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Analytic Study on Boundary Value Problem of Implicit Differential Equations via Composite Fractional Derivative

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Abstract
This paper is devoted to investigate implicit differential equations with boundary condition, which involves the composite fractional derivative in weighted space. The existence and uniqueness of the solution are obtained using the classic fixed point theorems. As an application, an example is presented to illustrate the main results.

Keywords Composite fractional derivative · Boundary value problem · Existence · Fixed point

Mathematics Subject Classification (2010) 26A33 · 33E12 · 34B10

1 Introduction

In recent years, fractional order calculus has been one of the most rapidly developing areas of mathematical analysis. Indeed, a natural phenomenon may depend not only on the current time but also on its previous time history. Fractional calculus facilitates modeling of

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such phenomena via nonlocal fractional differential and integral operators. Fractional order differential equations naturally appear in the mathematical modeling of systems with memory. One can find numerous applications of fractional calculus in diverse fields such as mathematics, physics, chemistry, optimal control theory, finance, biology, and engineering [10, 12, 15]. Since it is clear that dealing with Riemann-Liouville (R-L) derivative in various applied problems is very difficult, therefore, certain modifications were introduced to avoid the difficulties. In this regard, some new type fractional order derivative operators were introduced in literature like Caputo and Hadamard. Recently, Hilfer [12] initiated extended R-L fractional derivative, named Hilfer fractional derivative, which interpolates Caputo fractional derivative and R-L fractional derivative. This said operator arose in the theoretical simulation of dielectric relaxation in glass-forming materials (see [24–27]). Followed by the work, Sousa and Oliveira [20] introduced the composite fractional derivative (ψ-Hilfer fractional derivative) with respect to another function, in order to unify the wide number of fractional derivatives in a single fractional operator and consequently, open a window for new applications, we refer to [19, 21].

At present, a great deal of efforts were spent in linear and nonlinear fractional differential equations (FDEs). As an important issue for the theory of implicit FDEs, the existence, uniqueness, and stability of solutions for the nonlinear fractional initial value problems and fractional boundary value problems have attracted scholars' attention. For some recent work on the topic, see papers [1, 3, 5–8]. Existence and stability results for nonlinear implicit fractional differential equations with delay and impulses were discussed in [4]. Abbas et al. [2] investigated the asymptotic stability for implicit Hilfer fractional differential equations. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for FDEs with composite derivative [9, 27]. Inspired by the above works, in this paper, we are mainly concerned with the existence and uniqueness of solutions of implicit differential equations with composite fractional derivative given by

$$\left(D_{a^+}^{\alpha,\beta;\psi} y\right)(t) = f\left(t, y(t), \left(D_{a^+}^{\alpha,\beta;\psi} y\right)(t)\right) \quad \text{for each} \quad t \in (a, T],\ a > 0, \tag{1}$$

$$y(T) = c \in \mathbb{R}, \tag{2}$$

where $D_{a^+}^{\alpha,\beta;\psi}$ is the composite fractional derivative (to be defined later) of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ and $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function.

2 Preliminaries

In this section, we introduce some fundamental descriptions and lemmas which are used throughout this paper; for details, readers should study [13, 16, 17, 22, 23].

Let $0 < a < T$ and $J = [a, T]$. By $C(J, \mathbb{R})$, we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

We consider the weighted space of continuous functions

$$C_{\gamma,\psi}(J) = \left\{y : (a, T] \to \mathbb{R} : (\psi(t) - \psi(a))^\gamma\, y(t) \in C(J, \mathbb{R})\right\}, \quad 0 \le \gamma < 1,$$

and

$$C^n_{\gamma,\psi}(J) = \left\{y \in C^{n-1}(J) : y^{(n)} \in C_{\gamma,\psi}(J)\right\}, \quad n \in \mathbb{N},$$

$$C^0_{\gamma,\psi}(J) = C_{\gamma,\psi}(J)$$

with the norms

$$\|y\|_{C_{\gamma,\psi}} = \sup_{t\in J} \left|(\psi(t) - \psi(a))^{\gamma}\, y(t)\right|$$

and

$$\|y\|_{C^n_{1,\psi}} = \sum_{k=0}^{n-1} \left\|y^k\right\|_{\infty} + \left\|y^{(n)}\right\|_{C_{\gamma,\psi}}.$$

Consider the space $X^p_c(a,b)$, $(c \in \mathbb{R}, 1 \le p < \infty)$ of those complex-valued Lebesgue measurable functions f on J for which $\|f\|_{X^p_c} < \infty$, where the norm is defined by

$$\|f\|_{X^p_c} = \left(\int_a^b \left|t^c f(t)\right| dt\right)^{\frac{1}{p}}, \quad (1 \le p < \infty, c \in \mathbb{R}).$$

In particular, where $c = \frac{1}{p}$, the space $X^p_c(a,b)$ coincides with the $L_p(a,b)$ space $X^p_{\frac{1}{p}} = L_p(a,b)$.

Definition 1 [19] Let $\alpha \in \mathbb{R}_+$, $c \in \mathbb{R}$ and $g \in X^p_c(a,b)$. The left-sided fractional integral of order α and of a function f with respect to another function ψ on J is defined by

$$\left(I^{\alpha,\psi}_{a^+} g\right)(t) = \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1}\, \frac{g(s)}{\Gamma(\alpha)} ds, \quad t > a,$$

where $\Gamma(\dot{)}$ is the Euler Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \alpha > 0$.

Definition 2 [19] Let $\alpha \in \mathbb{R}_+/\mathbb{N}$ and $\psi'(t) \neq 0$ $(-\infty \le a < t < b \le \infty)$. The R-L fractional derivative of a function f with respect to ψ of order α corresponding to the R-L is defined by

$$\begin{aligned}\left(D^{\alpha,\psi}_{a^+} g\right)(t) &= \delta^n \left(I^{n-\alpha;\psi} g\right)(t) \\ &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \psi'(s)\,(\psi(t) - \psi(s))^{n-\alpha-1}\, \frac{g(s)}{\Gamma(n-\alpha)} ds, \quad t > a,\end{aligned}$$

where $n = [\alpha] + 1$ and $\delta^n = (\frac{1}{\psi'(t)}\frac{d}{dt})^n$.

Lemma 1 [18] *Let $\alpha > 0$ and $0 \le \gamma < 1$. Then, $I^{\alpha;\psi}_{a^+}$ is bounded from $C_{\gamma,\psi}(J)$ into $C_{\gamma,\psi}(J)$.*

Lemma 2 [18] *Let $0 < a < T < \infty$, $\alpha > 0$, $0 \le \gamma < 1$ and $y \in C_{\gamma,\psi}(J)$. If $\alpha > \gamma$, then $I^{\alpha;\psi}_{a^+} y$ is continuous on J and*

$$\left(I^{\alpha;\psi}_{a^+} y\right)(a) = \lim_{t\to a^+} \left(I^{\alpha;\psi}_{a^+} y\right)(t) = 0.$$

Lemma 3 [18] *Let $x > a$. Then, for $\alpha \ge 0$ and $\beta > 0$, we have*

$$\begin{aligned}\left[I^{\alpha;\psi}_{a^+} (\psi(s) - \psi(a))^{\beta-1}\right](t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1}, \\ \left[D^{\alpha;\psi}_{a^+} (\psi(s) - \psi(a))^{\alpha-1}\right](t) &= 0, \quad 0 < \alpha < 1.\end{aligned}$$

Lemma 4 [14] *Let* $\alpha > 0$, $0 \le \gamma < 1$ *and* $g \in C_{\gamma,\psi}[a,b]$. *Then,*

$$\left(D_{a^+}^{\alpha,\psi} I_{a^+}^{\alpha,\psi} g\right)(t) = g(t) \quad \textit{for all} \quad t \in (a,b].$$

Lemma 5 [14] *Let* $0 < \alpha < 1$, $0 \le \gamma < 1$. *If* $g \in C_{\gamma,\psi}[a,b]$ *and* $I_{a^+}^{1-\alpha;\psi} g \in C^1_{\gamma,\psi}[a,b]$, *then*

$$\left(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\psi} g\right)(t) = g(t) - \frac{\left(I_{a^+}^{\alpha;\psi} g\right)(a)}{\Gamma(\alpha)} (\psi(t) - \psi(a))^{\alpha-1} \quad \textit{for all} \quad (a,b].$$

Definition 3 [19] Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \le \beta \le 1$ with $n \in \mathbb{N}$. The composite fractional derivative of a function $g \in C_{\gamma,\psi}[a,b]$ is defined by

$$\left(D_{a^+}^{\alpha,\beta;\psi} g\right)(t) = \left(I_{a^+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} g\right)(t).$$

The composite fractional derivative as defined above can be written in the following:

$$\left(D_{a^+}^{\alpha,\beta;\psi} g\right)(t) = \left(I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} g\right)(t), \quad \gamma = \alpha + \beta - \alpha\beta.$$

Property 1 [18] The operator $D_{a^+}^{\alpha,\beta;\psi}$ can be written as

$$D_{a^+}^{\alpha,\beta;\psi} = I_{a^+}^{\beta(1-\alpha);\psi} D_{a^+}^{\gamma;\psi}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

Definition 4 [22] We consider the following parameters α, β, γ satisfying $\gamma = \alpha+\beta-\alpha\beta$, $0 < \alpha, \beta, \gamma < 1$. Thus, we define the spaces

$$C^{\alpha,\beta}_{1-\gamma,\psi}(J) = \left\{ y \in C_{1-\gamma,\psi}(J), D_{a^+}^{\alpha,\beta;\psi} y \in C_{1-\gamma,\psi}(J) \right\}$$

and

$$C^{\gamma}_{1-\gamma,\psi}(J) = \left\{ y \in C_{1-\gamma,\psi}(J), D_{a^+}^{\gamma;\psi} y \in C_{1-\gamma,\psi}(J) \right\}.$$

Since, $D_{a^+}^{\alpha,\beta;\psi} y = I_{a^+}^{\gamma(1-\alpha);\psi} D_{a^+}^{\gamma;\psi} y$, it follows from Lemma 1 that

$$C^{\gamma}_{1-\gamma,\psi}(J) \subset C^{\alpha,\beta}_{1-\gamma,\psi}(J) \subset C_{1-\gamma,\psi}(J).$$

Lemma 6 [23] *Let* $0 < \alpha < 1$, $0 \le \beta \le 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $y \in C^{\gamma}_{1-\gamma,\psi}(J)$, *then*

$$I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} = I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} y, \quad D_{a^+}^{\gamma;\psi} I_{a^+}^{\alpha;\psi} y = D_{a^+}^{\beta(1-\alpha)} y.$$

Theorem 1 [11, 28] ($PC_{1-\gamma,\psi}$ type Arzela-Ascoli Theorem) *Let* $A \subset PC_{1-\gamma,\psi}(J,\mathbb{R})$. *A is relatively compact (i.e.,* $\overline{A}$ *is compact) if*
1. *A is uniformly bounded, i.e., there exists* $M > 0$ *such that*

$$|f(x)| < M \quad \textit{for every} \quad f \in A \quad \textit{and} \quad x \in (t_k, t_{k+1}], k = 1, 2, \ldots, m.$$

2. *A is equicontinuous on* $(t_k, t_{k+1}]$, *i.e., for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $x, \overline{x} \in (t_k, t_{k+1}]$, $|x - \overline{x}| \le \delta$ *implies* $|f(x) - f(\overline{x})| \le \epsilon$ *for every* $f \in A$.

Theorem 2 (Krasnoselskii's fixed point theorem) *Let M be a closed, convex and nonempty subset of a Banach space X, and A, B be the operators such that*
1. $Ax + By \in M$ *for all* $x, y \in M$.

2. *A is compact and continuous.*
3. *B is a contraction mapping.*
   Then, there exists $z \in M$ such that $z = Az + Bz$.

3 Existence of Solutions

We consider the following linear FDEs of the form:

$$\left(D_{a^+}^{\alpha,\beta;\psi} y\right)(t) = \varphi(t), \quad t \in (a, T], \tag{3}$$

where $\varphi(\cdot) \in C_{1-\gamma,\psi}(J)$ with boundary condition

$$y(T) = c, \quad c \in \mathbb{R}. \tag{4}$$

The following theorem shows that (3)–(4) have a unique solution given by

$$\begin{aligned} y(t) &= (\psi(T) - \psi(a))^{1-\gamma} \\ &\times \left[c - \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(s)\,(\psi(T) - \psi(s))^{\alpha-1}\,\varphi(s)ds\right](\psi(t) - \psi(a))^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1}\,\varphi(s)ds. \end{aligned} \tag{5}$$

Theorem 3 *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $\varphi \in (a, T] \to \mathbb{R}$ is a function such that $\varphi(\cdot) \in C_{1-\gamma,\psi}(J)$, then y satisfies (3)–(4) if and only if it satisfies (5).*

Proof Let $y \in C^{\gamma}_{1-\gamma,\psi}(J)$ be a solution of (3)–(4). We prove that y is also a solution of (5). From the definition of $C^{\gamma}_{1-\gamma,\psi}(J)$, Lemma 1 and using Definition 2, we have

$$I_{a^+}^{1-\gamma;\psi} y \in C(J) \quad \text{and} \quad D_{a^+}^{\gamma;\psi} y = \delta I_{a^+}^{1-\gamma;\psi} y \in C_{1-\gamma,\psi}(J). \tag{6}$$

By the definition of the space $C^{n}_{1-\gamma,\psi}(J)$, it follows that

$$I_{a^+}^{1-\gamma;\psi} y \in C^{1}_{1-\gamma,\psi}(J).$$

Using Lemma 5, with $\alpha = \gamma$, we obtain

$$\left(I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} y\right)(t) = y(t) - \frac{\left(I_{a^+}^{1-\gamma;\psi} y\right)(a)}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1}, \tag{7}$$

where $t \in (a, T]$. By hypothesis, $y \in C^{\gamma}_{1-\gamma,\psi}(J)$ using Lemma 6 with (3), we have

$$\left(I_{a^+}^{\gamma;\psi} D_{a^+}^{\gamma;\psi} y\right)(t) = \left(I_{a^+}^{\alpha;\psi} D_{a^+}^{\alpha,\beta;\psi} y\right)(t) = \left(I_{a^+}^{\alpha;\psi} \varphi\right)(t). \tag{8}$$

Comparing (7) and (8), we see that

$$y(t) = \frac{\left(I_{a^+}^{1-\gamma;\psi} y\right)(a)}{\Gamma(\gamma)} (\psi(t) - \psi(a))^{\gamma-1} + \left(I_{a^+}^{\alpha;\psi} \varphi\right)(t).$$

Using (4), we obtain

$$\begin{aligned} y(t) &= (\psi(T) - \psi(a))^{1-\gamma} \\ &\times \left[c - \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(s)\,(\psi(T) - \psi(s))^{\alpha-1}\,\varphi(s)ds\right](\psi(t) - \psi(a))^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t) - \psi(s))^{\alpha-1}\,\varphi(s)ds, \end{aligned}$$

with $t \in (a, b]$, that is $y(\cdot)$ satisfies (5).

Conversely, let $y \in C^{\gamma}_{1-\gamma,\psi}(J)$ satisfying (5). We show that y also satisfies (3)–(4). Apply operator $D^{\gamma;\psi}_{a^+}$ on both sides of (5). Then, from Lemmas 3 and 6, we get

$$\left(D^{\gamma;\psi}_{a^+} y\right)(t) = \left(D^{\beta(1-\alpha);\psi}_{a^+} \varphi(t)\right). \tag{9}$$

By (6), we have $D^{\gamma;\psi}_{a^+} y \in C_{1-\gamma,\psi}(J)$; then, (9) implies

$$\left(D^{\gamma;\psi}_{a^+} y\right)(t) = \left(I^{(1-\beta)(1-\alpha);\psi}_{a^+} \varphi\right)(t) = \left(D^{\beta(1-\alpha);\psi}_{a^+} \varphi\right)(t) \in C_{1-\gamma,\psi}(J). \tag{10}$$

As $\varphi(\cdot) \in C_{1-\gamma,\psi}(J)$ and from Lemma 1, it follows

$$\left(I^{(1-\beta)(1-\alpha);\psi}_{a^+} \varphi(t)\right) \in C_{1-\gamma,\psi}(J). \tag{11}$$

From (10) and (11) and by the definition of the space $C^{n}_{1-\gamma,\psi}(J)$, we obtain

$$\left(I^{(1-\beta)(1-\alpha)}_{a^+} \varphi\right) \in C^{1}_{1-\gamma,\psi}(J).$$

Applying operator $I^{\beta(1-\alpha);\psi}_{a^+}$ on both sides of (10) and using Lemmas 2 and 5, we have

$$\begin{aligned}\left(I^{\beta(1-\alpha)}_{a^+} D^{\gamma;\psi}_{a^+} y\right)(t) &= \varphi(t) + \frac{\left(I^{1-\beta(1-\alpha);\psi}_{a^+} \varphi(t)\right)(a)}{\Gamma(\beta(1-\alpha))} (\psi(t)-\psi(a))^{\beta(1-\alpha)-1} \\ &= \left(D^{\alpha,\beta;\psi}_{a^+} y\right)(t) = \varphi(t).\end{aligned}$$

That is, (3) holds. Clearly, if $y \in C^{\gamma}_{1-\gamma,\psi}(J)$ satisfies (5), then it also satisfies (4). □

As a consequence of Theorem 3, we have the following theorem.

Theorem 4 *Let* $\gamma = \alpha+\beta-\alpha\beta$ *where* $0 < \alpha < 1$ *and* $0 \le \beta \le 1$, *let* $f : (a, T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *be a function such that* $f(\cdot, y(\cdot), u(\cdot)) \in C_{1-\gamma,\psi}(J)$ *for any* $y, u \in C_{1-\gamma,\psi}(J)$.

If $y \in C^{\gamma}_{1-\gamma,\psi}(J)$, *then* y *satisfies* (1)–(2) *if and only if* y *is the fixed point of the operator* $N : C_{1-\gamma,\psi}(J) \to C_{1-\gamma,\psi}(J)$ *defined by*

$$Ny(t) = M(\psi(t)-\psi(s))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(t)(\psi(t)-\psi(s))^{\alpha-1} g(s)ds, \quad t \in (a, T], \tag{12}$$

where

$$M := (\psi(T)-\psi(a))^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s)ds\right],$$

and $g : (a, T] \to \mathbb{R}$ *be a function satisfying the functional equation*

$$g(t) = f(t, y(t), g(t)).$$

Clearly, $g \in C_{1-\gamma,\psi}(J)$. *In addition, by Lemma 1,* $Ny \in C_{1-\gamma,\psi}(J)$.

Suppose that the function $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions

(H1) The function $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that $f(\cdot, u(\cdot), v(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\psi}$ for any $u, v \in C_{1-\gamma,\psi}(J)$.

(H2) There exist constants $K > 0$ and $L \in (0, 1)$ such that

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| \leq K |u - \overline{u}| + L |v - \overline{v}|$$

for any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in (a, T]$.

Now, we state and prove our existence result for (1) and (2) based on Banach's space fixed point.

Theorem 5 *Assume that* (H1) *and* (H2) *hold. If*

$$\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)} (\psi(T) - \psi(a))^{\alpha} < \frac{1}{2}, \tag{13}$$

then (1)–(2) *has a unique solution.*

Proof Step 1 We show that the operator N defined in (12) has a unique solution fixed point y^* in $C_{1-\gamma,\psi}(J)$.

Let $y, u \in C_{1-\gamma,\psi}(J)$ and $t \in (a, T]$, then we have

$$\begin{aligned} |Ny(t) - Nu(t)| \leq\ & \frac{1}{\Gamma(\alpha)} (\psi(T) - \psi(a))^{1-\gamma} (\psi(t) - \psi(a))^{\gamma-1} \\ & \times \int_a^T \psi'(s) (\psi(T) - \psi(s))^{\alpha-1} |g(s) - h(s)| \, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |g(s) - h(s)| \, ds, \end{aligned}$$

where $g, h \in C_{1-\gamma,\psi}(J)$ such that

$$g(t) = f(t, y(t), g(t)), \quad h(t) = f(t, u(t), h(t)).$$

By (H2), we have

$$|g(t) - h(t)| = |f(t, y(t), g(t)) - f(t, u(t), h(t))| \leq K |y(t) - u(t)| + L |g(t) - h(t)| .$$

Then,

$$|g(t) - h(t)| \leq \frac{K}{1-L} |y(t) - u(t)| .$$

Hence, for each $t \in (a, T]$

$$\begin{aligned} & |Ny(t) - Nu(t)| \\ \leq\ & \frac{K}{(1-L)} \frac{1}{\Gamma(\alpha)} (\psi(T) - \psi(a))^{1-\gamma} (\psi(t) - \psi(a))^{\gamma-1} \\ & \times \int_a^T \psi'(s) (\psi(T) - \psi(s))^{\alpha-1} |y(s) - u(s)| \, ds \\ & + \frac{K}{(1-L)} \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} |y(s) - u(s)| \, ds \\ \leq\ & \frac{K}{(1-L)} (\psi(T) - \psi(a))^{1-\gamma} (\psi(t) - \psi(a))^{\gamma-1} \|y - u\|_{C_{1-\gamma,\psi}} \\ & \times \left(I_{a^+}^{\alpha,\psi} (\psi(s) - \psi(a))^{\gamma-1} \right) (T) \\ & + \frac{K}{(1-L)} \left(I_{a^+}^{\alpha;\psi} (\psi(s) - \psi(a)) \right)^{\gamma-1} (t) \, \|y - u\|_{C_{1-\gamma,\psi}} . \end{aligned}$$

By Lemma 3, we have

$$|Ny(t) - Nu(t)| \leq \Bigg[\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}(\psi(t)-\psi(a))^{\gamma-1} + \frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(t)-\psi(a))^{\alpha+\gamma-1}\Bigg]\|y-u\|_{C_{1-\gamma,\psi}},$$

hence

$$\begin{aligned}&\left|(\psi(t)-\psi(a))^{1-\gamma}(Ny(t)-Nu(t))\right| \\ &\leq \left[\frac{K}{(1-L)}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha} + \frac{K}{(1-L)}\frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(t)-\psi(a))^{\alpha}\right] \\ &\quad\times\|y-u\|_{C_{1-\gamma,\psi}} \\ &\leq \frac{2K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}\|y-u\|_{C_{1-\gamma,\psi}},\end{aligned}$$

which implies that

$$\|Ny-Nu\|_{C_{1-\gamma,\psi}} \leq \frac{2K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}\|y-u\|_{C_{1-\gamma,\psi}}.$$

By (13), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $y^* \in C_{1-\gamma,\psi}(J)$.

Step 2 We show that a fixed point $y^* \in C_{1-\gamma,\psi}(J)$ is actually in $C^{\gamma}_{1-\gamma,\psi}(J)$.

Since y^* is the unique fixed point of operator N in $C_{1-\gamma,\psi}(J)$, then, for each $t \in (a, T]$, we have

$$\begin{aligned}y^* &= (\psi(T)-\psi(a))^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} f(s, y^*(s), g(s))ds\right] \\ &\quad\times(\psi(t)-\psi(a))^{\gamma-1} + I^{\alpha;\psi}_{a^+} f(s, y^*(s), g(s)).\end{aligned}$$

Applying $D^{\gamma;\psi}_{a^+}$ to both sides and by Lemmas 3 and 6, we have

$$D^{\gamma;\psi}_{a^+} y(t) = \left(D^{\gamma,\psi}_{a^+} I^{\alpha;\psi}_{a^+} f(s, y^*(s), g(s))\right)(t) = \left(D^{\beta(1-\alpha);\psi}_{a^+} I^{\alpha;\psi}_{a^+} f(s, y^*(s), g(s))\right)(t).$$

Since $\gamma \geq \alpha$, by (H1), the right-hand side is in $C_{1-\gamma,\psi}(J)$ and thus $D^{\gamma;\psi}_{a^+} y^* \in C_{1-\gamma,\psi}(J)$, which implies that $y^* \in C^{\gamma}_{1-\gamma,\psi}(J)$. As a consequence of Step 1 to 2 together with Theorem 4, we can conclude that (1)–(2) have a unique solution in $C^{\gamma}_{1-\gamma,\psi}(J)$. □

We present now the second result, which is based on Krasnoselskii fixed point theorem.

Theorem 6 *Assume that* (H1) *and* (H2) *hold. If*

$$\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha} < 1, \tag{14}$$

then (1)–(2) *have at least one solution.*

Proof Consider the set

$$B_{\eta^*} = \left\{y \in C_{1-\gamma,\psi}(J) : \|y\|_{C_{1-\gamma}} \leq \eta^*\right\},$$

where

$$\eta^* \leq \frac{(\psi(T)-\psi(a))^{1-\gamma}\left[|c| + \frac{f^*\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}\right]}{1-\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}}$$

and $f^* = \sup_{t\in J}|f(t,0,0)|$. We define the operator P and Q on B_{η^*}, by

$$\begin{aligned} Py(t) &= (\psi(T)-\psi(a))^{1-\gamma} \\ &\times\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\,(\psi(T)-\psi(a))^{\alpha-1}\,g(s)ds\right](\psi(t)-\psi(a))^{\gamma-1}, \quad (15)\\ Qy(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\,(\psi(t)-\psi(a))^{\alpha-1}\,g(s)ds. \end{aligned}$$

Then, the fractional integral (12) can be written as the operator equation

$$Ny(t) = Py(t) + Qy(t), \quad y \in C_{1-\gamma,\psi}(J).$$

The proof will be presented in several steps.

Step 1 We prove that $Py(t) + Qy(t) \in B_{\eta^*}$ for any $y, u \in B_{\eta^*}$. For any operator P, multiplying both sides of (15) by $(\psi(t)-\psi(a))^{1-\gamma}$, we have

$$\begin{aligned} &(\psi(t)-\psi(a))^{1-\gamma}\,Py(t) \\ &= (\psi(T)-\psi(a))^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,g(s)ds\right], \end{aligned}$$

then

$$\begin{aligned} &\left|(\psi(t)-\psi(a))^{1-\gamma}\,Py(t)\right| \\ &\leq (\psi(T)-\psi(a))^{1-\gamma}\left[|c| + \frac{1}{\Gamma(\alpha)}\int_a^T \psi'(s)\,(\psi(T)-\psi(s))^{\alpha-1}\,|g(s)|\,ds\right]. \quad (16) \end{aligned}$$

By (H3), we have for each $t \in (a, T]$

$$\begin{aligned} |g(t)| &\leq |f(t,y(t),g(t)) - f(t,0,0) + f(t,0,0)| \\ &\leq |f(t,y(t),g(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\leq K\,|y(t)| + L\,|g(t)| + f^*. \end{aligned}$$

Multiplying both sides of the above inequality by $(\psi(t)-\psi(a))^{1-\gamma}$, we get

$$\begin{aligned} &\left|(\psi(t)-\psi(a))^{1-\gamma}\,g(t)\right| \\ &\leq (\psi(t)-\psi(a))^{1-\gamma}\,f^* + K\left|(\psi(t)-\psi(a))^{1-\gamma}\,y(t)\right| + L\left|(\psi(t)-\psi(a))^{1-\gamma}\,g(t)\right| \\ &\leq (\psi(t)-\psi(a))^{1-\gamma}\,f^* + K\eta^* + L\left|(\psi(t)-\psi(a))^{1-\gamma}\,g(t)\right|. \end{aligned}$$

Then, for each $t \in (a, T]$, we have

$$\left|(\psi(t)-\psi(a))^{1-\gamma}\,g(t)\right| \leq \frac{(\psi(T)-\psi(a))^{1-\gamma}\,f^* + K\eta^*}{1-L} := M. \quad (17)$$

Thus, (16) and Lemma 3 imply

$$\left|(\psi(t)-\psi(a))^{1-\gamma}\,Py(t)\right| \leq (\psi(t)-\psi(a))^{1-\gamma}\left[|c| + \frac{M\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(t)-\psi(a))^{\alpha+\gamma-1}\right]. \quad (18)$$

Using (17) and Lemma 3, we have

$$
\begin{aligned}
&|Q(u)(t)| \\
&\leq \left[\frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{1-\gamma}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right](\psi(t)-\psi(a))^{\alpha+\gamma-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|(\psi(t)-\psi(a))^{1-\gamma}\, Qu(t)\right| \\
&\leq \left[\frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{1-\gamma}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right](\psi(T)-\psi(a))^{\alpha} \\
&\leq \frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{1-\gamma+\alpha}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|Qu\|_{C_{1-\gamma,\psi}} \\
&\leq \frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{1-\gamma+\alpha}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}. \qquad (19)
\end{aligned}
$$

Linking (18)–(19), for every $y, u \in B_{\eta^*}$, we obtain

$$
\begin{aligned}
&\|Py+Qu\|_{C_{1-\gamma,\psi}} \\
&\leq \max\left\{\|Py\|_{C_{1-\gamma,\psi}}, \|Qu\|_{C_{1-\gamma,\psi}}\right\} \\
&\leq (\psi(t)-\psi(a))^{1-\gamma}\left[|c|+\frac{M\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}(\psi(t)-\psi(a))^{\alpha+\gamma-1}\right] \\
&= \frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{1-\gamma+\alpha}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha} \\
&\quad +(\psi(T)-\psi(a))^{1-\gamma}|c|.
\end{aligned}
$$

Since

$$
\eta^* \geq \frac{(\psi(T)-\psi(a))^{1-\gamma}\left[|c|+\frac{\Gamma(\gamma) f^*}{(1-L)\Gamma(\alpha+\gamma)}(\psi(T)-\psi(a))^{\alpha}\right]}{1-\frac{K\Gamma(\gamma)}{(1-L)\Gamma(\alpha+\gamma)}}(\psi(T)-\psi(a))^{\alpha}.
$$

We have

$$
\|Py+Qu\|_{PC_{1-\gamma,\psi}} \leq \eta^*,
$$

which infers that $Py+Qu \in B_{\eta^*}$.

Step 2 P is a contraction.

Let $y, u \in C_{1-\gamma,\psi}(J)$ and $t \in (a, T]$, then we have

$$
\begin{aligned}
|Py(t)-Pu(t)| \leq\; &\frac{1}{\Gamma(\alpha)}(\psi(T)-\psi(a))^{1-\gamma}(\psi(t)-\psi(a))^{\gamma-1} \\
&\times\int_a^T \psi'(s)(\psi(T)-\psi(a))^{\alpha-1}|g(s)-h(s)|\,ds,
\end{aligned}
$$

where $g, h \in C_{1-\gamma,\psi}(J)$ such that

$$
g(t)=f(t, y(t), g(t)), \quad h(t)=f(t, u(t), h(t)).
$$

By (H2), we have

$|g(t)-h(t)| = |f(t,y(t),g(t)) - f(t,u(t),h(t))| \le K\,|y(t)-u(t)| + L\,|g(t)-h(t)|.$

Then,

$$|g(t)-h(t)| \le K\,|y(t)-u(t)| + L\,|g(t)-h(t)|.$$

Therefore, for each $t \in (a,T]$

$$\begin{aligned}|Py(t)-Pu(t)| &\le \frac{K}{(1-L)}\left(\psi(T)-\psi(a)\right)^{1-\gamma}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\\ &\quad\times\int_a^T \psi'(s)\left(\psi(T)-\psi(a)\right)^{\alpha-1}|y(s)-u(s)|\,ds\\ &\le \frac{K}{(1-L)}\left(\psi(T)-\psi(a)\right)^{1-\gamma}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\|y-u\|_{C_{1-\gamma}}\\ &\quad\times\left(I_{a^+}^{\alpha;\psi}\left(\psi(T)-\psi(a)\right)^{\gamma-1}\right)(T).\end{aligned}$$

By Lemma 3, we have

$$|Py(t)-Pu(t)| \le \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\psi(T)-\psi(a)\right)^{\alpha}\left(\psi(t)-\psi(a)\right)^{\gamma-1}\|y-u\|_{C_{1-\gamma,\psi}},$$

hence,

$$\left|\left(\psi(t)-\psi(a)\right)^{1-\gamma}\left(Py(t)-Pu(t)\right)\right| \le \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\psi(T)-\psi(a)\right)^{\alpha}\|y-u\|_{C_{1-\gamma,\psi}}.$$

By (14), the operator P is contraction.

Step 3 Q is compact and continuous.

The continuity of Q follows from the continuity of f. Next, we prove that Q is uniformly bounded on B_{η^*}.

Let any $u \in B_{\eta^*}$. Then, by (19), we have

$$\begin{aligned}&\|Qu\|_{PC_{1-\gamma,\psi}}\\ &\le \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\psi(T)-\psi(a)\right)^{1-\gamma+\alpha} + \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\psi(T)-\psi(a)\right)^{\alpha}.\end{aligned}$$

This means that Q is uniformly bounded on B_{η^*}. Next, we show that QB_{η^*} is equicontinuous.

Let any $u \in B_{\eta^*}$ and $0 < \alpha < \tau_1 < \tau_2 \le T$. Then,

$$\begin{aligned}&\left|\left(\psi(\tau_2)-\psi(a)\right)^{1-\gamma}Q(y)(\tau_2) - \left(\psi(\tau_1)-\psi(a)\right)^{1-\gamma}Q(y)(\tau_1)\right|\\ &\le \left(\psi(\tau_2)-\psi(a)\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\psi'(s)\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}|g(s)|\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_a^{\tau_1}\Big[\psi'(s)\left(\psi(\tau_2)-\psi(a)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\\ &\quad-\psi'(s)\left(\psi(\tau_1)-\psi(a)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\Big]|g(s)|\,ds\\ &\le \frac{M\Gamma(\gamma)\left(\psi(\tau_2)-\psi(a)\right)^{1-\gamma}}{\Gamma(\alpha+\gamma)}\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\alpha+\gamma-1}\\ &\quad+\frac{M}{\Gamma(\alpha)}\int_a^{\tau_1}\Big|\psi'(s)\left(\psi(\tau_2)-\psi(a)\right)^{1-\gamma}\left(\psi(\tau_2)-\psi(s)\right)^{\alpha-1}\\ &\quad-\psi'(s)\left(\psi(\tau_1)-\psi(a)\right)^{1-\gamma}\left(\psi(\tau_1)-\psi(s)\right)^{\alpha-1}\Big|\left(\psi(s)-\psi(a)\right)^{\gamma-1}ds.\end{aligned}$$

Note that

$$\left|(\psi(\tau_2) - \psi(a))^{1-\gamma}\, Q(y)(\tau_2) - (\psi(\tau_1) - \psi(a))^{1-\gamma}\, Q(y)(\tau_1)\right| \to 0 \quad \text{as} \quad \tau_2 - \tau_1.$$

This shows that Q is equicontinuous on J. Therefore, Q is relatively compact on B_{η^*}. By $C_{1-\gamma,\psi}$ type Arzela-Ascoli Theorem Q is compact on B_{η^*}.

As a consequence of Krasnoselskii's fixed point theorem, we conclude that N has at least a fixed point $y \in C_{1-\gamma,\psi}(J)$ and by the same way of the proof of Theorem 5, we can easily show that $y^* \in C^{\gamma}_{1-\gamma,\psi}(J)$. Using Lemma 5, we conclude that (1)–(2) have at least one solution in the space $C^{\gamma}_{1-\gamma,\psi}(J)$. □

4 An Application

Consider the BVP

$$D^{\alpha,\beta;\psi}_{1^+} y(t) = \frac{2 + |y(t)| + \left|D^{\alpha,\beta;\psi}_{1^+} y(t)\right|}{107e^{-t+4}\left(1 + |y(t)| + \left|D^{\alpha,\beta;\psi}_{+1} y(t)\right|\right)} + \frac{\log\left(\sqrt{t}+1\right)}{3\sqrt{t}-1}, \quad t \in (1,3], \tag{20}$$

$$y(3) = c \in \mathbb{R}, \tag{21}$$

where $\alpha = \frac{1}{2}$, $\beta = 0$ and $\gamma = \frac{1}{2}$. Set

$$f(t,u,v) = \frac{2+u+v}{107e^{-t+4}(1+u+v)} + \frac{\log\left(\sqrt{t}+1\right)}{3\sqrt{t}-1}, \quad t \in (1,3],\, u,v \in (0,\infty].$$

We have

$$C^{\beta(1-\alpha)}_{1-\gamma,\psi}([1,3]) = C^{0}_{\alpha,\beta;\psi}([1,3]) = \left\{h : (1,3] \to \mathbb{R} : \sqrt{3}\left(\sqrt{t}-1\right)^{\frac{1}{2}} h \in C[1,3]\right\}.$$

Clearly, the function $f \in C_{\alpha,\beta;\psi}([1,3])$. Hence the condition (H1) is satisfied. For each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in (1,3]$

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \le \frac{1}{107e^{-t+4}}\left(|u-\overline{u}| + |v-\overline{v}|\right) \le \frac{1}{107e}\left(|u-\overline{u}| + |v-\overline{v}|\right).$$

Therefore, (H2) is verified with $K = L = \frac{1}{107e}$. The condition (14) is also satisfied with $T = 3$ and $a = 1$. It follows from Theorem 6 that (20)–(21) have a solution in the space $C^{\gamma}_{\alpha,\beta;\psi}([1,3])$.

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