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EXISTENCE AND UNIQUENESS RESULTS FOR SEQUENTIAL ψ-HILFER FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study multi-point boundary value problems for sequential fractional differential equations involving ψ-Hilfer fractional derivative. Existence and uniqueness results are obtained by using the classical fixed point theorems of Banach, Krasnoselskii, and the nonlinear alternative of Leray-Schauder. Examples illustrating our results are also presented.

1. INTRODUCTION

In recent few decades, fractional differential equations with initial/boundary conditions have been studied by many researchers. This is because fractional differential equations describe many real world processes related to memory and hereditary properties of various materials more accurately comparing to classical order differential equations. Therefore, the fractional-order models become more practical and realistic comparing to the integer-order models. Fractional differential equations arise in lots of engineering and clinical disciplines which include biology, physics, chemistry, economics, signal, and image processing, control theory and so on; see the monographs as [**3, 11, 14, 16, 17, 19, 29**].

In the literature, there exist several different definitions of fractional integrals and derivatives. The most popular are the Riemann-Liouville fractional derivative of order $\alpha > 0$ defined for a continuous function by

$$^{RL}D^{\alpha}u(t) := D^{n}I^{n-\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{a}^{t}(t-s)^{n-\alpha-1}u(s)\mathrm{d}s,$$

$n-1<\alpha<n$, and the Caputo fractional derivative of order $\alpha>0$, defined by

$$^{C}D^{\alpha}u(t) := I^{n-\alpha}D^{n}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{n}u(s)\mathrm{d}s,$$

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$n-1<\alpha<n$. In the above definitions, I^α is the Riemann-Liouville fractional integral of order $\alpha>0$ defined by

$$I^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\mathrm{d}s, \qquad n-1<\alpha<n,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a,∞).

In the literature, other known definitions of fractional integrals and derivatives are the Hadamard fractional derivative, the Erdeyl-Kober fractional derivative, and so on. A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [**7**], known as *the Hilfer fractional derivative* of order α and type $\beta\in[0,1]$, which interpolates between the Riemann-Liouville and Caputo derivatives, since it is reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta=0$ and $\beta=1$, respectively. The Hilfer fractional derivative of order α and parameter β of a function is defined by

$${}^H D^{\alpha,\beta}u(t)=I^{\beta(n-\alpha)}D^n I^{(1-\beta)(n-\alpha)}u(t), \qquad t>a,$$

where $n-1<\alpha<n,\ \ 0\le\beta\le 1,\ \ D=\dfrac{\mathrm{d}}{\mathrm{d}t}$. Some properties and applications of the Hilfer derivative are given in [**8, 9**], and references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see, for example, [**4, 6, 28**] and references therein. In [**1**], the authors initiated the study of nonlocal boundary value problems for Hilfer fractional derivative, by studying boundary value problem of Hilfer-type fractional differential equations with nonlocal integral boundary conditions:

$$(1)\qquad {}^H D^{\alpha,\beta}x(t)=f(t,x(t)), \qquad t\in[a,b],\ 1<\alpha<2,\ 0\le\beta\le 1,$$

$$(2)\qquad x(a)=0,\quad x(b)=\sum_{i=1}^m \delta_i I^{\varphi_i}x(\xi_i), \qquad \varphi_i>0,\ \delta_i\in\mathbb{R},\ \xi_i\in[a,b],$$

where ${}^H D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, $1<\alpha<2$, and parameter β, $0\le\beta\le 1$, I^{φ_i} is the Riemann-Liouville fractional integral of order $\varphi_i>0$, $\xi_i\in[a,b]$, $a\ge 0$ and $\delta_i\in\mathbb{R}$. Several existence and uniqueness results were proved by using a variety of fixed point theorems.

In [**18**], the existence and uniqueness of solutions were studied for a new class of system of Hilfer-Hadamard sequential fractional differential equations

$$(3)\qquad \begin{cases} ({}_H D_{1^+}^{\alpha_1,\beta_1}+k_1\,{}_H D_{1^+}^{\alpha_1-1,\beta_1})u(t)=f(t,u(t),v(t)), & 1<\alpha_1\le 2,\ t\in[1,e],\\ ({}_H D_{1^+}^{\alpha_2,\beta_2}+k_2\,{}_H D_{1^+}^{\alpha_2-1,\beta_2})v(t)=g(t,u(t),v(t)), & 1<\alpha_2\le 2,\ t\in[1,e],\end{cases}$$

with two point boundary conditions

$$(4)\qquad \begin{cases} u(1)=0, \quad u(e)=A_1,\\ v(1)=0, \quad v(e)=A_2,\end{cases}$$

where ${}_H D^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order $\alpha_i \in (1,2]$ and type $\beta_i \in [0,1]$ for $i \in \{1,2\}$, $k_1, k_2, A_1, A_2 \in \mathbb{R}_+$, and $f, g\colon [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

The fractional derivative with another function, in the Hilfer sense, called ψ-Hilfer fractional derivative, was introduced in [**20**]. For some recent results on existence and uniqueness of initial value problems and results on Ulam-Hyers-Rassias stability, see [**10, 21, 22, 23, 24, 25, 26, 13, 27**] and references therein. Recently, in [**15**], the authors extended the results from [**1**] to ψ-Hilfer nonlocal implicit fractional boundary value problems.

Motivated by the research going on in this direction, in this paper, we initiate the study of existence and uniqueness of solutions for a new class of boundary value problems of sequential ψ-Hilfer-type fractional differential equations with multi-point boundary conditions of the form

$$\left({}^H D^{\alpha,\beta;\psi}_{a^+} + k\ {}^H D^{\alpha-1,\beta;\psi}_{a^+}\right) x(t) = f(t, x(t)), \qquad t \in [a,b], \tag{5}$$

$$x(a) = 0, \qquad x(b) = \sum_{i=1}^{m} \lambda_i\ x(\theta_i), \tag{6}$$

where ${}^H D^{\alpha,\beta;\psi}_{a^+}$ is the ψ-Hilfer fractional derivative of order α, $1 < \alpha < 2$, and parameter β, $0 \le \beta \le 1$, $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a < b$, $k, \lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and $a < \theta_1 < \theta_2 < \cdots < \theta_m < b$.

Existence and uniqueness results are proved by using classical fixed point theorems. We make use of Banach's fixed point theorem to obtain the uniqueness result, while nonlinear alternative of Leray-Schauder type [**5**] and Krasnoselskii's fixed point theorem [**12**] are applied to obtain the existence results for the problem (5)–(6). The main results are presented in Section 3. Examples are constructed to illustrate the main results. In Section 2, some notations, definitions, and known results from fractional calculus are recalled.

2. Preliminaries

Let $\gamma = \alpha + 2\beta - \alpha\beta$, $1 < \alpha < 2$, $0 \le \beta \le 1$. Then $1 < \gamma \le 2$. Let $\psi \in C^1([a,b], \mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$.

Definition 2.1 ([**11**])**.** Let $\alpha > 0, \alpha \in \mathbb{R}$, and $g \in L^1([a,b], \mathbb{R})$. The ψ-Riemann-Liouville fractional derivative of a function g with resepect to ψ is defined by

$$I^{\alpha;\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s) \mathrm{d}s.$$

Definition 2.2 ([**20**])**.** Let $n - 1 < \alpha < n, n \in \mathbb{N}$ and $g \in C^n([a,b], \mathbb{R})$. The ψ-Hilfer fractional derivative ${}^H D^{\alpha,\beta;\psi}(\cdot)$ of a function g of order α and type $0 \le \beta \le 1$, is defined by

$${}^H D^{\alpha,\beta;\psi}_{a^+} g(t) = I^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\right)^n I^{(1-\beta)(n-\alpha);\psi} g(t).$$

Lemma 2.3 ([**20**])**.** *Let $\alpha, \chi > 0$ and $\delta > 0$. Then*

(i) $I^{\alpha;\psi} I^{\chi;\psi} h(t) = I^{\alpha+\chi;\psi} h(t)$,

(ii) $I^{\alpha;\psi} \left(\psi(t) - \psi(a)\right)^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t) - \psi(a))^{\alpha+\delta-1}$.

We note also that ${}^{H}D^{\alpha,\beta;\psi}_{a^+} (\psi(t) - \psi(a))^{\gamma-1} = 0$.

The following lemma contains the compositional property of Riemann-Liouville fractional integral operator with the ψ-Hilfer fractional derivative operator.

Lemma 2.4 ([**20**])**.** *Let $f \in L(a,b)$, $n-1 < \alpha \le n, n \in \mathbb{N}$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $I^{(n-\alpha)(1-\beta)} f \in AC^k[a,b]$. Then*

$$\begin{aligned}&\left(I^{\alpha;\psi};\psi\ {}^{H}D^{\alpha,\beta;\psi}_{a^+} f\right)(t)\\ &= f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f^{[n-k]}_{\psi} \lim_{t\to a^+} \left(I^{(1-\beta)(n-\alpha);\psi} f\right)(t),\end{aligned}$$

where $f^{[n-k]}_{\psi} = \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-k} f(t)$.

For convenience of the reader, we collect the fixed point theorems used to prove the results in this paper.

Lemma 2.5 (Banach fixed point theorem, [**2**])**.** *Let X be a Banach space, $D \subset X$ closed, and $F\colon D \to D$ a strict contraction, i.e., $|Fx - Fy| \le k|x-y|$ for some $k \in (0,1)$ and all $x, y \in D$. Then F has a fixed point in D.*

Lemma 2.6 (Krasnoselskii's fixed point theorem, [**12**])**.** *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) *$Ax + By \in M$ whenever $x, y \in M$,* (b) *A is compact and continuous,* (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 2.7 (Nonlinear alternative for single valued maps, [**5**])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A}\colon \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$, or*

(ii) *there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.*

3. Main results

We first prove an auxiliary lemma concerning a linear variant of the boundary value problem (5)–(6).

Lemma 3.1. *Let $a < b$, $1 < \alpha < 2$, $\gamma = \alpha + 2\beta - \alpha\beta$, $h \in C([a,b],\mathbb{R})$, and*

$$\Lambda := (\psi(b) - \psi(a))^{\gamma-1} - \sum_{i=1}^{m} \lambda_i (\psi(\theta_i) - \psi(a))^{\gamma-1} \neq 0. \tag{7}$$

Then the function $x \in C([a,b],\mathbb{R})$ is a solution of the boundary value problem

$$\left({}^{H}D_{a^+}^{\alpha,\beta;\psi} + k\ {}^{H}D_{a^+}^{\alpha-1,\beta;\psi}\right)x(t) = h(t), \quad t \in [a,b],\ 1<\alpha<2,\ 0 \le \beta \le 1, \tag{8}$$

$$x(a) = 0, \qquad x(b) = \sum_{i=1}^{m} \lambda_i\ x(\theta_i), \tag{9}$$

if and only if

$$\begin{aligned} x(t) = I^{\alpha;\psi}h(t) - kI^{1;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[&- k\sum_{i=1}^{m}\lambda_i I^{1;\psi}x(\theta_i) \\ - I^{\alpha;\psi}h(b) + \sum_{i=1}^{m}\lambda_i I^{\alpha;\psi}h(\theta_i) + kI^{1;\psi}x(b)\Big], &\qquad t \in [a,b]. \end{aligned} \tag{10}$$

Proof. Assume that x is a solution of the nonlocal boundary value problem (8)–(9). Operating fractional integral $I^{\alpha;\psi}$ on both sides of equation (8) and using Lemma 2.4, we obtain for $t \in [a,b]$,

$$x(t) - \sum_{k=1}^{2}\frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} x_{\psi}^{[2-k]} \lim_{t\to a^+}\left(I^{(1-\beta)(2-\alpha);\psi}x\right)(t) + kI^{1;\psi}x(t) = I^{\alpha;\psi}h(t).$$

Hence, using the fact that $(1-\beta)(2-\alpha) = 2-\gamma$, we have

$$\begin{aligned} x(t) &= \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\Big(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\Big)I^{2-\gamma;\psi}x(t)\Big|_{t=a} \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}I^{2-\gamma;\psi}x(t)\Big|_{t=a} - kI^{1;\psi}x(t) + I^{\alpha;\psi}h(t) \\ &= \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}\ {}^{H}D_{a^+}^{\gamma-1,\beta;\psi}x(t)\Big|_{t=a} \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}I^{2-\gamma;\psi}x(t)\Big|_{t=a} - kI^{1;\psi}x(t) + I^{\alpha;\psi}h(t) \\ &= c_1\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} + c_2\frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)} - kI^{1;\psi}x(t) + I^{\alpha;\psi}h(t), \end{aligned}$$

where $c_1 = {}^{H}D_{a^+}^{\gamma-1,\beta;\psi}x(t)\big|_{t=a}$ and $c_2 = I^{2-\gamma;\psi}x(t)\big|_{t=a}$.

From the first boundary condition $x(a) = 0$, we can obtain $c_2 = 0$. Then we get

$$x(t) = c_1\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} - kI^{1;\psi}x(t) + I^{\alpha;\psi}h(t), \qquad t \in [a,b]. \tag{11}$$

From $x(b) = \sum_{i=1}^{m}\lambda_i\ x(\theta_i)$, we have

$$c_1 = \frac{\Gamma(\gamma)}{\Lambda}\Big[- k\sum_{i=1}^{m}\lambda_i I^{1;\psi}x(\theta_i) - I^{\alpha;\psi}h(b) + \sum_{i=1}^{m}\lambda_i I^{\alpha;\psi}h(\theta_i) + kI^{1;\psi}x(b)\Big].$$

Substituting the values of c_1 in (11), we obtain the solution (10).

Conversely, suppose that x is the solution of the fractional integral equation (10). Operating fractional derivative ${}^H D_{a^+}^{\alpha,\beta;\psi}$ on both sides of equation (10) and using Lemma 2.3, we obtain

$$\begin{aligned}
{}^H D_{a^+}^{\alpha,\beta;\psi} x(t) &= h(t) - \left({}^H D_{a^+}^{\alpha,\beta;\psi} k\right) I^{1;\psi} x(t) \\
&\quad + \frac{1}{\Lambda}\Big[-k \sum_{i=1}^{m} \lambda_i I^{1;\psi} x(\theta_i) - I^{\alpha;\psi} h(b) + \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) \\
&\quad + k I^{1;\psi} x(b)\Big] {}^H D_{a^+}^{\alpha,\beta;\psi} (\psi(t)-\psi(a))^{\gamma-1} \\
&= h(t) - k D_{a^+}^{\alpha-1,\beta;\psi} h(t), \ t \in [a,b].
\end{aligned} \tag{12}$$

Now we prove that x satisfies the boundary condition (9). Obviously $x(a) = 0$. For each $i (i = 1, \ldots, m)$, from equation (10), we have

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i x(\theta_i) &= \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) - \sum_{i=1}^{m} \lambda_i k I^{1;\psi} x(\theta_i) \\
&\quad + \sum_{i=1}^{m} \lambda_i \frac{(\psi(\theta_i)-\psi(a))^{\gamma-1}}{\Lambda} \Big[-k \sum_{i=1}^{m} \lambda_i I^{1;\psi} x(\theta_i) \\
&\quad - I^{\alpha;\psi} h(b) + \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) + k I^{1;\psi} x(b)\Big] \\
&= \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) - \sum_{i=1}^{m} \lambda_i k I^{1;\psi} x(\theta_i) \\
&\quad + \left[\frac{(\psi(b)-\psi(a))^{\gamma-1}}{\Lambda} - 1\right] \Big[-k \sum_{i=1}^{m} \lambda_i I^{1;\psi} x(\theta_i) \qquad \text{[by (7)]} \\
&\quad - I^{\alpha;\psi} h(b) + \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) + k I^{1;\psi} x(b)\Big] \\
&= I^{\alpha;\psi} h(b) - k I^{1;\psi} x(b) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{\Lambda} \Big[-k \sum_{i=1}^{m} \lambda_i I^{1;\psi} x(\theta_i) \\
&\quad - I^{\alpha;\psi} h(b) + \sum_{i=1}^{m} \lambda_i I^{\alpha;\psi} h(\theta_i) + k I^{1;\psi} x(b)\Big] \\
&= x(b).
\end{aligned}$$

This completes the proof. □

Let $C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. In view of Lemma 3.1, we define an operator $\mathcal{A}\colon C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$\begin{aligned}(\mathcal{A}x)(t) &= I^{\alpha;\psi} f(t,x(t)) - kI^{1;\psi}x(t)\\ &\quad+ \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[-k\sum_{i=1}^{m}\lambda_i I^{1;\psi}x(\theta_i)\\ &\quad+\sum_{i=1}^{m}\lambda_i I^{\alpha;\psi} f(\theta_i,x(\theta_i)) + kI^{1;\psi}x(b) - I^{\alpha;\psi} f(b,x(b))\Big], \quad t\in[a,b].\end{aligned} \tag{13}$$

It should be noted that the sequential boundary value problem (5)–(6) has solution if and only if the operator $\mathcal{A}$ has fixed points.

In the following, for the sake of convenience, we set constants

$$\Omega = \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\alpha+1)}\Big[\sum_{i=1}^{m}|\lambda_i|(\theta_i-a)^{\alpha} + (\psi(b)-\psi(a))^{\alpha}\Big] + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}, \tag{14}$$

and

$$\Omega_1 = |k|(b-a) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[|k|\sum_{i=1}^{m}|\lambda_i|(\theta_i-a) + |k|(b-a)\Big]. \tag{15}$$

In the following subsections, we prove existence, as well as existence and uniqueness results for the sequential boundary value problem (5)–(6) by using classical fixed point theorems.

3.1. Existence and uniqueness result

Our first result is an existence and uniqueness result, based on Banach's fixed point theorem.

Theorem 3.2. *Assume that*

(H_1) *there exists a constant $L > 0$ such that*

$$|f(t,x) - f(t,y)| \le L|x-y| \qquad \text{for each } t\in[a,b] \text{ and } x,y\in\mathbb{R}.$$

If

$$L\Omega + \Omega_1 < 1, \tag{16}$$

where Ω and Ω_1 are defined by (14) *and* (15)*, respectively, then the boundary value problem* (5)–(6) *has a unique solution on* $[a,b]$.

Proof. We transfrom the boundary value problem (5)–(6) into a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined as in (13). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (5)–(6). Applying the Banach contraction mapping principle, we show that $\mathcal{A}$ has a unique fixed point.

Let $\sup_{t\in[a,b]}|f(t,0)| = M < \infty$, and choose

$$r \ge \frac{M\Omega}{1 - L\Omega - \Omega_1}. \tag{17}$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in C([a,b],\mathbb{R}) : \|x\| \leq r\}$. By using the assumption (H_1), we have

$$\begin{aligned}|f(t,x(t))| &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)|\\ &\leq L|x(t)| + M \leq L\|x\| + M \leq Lr + M.\end{aligned}$$

For any $x \in B_r$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t)|\\
&\leq \sup_{t\in[a,b]}\Bigg\{I^{\alpha;\psi}|f(t,x(t))| + |k|I^{1;\psi}|x(t)|\\
&\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big(\sum_{i=1}^{m}|\lambda_i|I^{\alpha;\psi}|f(\theta_i,x(\theta_i))| + I^{\alpha;\psi}|f(b,x(b))|\\
&\quad + |k|I^{1;\psi}|x(b)| + |k|\sum_{i=1}^{m}|\lambda_i|I^{1;\psi}|x(\theta_i)|\Big)\Bigg\}\\
&\leq I^{\alpha;\psi}(|f(t,x(t)) - f(t,0)| + |f(t,0)|) + |k|I^{1;\psi}|x(b)|\\
&\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big(\sum_{i=1}^{m}|\lambda_i|I^{\alpha;\psi}|f(\theta_i,x(\theta_i)) - f(\theta_i,0)| + |f(\theta_i,0)|)\\
&\quad + I^{\alpha;\psi}(|f(b,x(b)) - f(b,0)| + |f(b,0)|) + |k|I^{1;\psi}|x(b)|\\
&\quad + |k|\sum_{i=1}^{m}|\lambda_i|I^{1;\psi}|x(\theta_i)|\Big)\\
&\leq \Bigg\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^{m}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\Bigg]\Bigg\}(L\|x\| + M)\\
&\quad + \Bigg\{|k|(b-a) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[|k|\sum_{i=1}^{m}|\lambda_i|(\theta_i - a) + |k|(b-a)\Big]\Bigg\}\|x\|\\
&\leq (Lr+M)\Omega + \Omega_1 r \leq r,
\end{aligned}$$

which implies that $\mathcal{A}B_r \subset B_r$.

Next, let $x, y \in C([a,b],\mathbb{R})$. Then for $t \in [a,b]$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)|\\
&\leq I^{\alpha;\psi}|f(t,x(t)) - f(t,y(t))| + |k|I^{1;\psi}|x(t) - y(t)|\\
&\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big(\sum_{i=1}^{m}|\lambda_i|I^{\alpha;\psi}|f(\theta_i,x(\theta_i)) - f(\theta_i,y(\theta_i))| + I^{\alpha;\psi}|f(b,x(b))\\
&\quad - f(b,y(b))| + |k|I^{1;\psi}|x(b) - y(b)| + |k|\sum_{i=1}^{m}|\lambda_i|I^{1;\psi}|x(\theta_i) - y(\theta_i)|\Big)
\end{aligned}$$

$$\leq L\left\{\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\left[\sum_{i=1}^{m}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right.\right.$$
$$\left.\left.+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right]\right\}\|x-y\|$$
$$+\left\{|k|(b-a)+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[|k|\sum_{i=1}^{m}|\lambda_i|(\theta_i-a)+|k|(b-a)\Big]\right\}\|x-y\|$$
$$=(L\Omega+\Omega_1)\|x-y\|,$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\|\leq(L\Omega+\Omega_1)\|x-y\|$. As $L\Omega+\Omega_1<1$, $\mathcal{A}$ is a contraction. Therefore, we deduce by the Banach's contraction mapping principle, that $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (5)–(6). The proof is completed. □

Example 3.3. Consider the multi-point boundary value problem with sequential ψ-Hilfer fractional differential equation

$$\left(D_{1^+}^{\frac{3}{2},\frac{1}{2};t}+\frac{1}{6}D_{1^+}^{\frac{1}{2},\frac{1}{2};t}\right)x(t)=\frac{|x(t)|}{8(t+1)^2\,(1+|x(t)|)},\qquad t\in[1,3],\tag{18}$$

$$x(1)=0,\quad x(3)=\frac{1}{8}x\left(\frac{3}{2}\right)+\frac{3}{7}x(2)+\frac{2}{15}x\left(\frac{5}{2}\right).\tag{19}$$

Here $\alpha=3/2$, $\beta=1/2$, $k=1/6$, $\lambda_1=1/8$, $\lambda_2=3/7$, $\lambda_3=2/15$, $\theta_1=3/2$, $\theta_2=2$, $\theta_3=5/2$, $\psi(t)=t$, and $f(t,x)=\frac{1}{8(t+1)^2}\frac{|x|}{(1+|x|)}$, $t\in[1,3]$, $x\in\mathbb{R}$.

For any $x,y\in\mathbb{R}$ and $t\in[1,3]$,

$$|f(t,x)-f(t,y)|\leq\frac{1}{32}\,|x-y|\,.$$

Hence condition (H_1) is satisfied with $L=1/32$. With the given data, we find $\Lambda\approx 2.679969$, $\Omega\approx 8.657332$, and $\Omega_1\approx 0.687071$. Therefore $L\Omega+\Omega_1\approx 0.957612<1$.

It follows from Theorem 3.2 that the problem (18), (19) has a unique solution.

3.2. Existence results

In this subsection, we present two existence results. The first is based on the well-known Krasnoselskii's fixed point theorem.

Theorem 3.4. *Let $f\colon[a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that:*

(H_2) $|f(t,x)|\leq\varphi(t)$ *for all* $(t,x)\in[a,b]\times\mathbb{R}$ *and* $\varphi\in C([a,b],\mathbb{R}^+)$.

Then the boundary value problem (5)–(6) *has at least one solution on* $[a,b]$, *provided*

$$\Omega_1:=|k|(b-a)+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[|k|\sum_{i=1}^{m}|\lambda_i|(\theta_i-a)+|k|(b-a)\Big]<1.\tag{20}$$

Proof. Setting $\sup_{t\in[a,b]}\varphi(t)=\|\varphi\|$ and choosing

$$\rho\geq\frac{\|\varphi\|\Omega}{1-\Omega_1},\tag{21}$$

we consider $B_\rho = \{x \in C([a,b],\mathbb{R}) : \|x\| \leq \rho\}$. We define the operators $\mathcal{A}_1$, $\mathcal{A}_2$ on B_ρ by

$$\begin{aligned}\mathcal{A}_1 x(t) = I^{\alpha;\psi} f(t,x(t)) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[\sum_{i=1}^m \lambda_i I^{\alpha;\psi} f(\theta_i, x(\theta_i)) \\ - I^{\alpha;\psi} f(b,x(b))\Big], \qquad t \in [a,b],\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_2 x(t) = \; -kI^{1;\psi}x(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\Big[-k\sum_{i=1}^m \lambda_i I^{1;\psi} x(\theta_i) \\ + kI^{1;\psi}x(b)\Big], \qquad t \in [a,b].\end{aligned}$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}&|(\mathcal{A}_1 x)(t) + (\mathcal{A}_2 y)(t)| \\ &\leq \sup_{t\in[a,b]}\Bigg\{ I^{\alpha;\psi}|f(t,x(t))| + |k| I^{1;\psi}|y(t)| \\ &\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big(\sum_{i=1}^m |\lambda_i| I^{\alpha;\psi}|f(\theta_i,x(\theta_i))| + I^{\alpha;\psi}|f(b,x(b))| \\ &\quad + |k| I^{1;\psi}|y(b)| + |k|\sum_{i=1}^m |\lambda_i| I^{1;\psi}|y(\theta_i)|\Big)\Bigg\} \\ &\leq \Bigg\{\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|\varphi\| \\ &\quad + \Bigg\{|k|(b-a) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[|k|\sum_{i=1}^m |\lambda_i|(\theta_i - a) + |k|(b-a)\Big]\Bigg\}\|y\| \\ &\leq \|\varphi\|\Omega + \Omega_1\rho \leq \rho.\end{aligned}$$

This shows that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_\rho$. By using (20), it is easy to see that $\mathcal{A}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Also, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\begin{aligned}\|\mathcal{A}_1 x\| \leq \Bigg\{\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \\ + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\Bigg]\Bigg\}\|\varphi\|.\end{aligned}$$

Now we prove the compactness of the operator $\mathcal{A}_1$.

We define $\sup_{(t,x)\in[a,b]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently, we have

$$
\begin{aligned}
&|(\mathcal{A}_1x)(t_2) - (\mathcal{A}_1x)(t_1)| \\
&= \frac{1}{\Gamma(\alpha)}\bigg|\int_a^{t_1} \psi'(s)[(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_1)-\psi(s))^{\alpha-1}]f(s,x(s))\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} f(s,x(s))\mathrm{d}s\bigg| \\
&\quad + \frac{(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\sum_{i=1}^m |\lambda_i| I^{\alpha;\psi}|f(\theta_i, x(\theta_i))| \\
&\quad + |I^{\alpha;\psi} f(b,x(b))|\Big] \\
&\le \frac{\bar{f}}{\Gamma(\alpha+1)}[2(\psi(t_2)-\psi(t_1))^\alpha + |(\psi(t_2)-\psi(a))^\alpha - (\psi(t_1)-\psi(a))^\alpha|] \\
&\quad + \bar{f}\frac{(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|}\Big[\sum_{i=1}^m |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\Big],
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{A}_1$ is equicontinuous. So $\mathcal{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 2.6 are satisfied. So the conclusion of Lemma 2.6 implies that the boundary value problem (5)–(6) has at least one solution on $[a,b]$. □

The Leray-Schauder's Nonlinear Alternative is used for proving our second existence result.

Theorem 3.5. *Let $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that* (20) *holds. In addition we suppose that:*

(H$_3$) *there exist a continuous, nondecreasing function $\psi\colon [0,\infty)\to(0,\infty)$ and a function $p\in C([a,b],\mathbb{R}^+)$ such that*

$$|f(t,u)| \le p(t)\psi(|u|) \quad \text{for each} \quad (t,u)\in[a,b]\times\mathbb{R},$$

(H$_4$) *there exists a constant $K>0$ such that*

$$\frac{(1-\Omega_1)K}{\psi(K)\|p\|\Omega} > 1.$$

Then the boundary value problem (5)–(6) *has at least one solution on* $[a,b]$.

Proof. Let the operator $\mathcal{A}$ be defined by (13). First, we show that $\mathcal{A}$ maps bounded sets (balls) into bounded set in $C([a,b],\mathbb{R})$. For a number $r>0$, let $B_r = \{x\in C([a,b],\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C([a,b],\mathbb{R})$.

Then for $t \in [a,b]$, we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t)| \\
&\leq \sup_{t\in[a,b]} \bigg\{ I^{\alpha;\psi}|f(t,x(t))| + |k| I^{1;\psi}|x(t)| \\
&\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|} \Big(\sum_{i=1}^{m} |\lambda_i| I^{\alpha;\psi}|f(\theta_i, x(\theta_i))| \\
&\quad + I^{\alpha;\psi}|f(b,x(b))| + |k| I^{1;\psi}|x(b)| + |k| \sum_{i=1}^{m} |\lambda_i| I^{1;\psi}|x(\theta_i)| \Big) \bigg\} \\
&\leq \bigg\{ \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|} \Big[\sum_{i=1}^{m} |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \Big] \bigg\} \|p\| \psi(\|x\|) \\
&\quad + \bigg\{ |k|(b-a) + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|} \Big[|k| \sum_{i=1}^{m} |\lambda_i| (\theta_i - a) + |k|(b-a) \Big] \bigg\} \|x\|,
\end{aligned}
$$

and consequently,

$$\|\mathcal{A}x\| \leq \psi(r)\|p\|\Omega + \Omega_1 r.$$

Next, we show that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $C([a,b], \mathbb{R})$. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $x \in B_r$. Then we have

$$
\begin{aligned}
&|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| \\
&= \frac{1}{\Gamma(\alpha)} \bigg| \int_a^{t_1} \psi'(s)[(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_1)-\psi(s))^{\alpha-1}] f(s,x(s)) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} f(s,x(s)) \mathrm{d}s \bigg| + |k| r (t_2 - t_1) \\
&\quad + \frac{(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|} \Big[\sum_{i=1}^{m} |\lambda_i| I^{\alpha;\psi}|f(\theta_i, x(\theta_i))| \\
&\quad + |I^{\alpha;\psi} f(b,x(b))| + |k| \sum_{i=1}^{m} |\lambda_i| (\theta_i - a) r + |k|(b-a) r \Big] \\
&\leq \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)} [2(\psi(t_2)-\psi(t_1))^{\alpha} + |(\psi(t_2)-\psi(a))^{\alpha} - (\psi(t_1)-\psi(a))^{\alpha}|] \\
&\quad + |k| r (t_2 - t_1) \\
&\quad + \|p\|\psi(r) \frac{(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}}{|\Lambda|} \Big[\sum_{i=1}^{m} |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + |k| \sum_{i=1}^{m} |\lambda_i| (\theta_i - a) r + |k|(b-a) r \Big].
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{A}\colon C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is completely continuous.

The result follows from the Leray-Schauder nonlinear alternative (Lemma 2.7) once we have proved the boundedness of the set of all solutions to equations $x = \lambda\mathcal{A}x$ for $\lambda \in (0,1)$.

Let x be a solution. Then, for $t \in [a,b]$, and following the similar computations as in the first step, we have

$$|x(t)| \le \psi(\|x\|)\|p\|\Omega + \Omega_1\|x\|,$$

which leads to

$$\frac{(1-\Omega_1)\|x\|}{\psi(\|x\|)\|p\|\Omega} \le 1.$$

In view of (H_4), there exists K such that $\|x\| \ne K$. Let us set

$$U = \{x \in C([a,b],\mathbb{R}) : \|x\| < K\}.$$

We see that the operator $\mathcal{A}\colon \bar{U} \to C([a,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda\mathcal{A}x$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 2.7), we deduce that $\mathcal{A}$ has a fixed point $x \in \bar{U}$ which is a solution of the boundary value problem (5)–(6). This completes the proof. $\square$

Example 3.6. Consider the multi-point boundary value problem with sequential ψ-Hilfer fractional differential equation

$$\left(D_{0+}^{\frac{5}{4},\frac{3}{4};t} + \frac{3}{8}D_{0+}^{\frac{1}{4},\frac{3}{4};t}\right)x(t) = \frac{1}{5\sqrt{\pi}}\left(\sin t\tan^{-1}x + \frac{\pi}{2}\right), \quad t \in [0,1], \tag{22}$$

$$x(0) = 0, \quad x(1) = \frac{1}{9}x\left(\frac{2}{5}\right) + \frac{1}{5}x\left(\frac{1}{2}\right). \tag{23}$$

Here $\alpha = 5/4$, $\beta = 3/4$, $k = 3/8$, $\lambda_1 = 1/9$, $\lambda_2 = 1/5$, $\theta_1 = 2/5$, $\theta_2 = 1/2$, and $\psi(t) = t$. Set $f(t,x) = \frac{1}{5\sqrt{\pi}}\left(\sin t\tan^{-1}x + \frac{\pi}{2}\right)$. Obviously, $|f(t,x)| \le \frac{\sqrt{\pi}}{5}$. With the given data it is found that $\Omega_1 \approx 0.8815 < 1$ with $\Lambda \approx 0.8333$. Clearly, all the conditions of Theorem 3.4 are satisfied. Hence by the conclusion of the Theorem 3.4, it follows that problem (22)–(23) has at least one solution on $[0,1]$.

Example 3.7. Consider the multi-point boundary value problem with sequential ψ-Hilfer fractional differential equation

$$\begin{aligned}\left(D_{0+}^{\frac{6}{5},\frac{3}{5};t} + \frac{1}{7}D_{0+}^{\frac{1}{5},\frac{3}{5};t}\right)x(t)& \\ = \frac{1}{10}\left(\frac{1}{6}|x| + \frac{1}{8}\cos x + \frac{|x|}{4(1+|x|)} + \frac{1}{16}\right)&, \quad t \in [0,1],\end{aligned} \tag{24}$$

$$x(0) = 0, \quad x(1) = \frac{11}{8}x\left(\frac{1}{2}\right) + \frac{14}{9}x\left(\frac{2}{3}\right), \tag{25}$$

where $f(t,x) = \frac{1}{10}\left(\frac{1}{6}|x| + \frac{1}{8}\cos x + \frac{|x|}{4(1+|x|)} + \frac{1}{16}\right)$ and $|f(t,x)| \le \frac{1}{10}\left(\frac{1}{6}|x| + \frac{7}{16}\right)$.

Here $\alpha = 6/5$, $\beta = 3/5$, $k = 1/7$, $\lambda_1 = 11/8$, $\lambda_2 = 14/9$, $\theta_1 = 1/2$, $\theta_2 = 2/3$, and $\psi(t) = t$. We have $\|p\| = \frac{1}{10}$, $\psi(|x|) = \frac{1}{6}|x| + \frac{7}{6}$. We find that the condition (H_4) holds for $K > 0.7496$. Hence, by the conclusion of the Theorem 3.5, it follows that problem (24)–(25) has at least one solution on $[0, 1]$.

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