ be continuous. A function u is a solution of the ψ -fractional integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, u(s), \mathfrak{H}u(s)) ds \\ - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} \mathfrak{F}(s, u(s), \mathfrak{H}u(s)) ds - c \right]$$

if and only if u is a solution of the ψ -fractional BVP

$${}^c\mathfrak{D}^{\alpha; \psi}u(t) = \mathfrak{F}(t, u(t), \mathfrak{H}u(t)), \text{ for each } t \in \mathfrak{I} := [0, T], \alpha \in (0, 1), \\ au(0) + bu(T) = c.$$

We impose the following assumptions:

(A1) The function $\mathfrak{F}: \mathcal{I} \times \mathfrak{R} \rightarrow \mathfrak{R}$ is continuous.

(A2) There exists a constant $\mathfrak{K} > 0$ such that

$$|\mathfrak{F}(t, \mathfrak{X}) - \mathfrak{F}(t, \bar{\mathfrak{X}})| \leq \mathfrak{K}|\mathfrak{X} - \bar{\mathfrak{X}}|, \text{ for each } t \in \mathcal{I}, \forall \mathfrak{X}, \bar{\mathfrak{X}} \in \mathfrak{R}.$$

(A3) There exists a constant $\mathfrak{M} > 0$ such that

$$|\mathfrak{F}(t, \mathfrak{X}, \bar{\mathfrak{X}})| \leq \mathfrak{M} \text{ for each } t \in \mathcal{I} \text{ and } \forall \mathfrak{X}, \bar{\mathfrak{X}} \in \mathfrak{R}.$$

(A4) The function $\mathfrak{h}: \Delta \times \mathfrak{R} \rightarrow \mathfrak{R}$ is continuous and there exists a constant $\mathfrak{H}_1 > 0$ such that

$$|\mathfrak{h}(t, s, \mathfrak{X}) - \mathfrak{h}(t, s, \bar{\mathfrak{X}})| \leq \mathfrak{H}_1|\mathfrak{X} - \bar{\mathfrak{X}}|, \quad \forall \mathfrak{X}, \bar{\mathfrak{X}} \in \mathfrak{R}.$$

Our first result is derived from the Banach fixed point theorem.

THEOREM 3.1. *Suppose (A1), (A2) and (A4) hold. If*

$$(3.1) \quad \frac{\mathfrak{K}(1 + \mathfrak{H}_1)(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|}\right) < 1,$$

then the BVP (1.1)–(1.2) has only one solution on \mathcal{I} .

PROOF. Convert the problem (1.1)–(1.2) into a fixed point problem. Let $\Phi = C(\mathcal{I}, \mathfrak{R})$. Consider the operator $\mathfrak{P}: \Phi \rightarrow \Phi$ defined by

$$(3.2) \quad \begin{aligned} \mathfrak{P}\mathfrak{u}(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s)) ds \\ & - \frac{1}{a + b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} f(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s)) ds - c \right]. \end{aligned}$$

Noticeably, the fixed points of the operator \mathfrak{P} are solution of the problem (1.1)–(1.2). We shall employ the Banach contraction principle to verify that \mathfrak{P} defined by (3.2) has a fixed point. We shall demonstrate that \mathfrak{P} is a contraction.

Let $\mathfrak{u}, \mathfrak{v} \in \Phi$. Then, for each $t \in \mathcal{I}$ we have

$$\begin{aligned} |\mathfrak{P}(\mathfrak{u})(t) - \mathfrak{P}(\mathfrak{v})(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s)) - \mathfrak{F}(s, \mathfrak{v}(s), \mathfrak{H}\mathfrak{v}(s))| ds \\ & \quad - \frac{|b|}{|a + b|\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s)) - \mathfrak{F}(s, \mathfrak{v}(s), \mathfrak{H}\mathfrak{v}(s))| ds \\ & \leq \frac{\mathfrak{K}(1 + \mathfrak{H}_1)\|\mathfrak{u} - \mathfrak{v}\|_\infty}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds \\ & \quad - \frac{|b|\mathfrak{K}(1 + \mathfrak{H}_1)\|\mathfrak{u} - \mathfrak{v}\|_\infty}{|a + b|\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds \\ & \leq \left[\frac{\mathfrak{K}(1 + \mathfrak{H}_1)(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|}\right) \right] \|\mathfrak{u} - \mathfrak{v}\|_\infty. \end{aligned}$$

Therefore,

$$\|\mathfrak{P}(\mathfrak{u}) - \mathfrak{P}(\mathfrak{v})\|_\infty \leq \left[\frac{\mathfrak{K}(1 + \mathfrak{H}_1)(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|} \right) \right] \|\mathfrak{u} - \mathfrak{v}\|_\infty.$$

As a result by (3.1), \mathfrak{P} is a contraction. Due to a result of the Banach fixed point theorem, we deduce that \mathfrak{P} has a fixed point which is a solution of the problem (1.1)–(1.2). \square

The next result is derived from Schaefer's fixed point theorem.

THEOREM 3.2. *Suppose (A1), (A3) and (A4) hold. Then the BVP (1.1)–(1.2) has at least one solution on \mathfrak{J} .*

PROOF. We shall make use of Schaefer's fixed point theorem to verify that \mathfrak{P} defined by (3.2) has a fixed point. The proof will be given in some steps.

CLAIM 1: The operator \mathfrak{P} is continuous. Let $\{\mathfrak{u}_n\}$ be a sequence such that $\mathfrak{u}_n \rightarrow \mathfrak{u}$ in Φ . Then for each $t \in \mathfrak{J}$

$$\begin{aligned} & |\mathfrak{P}(\mathfrak{u}_n)(t) - \mathfrak{P}(\mathfrak{u})(t)| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}_n(s), \mathfrak{H}\mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s))| ds \\ & + \frac{|b|}{|a + b|\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}_n(s), \mathfrak{H}\mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s))| ds \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \sup_{s \in \mathfrak{J}} |\mathfrak{F}(s, \mathfrak{u}_n(s), \mathfrak{H}\mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s))| ds \\ & + \frac{|b|}{|a + b|\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} \sup_{s \in \mathfrak{J}} |\mathfrak{F}(s, \mathfrak{u}_n(s), \mathfrak{H}\mathfrak{u}_n(s)) - \mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s))| ds \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \right. \\ & \quad \left. + \frac{|b|}{|a + b|} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds \right] \|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot), \mathfrak{H}\mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot), \mathfrak{H}\mathfrak{u}(\cdot))\|_\infty \\ & \leq \frac{(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|} \right) \|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot), \mathfrak{H}\mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot), \mathfrak{H}\mathfrak{u}(\cdot))\|_\infty. \end{aligned}$$

Since \mathfrak{F} is a continuous function, we have

$$\|\mathfrak{P}(\mathfrak{u}_n) - \mathfrak{P}(\mathfrak{u})\|_\infty \leq \frac{(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|} \right) \|\mathfrak{F}(\cdot, \mathfrak{u}_n(\cdot), \mathfrak{H}\mathfrak{u}_n(\cdot)) - \mathfrak{F}(\cdot, \mathfrak{u}(\cdot), \mathfrak{H}\mathfrak{u}(\cdot))\|_\infty.$$

CLAIM 2: The operator \mathfrak{P} maps bounded sets into bounded sets in Φ . In fact, it is sufficient to prove that for any $q > 0$, there exists a positive constant ζ such that for each $\mathfrak{u} \in \mathfrak{D}_q = \{\mathfrak{u} \in \Phi : \|\mathfrak{u}\|_\infty \leq q\}$, we have $\|\mathfrak{P}(\mathfrak{u})\|_\infty \leq \zeta$.

By (A3) we have for each $t \in \mathfrak{J}$

$$|\mathfrak{P}(\mathfrak{u})(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathfrak{u}(s), \mathfrak{H}\mathfrak{u}(s))| ds$$

$$\begin{aligned}
& + \frac{|b|}{|a+b|} \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, \mathbf{u}(s), \mathfrak{H}\mathbf{u}(s))| ds + \frac{|c|}{|a+b|} \\
& \leq \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\
& + \frac{|b|}{|a+b|} \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} ds + \frac{|c|}{|a+b|} \\
& \leq \frac{\mathfrak{M}}{\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{\mathfrak{M}|b|}{|\alpha+1|\Gamma(\alpha+b)} (\psi(T))^\alpha + \frac{|c|}{|a+b|}.
\end{aligned}$$

Therefore,

$$\|\mathfrak{P}(\mathbf{u})\|_\infty \leq \frac{\mathfrak{M}}{\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{\mathfrak{M}|b|}{|a+b|\Gamma(\alpha+1)} (\psi(T))^\alpha + \frac{|c|}{|a+b|} := \zeta.$$

CLAIM 3: The operator \mathfrak{P} maps bounded sets into equicontinuous sets of Φ . Let $t_1, t_2 \in \mathfrak{I}$, $t_1 < t_2$, \mathfrak{D}_q be a bounded set of Φ as in Claim 2, and let $\mathbf{u} \in \mathfrak{D}_q$. Then

$$\begin{aligned}
& |\mathfrak{P}(\mathbf{u})(t_2) - \mathfrak{P}(\mathbf{u})(t_1)| \\
& \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] \mathfrak{F}(s, \mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds \right. \\
& \quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds \right| \\
& \leq \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)[(\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}] ds \\
& \quad + \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} ds \\
& \leq \frac{\mathfrak{M}}{\Gamma(\alpha+1)} (\psi(t_2) - \psi(t_1))^\alpha + \frac{\mathfrak{M}}{\Gamma(\alpha+1)} ((\psi(t_1))^\alpha - (\psi(t_2))^\alpha).
\end{aligned}$$

Since $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to 0. Because, a result of Stage 1 to 3 together with the Arzela–Ascoli theorem, we can finish that $\mathfrak{F}: \Phi \rightarrow \Phi$ is continuous and completely continuous.

CLAIM 4: A priori bounds. Now it remains to prove that the set

$$\kappa = \{\mathbf{u} \in \Phi :=: \mathbf{u} = \lambda \mathfrak{P}(\mathbf{u}) \text{ for some } \lambda \in (0, 1)\}$$

is bounded.

Let $\mathbf{u} \in \kappa$, then $\lambda \mathfrak{P}(\mathbf{u})$ for some $\lambda \in (0, 1)$. Hence, for each $t \in \mathfrak{I}$ we have

$$\begin{aligned}
\mathbf{u}(t) = \lambda \left[\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds \right. \\
\left. - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathbf{u}(s), \mathfrak{H}\mathbf{u}(s)) ds - c \right] \right].
\end{aligned}$$

We complete this stage by considering the estimation in Claim 2. The same as a result of Schaefer's fixed point theorem, we finish the proof that \mathfrak{P} has fixed point which is the solution of the problem (1.1)–(1.2). \square

4. Ulam–Hyers–Rassias stability

In this part, we study the Ulam stability of BVP for ψ -fractional differential equations (1.1)–(1.2). There are many works on the Ulam-stability of solutions for fractional differential equations. We mention here some works [3, 14–19, 24]; also see the references cited therein. A similar idea can be found in [7]. But there is no work on the Ulam stability results for ψ -fractional integro-differential equations with boundary conditions. Now we consider the Ulam stability for the following problem

$$(4.1) \quad {}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) = \mathfrak{F}(t, \mathfrak{u}(t), \mathfrak{H}\mathfrak{u}(t)), \quad t \in \mathfrak{I} := [0, T],$$

and the following inequalities:

$$(4.2) \quad |{}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t, \mathfrak{z}(t), \mathfrak{H}\mathfrak{z}(t))| \leq \epsilon, \quad t \in \mathfrak{I},$$

$$(4.3) \quad |{}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t, \mathfrak{z}(t), \mathfrak{H}\mathfrak{z}(t))| \leq \epsilon\varphi(t), \quad t \in \mathfrak{I},$$

$$(4.4) \quad |{}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) - \mathfrak{F}(t, \mathfrak{z}(t), \mathfrak{H}\mathfrak{z}(t))| \leq \varphi(t), \quad t \in \mathfrak{I}.$$

DEFINITION 4.1. Equation (4.1) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{z} \in \Phi$ of inequality (4.2) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \leq C_f\epsilon, \quad t \in J.$$

DEFINITION 4.2. Equation (4.1) is generalized Ulam–Hyers stable if there exists $\psi_f \in C([0, \infty), [0, \infty))$, $\psi_f(0) = 0$ such that for each solution $\mathfrak{z} \in \Phi$ of inequality (4.2) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \leq \psi_f\epsilon, \quad t \in J.$$

DEFINITION 4.3. The equation (4.1) is Ulam–Hyers–Rassias stable with respect to $\varphi \in \Phi$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathfrak{z} \in \Phi$ of inequality (4.3) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \leq C_f\epsilon\varphi(t), \quad t \in J.$$

DEFINITION 4.4. Equation (4.1) is generalized Ulam–Hyers–Rassias stable with respect to $\varphi \in \Phi$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{z} \in \Phi$ of inequality (4.4) there exists a solution $\mathfrak{u} \in \Phi$ of equation (4.1) with

$$|\mathfrak{z}(t) - \mathfrak{u}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

REMARK 4.1. A function $\mathfrak{z} \in \Phi$ is a solution of (4.2) if and only if there exists a function $g \in \Phi$ (which depend on \mathfrak{z}) such that

$$(1) |g(t)| \leq \epsilon, \quad t \in \mathfrak{I}; \quad (2) {}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{z}(t) = \mathfrak{F}(t, \mathfrak{z}(t), \mathfrak{H}\mathfrak{z}(t)) + g(t), \quad t \in \mathfrak{I}.$$

REMARK 4.2. Let $\alpha \in (0, 1)$, if $\mathfrak{Z} \in \Phi$ is a solution of the inequality (4.2), then \mathfrak{Z} is a solution of the following inequality

$$\left| \mathfrak{Z}(t) - \mathfrak{A}_3 - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \right| \leq \epsilon \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right).$$

In fact, by Remark 4.1, we have that ${}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{Z}(t) = \mathfrak{F}(t, \mathfrak{Z}(t), \mathfrak{H}\mathfrak{Z}(t)) + g(t)$, $t \in \mathcal{J}$. Then

$$\begin{aligned} \mathfrak{Z}(t) &= \mathfrak{A}_3 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) ds \\ &\quad - \left(\frac{b}{a+b} \right) \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) ds, \quad t \in \mathcal{J}, \end{aligned}$$

with

$$\mathfrak{A}_3 = \frac{1}{a+b} \left[c - \frac{b}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \right].$$

From this it follows that

$$\begin{aligned} &\left| \mathfrak{Z}(t) - \mathfrak{A}_3 - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) ds \right. \\ &\quad \left. - \left(\frac{b}{a+b} \right) \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |g(s)| ds \\ &\quad - \left(\frac{b}{a+b} \right) \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |g(s)| ds \\ &\leq \epsilon \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right). \end{aligned}$$

REMARK 4.3. Undoubtedly,

Definition 4.1 \Rightarrow Definition 4.2 and Definition 4.3 \Rightarrow Definition 4.4.

REMARK 4.4. A solution of the ψ -fractional differential equations with boundary condition inequality (4.2) is called an ϵ -solution of problem (4.1).

THEOREM 4.1. Suppose (A1), (A2), (A4) and (3.1) hold. Then, the problem (1.1)–(1.2) is Ulam–Hyers stable.

PROOF. Let $\epsilon > 0$ and let $\mathfrak{Z} \in \Phi$ be a function which satisfies inequality (4.2) and let $\mathfrak{u} \in \Phi$ be the unique solution of the following problem

$$\begin{aligned} {}^c\mathfrak{D}^{\alpha;\psi}\mathfrak{u}(t) &= \mathfrak{F}(t, \mathfrak{u}(t), \mathfrak{H}\mathfrak{u}(t)), \quad t \in \mathcal{J}, \quad \alpha \in (0, 1), \\ \mathfrak{u}(0) &= \mathfrak{Z}(0), \quad \mathfrak{u}(T) = \mathfrak{Z}(T). \end{aligned}$$

Using Lemma 3.2, we obtain

$$u(t) = \mathfrak{A}_u + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, u(s), \mathfrak{H}u(s)) ds.$$

Alternatively, if $u(0) = \mathfrak{Z}(0)$, $u(T) = \mathfrak{Z}(T)$, then $\mathfrak{A}_u = \mathfrak{A}_\mathfrak{Z}$. In fact,

$$\begin{aligned} |\mathfrak{A}_u - \mathfrak{A}_\mathfrak{Z}| &\leq \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, u(s), \mathfrak{H}u(s)) \\ &\quad - \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s))| ds \\ &\leq \frac{\mathfrak{K}(1 + \mathfrak{H}_1)|b|}{|a+b|} I^{\alpha; \psi} |u(T) - \mathfrak{Z}(T)| = 0. \end{aligned}$$

Therefore, $\mathfrak{A}_u = \mathfrak{A}_\mathfrak{Z}$. We have

$$u(t) = \mathfrak{A}_\mathfrak{Z} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, u(s), \mathfrak{H}u(s)) ds.$$

By integration of inequality (4.2) and using Remark 4.2, we obtain

$$\begin{aligned} \left| \mathfrak{Z}(t) - \mathfrak{A}_\mathfrak{Z} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \right| \\ \leq \epsilon \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right). \end{aligned}$$

We have for any $t \in \mathfrak{J}$

$$\begin{aligned} |\mathfrak{Z}(t) - u(t)| &\leq \left| \mathfrak{Z}(t) - \mathfrak{A}_\mathfrak{Z} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s)) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{F}(s, u(s), \mathfrak{H}u(s)) \\ &\quad - \mathfrak{F}(s, \mathfrak{Z}(s), \mathfrak{H}\mathfrak{Z}(s))| ds \\ &\leq \epsilon \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) \\ &\quad + \frac{\mathfrak{K}(1 + \mathfrak{H}_1)}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\mathfrak{Z}(s) - u(s)| ds \end{aligned}$$

Using the Gronwall inequality, Lemma 2.1 and Remark 2.1, we obtain

$$|\mathfrak{Z}(t) - u(t)| \leq \left(1 + \frac{|b|}{|a+b|} \right) \frac{\epsilon(\psi(T))^\alpha}{\Gamma(\alpha+1)} E_{\alpha; \psi}(\mathfrak{K}(1 + \mathfrak{H}_1)(\psi(T))^\alpha).$$

Thus, the problem (1.1)–(1.2) is Ulam–Hyers stable. \square

THEOREM 4.2. *Suppose (A1)–(A2), inequality (3.1) and*

(A4) there exists an increasing function $\varphi \in \Phi$ and $\lambda_\varphi > 0$ such that

$$I^{\alpha; \psi} \varphi(t) \leq \lambda_\varphi \varphi(t), \quad \text{for each } t \in \mathfrak{J}$$

hold. Then problem (1.1)–(1.2) is Ulam–Hyers–Rassias stable.

REMARK 4.5. Under the assumptions of Theorem 4.1, we consider problem (1.1)–(1.2) and inequality (4.4). One can repeat the same process to verify that problem (1.1)–(1.2) is Ulam–Hyers–Rassias stable.

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