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Fractional neutral stochastic integrodifferential equations with Caputo fractional derivative: Rosenblatt process, Poisson jumps and Optimal control

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Abstract

The objective of this paper is to investigate the existence of mild solutions and optimal controls for a class of fractional neutral stochastic integrodifferential equations driven by Rosenblatt process and Poisson jumps in Hilbert spaces. First we establish a new set of sufficient conditions for the existence of mild solutions of the aforementioned fractional systems by using the successive approximation approach. The results are formulated and proved by using the fractional calculus, solution operator and stochastic analysis techniques. The existence of optimal control pairs of system governed by fractional neutral stochastic differential equations driven by Rosenblatt process and poisson jumps is also been presented. An example is provided to illustrate the theory.

Keywords: *Fractional neutral stochastic integrodifferential system, Rosenblatt process, Poisson jumps, Optimal control, Successive approximation.*

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1. Introduction

Fractional differential equations (FDEs) is about to generalization of the integer order and derivative to arbitrary order. The potential applications of FDEs are in many fields of science and including fluid flow, electrical networks and control theory, see [20, 21, 22, 11, 1, 2, 3, 4]. It is well known that many real world problems in science and engineering are modeled as stochastic differential equations [6]. Since fractional stochastic differential equations describe a physical dynamical system more accurately, it seems necessary to discuss the qualitative properties for such systems.

Nowadays various real-life situations can be modeled by using Poisson jumps. For example, if a system jumps from a "normal state" to " a other state", the strength of systems is random. In order to make more realistic model, a jump term is included in any dynamical systems. The study of stochastic differential equations driven by Poisson jumps has considerable attentions [9, 8, 5, 10]. Recently, Tamilalagan et al. [10] have investigated the stochastic fractional evolution inclusions driven by Poisson jumps in a Hilbert space. Very recently Rihan et al. [11] extended to study the existence of fractional SDEs with Hilfer fractional derivative and Poisson jumps. In [12] Balasubramaniam et al. studied a class of Hilfer fractional stochastic integrodifferential equations with Poisson jumps through the fixed point technique.

The fractional Brownian motion is the usual candidate to model phenomena due to its self-similarity of increments and long-range dependence. This fractional Brownian w^{H} is the continuous centered Gaussian process with covariance function described by

$$R^{\mathrm{H}}(t,s) \;:=\; \mathbf{E}\left[w^{\mathrm{H}}(t)w^{\mathrm{H}}(s)\right] = \frac{1}{2}(t^{2\mathrm{H}} + s^{2\mathrm{H}} - |t-s|^{2\mathrm{H}}).$$

The parameter H characterizes all the important properties of the process, when $\mathrm{H} < \frac{1}{2}$ the increments are negatively correlated and the correlation decays more slowly than quadratically; when $\mathrm{H} > \frac{1}{2}$, the increments are positively correlated and the correlation decays so slowly that they are not summable, a situation which is commonly known as the long memory property. Natural candidates are the Hermite processes, these non-Gaussian stochastic processes appear as limits are called Non-Central Limit theorem [15]. The fractional Brownian motion can be expressed as a Wiener integral with respect to the standard Wiener process, i.e. the integral of a deterministic kernel with respect to a standard Brownian motion, the Hermite process of order 1 is fractional Brownian motion and of order 2 is the Rosenblatt process.

Frequently, the optimal control is largely applied to biomedicine, namely, to model the cancer chemotherapy, and recently applied to epidemiological models and medicine (see [17, 18] and references therein). The main goal of optimal control is to find, in an open-loop control, the optimal values of the control variables for the dynamic system which maximize or minimize a given performance index. If a fractional differential equation describes the performance index and system dynamics, then an optimal control problem is known as a fractional optimal control problem. Using the fractional variational principle and Lagrange multiplier technique, Agrawal [13] discussed the general formulation and solution scheme for Riemann-Liouville fractional optimal control problems. It is remarkable that the fixed point technique, which is used to establish the existence results for abstract fractional differential equations, could be extended to address the fractional optimal control problems. Recently, Aicha Harrat et al. [23] studied the optimal controls of impulsive fractional system with Clarke subdifferential. Very recently, Using the Leray-Schauder fixed point theorem, Balasubramaniam et al. [3] studied the solvability and optimal controls for impulsive fractional stochastic integrodifferential equations. Tamilalagan et al. [19] investigated the solvability and optimal controls for fractional stochastic differential equations driven by Poisson jumps in Hilbert space via analytic resolvent operators and Banach contraction mapping principle. Ramkumar et al. [28] investigated the existence of mild solutions and optimal control for a class of fractional neutral stochastic differential equation driven by fractional Brownian motion and Poisson jumps in Hilbert spaces via successive approximation method.

Motivated by the aforementioned research works, in this manuscript we derive the sufficient conditions for the existence of solutions of the following class of optimal control for fractional neutral stochastic integrodifferential system driven by Rosenblatt process with Poisson jumps

$$\begin{aligned} D_t^{\beta}\left[x(t)-f(t,x_t)\right] &= A\Big[x(t)-f(t,x_t)\Big]+\Gamma_t^{1-\beta}\Big[\int_0^t g(s,x_s)dZ_{\mathrm{H}}(s)+\int_Z h(t,x_t,\eta)\widetilde{N}(dt,d\eta)\Big], \quad t\in[0,b], \\ x(t) &= \phi(t), \quad -r\le t\le 0, \end{aligned} \tag{1.1}$$

where D_t^{β} is the Caputo fractional derivative of order β, $0<\beta<1$. $\Gamma_t^{1-\beta}(.)$ denotes the $1-\beta$ order fractional integral. Let $A:\mathcal{D}(A)\subset\mathcal{X}\to\mathcal{X}$ is an infinitesimal generator of solution operator $\{S_\alpha(t)\}_{t\ge0}$ defined on a

Hilbert space $\mathcal{X}$ with an inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $\mathcal{Y}$ be another separable Hilbert space. The functions $f : [0, b] \times \mathcal{C} \to \mathcal{X}$, $g : [0, b] \times \mathcal{C} \to \mathcal{L}_2^0(Q^{1/2}\mathcal{Y}, \mathcal{X})$ and $h : [0, b] \times \mathcal{C} \times Z \to \mathcal{X}$ are nonlinear, $\phi(0)$ is $\Im_0$-measurable X-valued stochastic process independent of the Rosenblatt process Z_{H} with finite second moment. Let $\mathcal{C} = \mathcal{C}([-r, 0]; \mathcal{X})$ is the Banach space of all continuous functions $\phi : [-r, 0] \to \mathcal{X}$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$. Also, for $x \in ([-r, b]; \mathcal{X})$, we have $x_t \in \mathcal{C}$ for $t \in [0, b]$, $x_t(s) = x(t + s)$ for $s \in [-r, 0]$. In $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - dt(\lambda d\eta)$ the Poisson measure $\widetilde{N}(dt, d\eta)$ denotes the Poisson counting measure.

Let $(\Omega, \Im, \mathbf{P})$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\Im_t, t \in [0, b]\}$ satisfying $\Im_t \in \Im$. $\Im_t$ denotes the σ-field generated by $\left\{Z_{\mathrm{H}(s), s\in[0,t]}\right\}$ and the $\mathbf{P}$-null sets. $Z_{\mathrm{H}}(t)$ is a Rosenblatt process with parameter $\mathrm{H} \in (\frac{1}{2}, 1)$ on a real separable Hilbert space $\mathcal{Y}$. The collection of all strongly measurable, square integrable $\mathcal{X}$-valued random variable is denoted by $\mathcal{L}_2(\Omega, \Im, \mathcal{X}) \equiv \mathcal{L}_2(\Omega, \mathcal{X})$ which is a Banach space equipped with norm

$$\|x(\cdot)\|_{\mathcal{L}_2} = \left(\mathbf{E}\,\|x(t)\|^2\right)^{1/2},$$

It is easy to verify that $\mathcal{L}_2(\Omega, \mathcal{X})$ is a Banach space equipped with the above norm.

Let $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ denotes the space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$, whenever $\mathcal{X} = \mathcal{Y}$, we simply denote $\mathcal{L}(\mathcal{Y})$. $Q \in \mathcal{Y}$ represents a non-negative self adjoint operator. We introduce the subspace $\mathcal{Y}_0 = Q^{\frac{1}{2}}\mathcal{Y}$ of $\mathcal{Y}$ which is endowed with the inner product $< u, v >_{\mathcal{Y}_0} = < Q^{\frac{1}{2}}u, Q^{\frac{1}{2}}v >_{\mathcal{Y}}$ is a Hilbert space. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathcal{Y}_0, \mathcal{X})$ be the space of all Hilbert-Schmit operators from $\mathcal{Y}_0$ into $\mathcal{X}$, $\phi \in \mathcal{L}_2^0$ is called a Q-Hilbert-Schmidt operator, if

$$\|\phi\|_{\mathcal{L}_2^0}^2 = \sum_{n=1}^{\infty} \left\|Q^{\frac{1}{2}} e_n \phi\right\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with inner product $< \phi, \psi >_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} < \phi e_n, \psi e_n >$ is a separable Hilbert space. Also if $\phi = \psi$, then $\|\phi\|_{\mathcal{L}_2^0}^2 = \left\|\phi Q^{\frac{1}{2}}\right\|^2 = Tr(\phi Q \phi^*)$.

2. Preliminaries

In this section, we recollect basic concepts, definitions and Lemmas which will be used in the sequel to obtain the main results.

Definition 2.1. *[15] The basic concepts of the Rosenblatt process as far as Wiener integral, let $Z_{\mathrm{H}}(t)$ be a one-dimensional Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$. Hence the Rosenblatt process with parameter $H > \frac{1}{2}$ representation as*

$$Z_{\mathrm{H}}(t) \;=\; c(H)\int_0^t\int_0^t\left[\int_{y_1\vee y_2}^t \frac{\partial K^{\mathrm{H}'}}{\partial v}(v,y_1)\frac{\partial K^{\mathrm{H}'}}{\partial v}(v,y_2)dv\right]dw(y_1)dw(y_2),$$

(2.1)

where $K^{\mathrm{H}}(t,s)$ is defined as

$$K^{\mathrm{H}}(t,s) \;=\; C_{\mathrm{H}}s^{\frac{1}{2}-\mathrm{H}}\int_s^t (v-s)^{\mathrm{H}-\frac{3}{2}}v^{\mathrm{H}-\frac{1}{2}}dv, \quad t>s$$

with $c_{\mathrm{H}} = \sqrt{\frac{\mathrm{H}(2\mathrm{H}-1)}{\beta(2-2\mathrm{H},\mathrm{H}-\frac{1}{2})}})$.

For basic preliminaries and fundamental results on Rosenblatt process one can refer to [15, 3].

A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_c \frac{\mu^{\alpha-\beta}e^{\mu}}{\mu^{\alpha}-z}d\mu, \quad \alpha,\beta>0, \quad z\in\mathbf{C}$$

where c is a contour that starts and ends with $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^{\infty} e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^{\alpha})dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}-\omega}, \quad Re\lambda > \omega^{1/\alpha}, \quad \omega>0$$

Definition 2.2. *The Caputo derivative of order α with the lower limit 0 for a function f can be written as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^n(t), \quad t > 0, \quad 0 \leq n-1 < \alpha < n.$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$\mathcal{L}\left\{D_t^\alpha f(t); \lambda\right\} = \lambda^\alpha \tilde{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0), \quad n-1 < \alpha < n.$$

Definition 2.3. *A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbf{R}, \theta \in [\pi/2, \pi], M > 0$ such that the following two constants are satisfied*
(i) $\rho(A) \subset \sum_{(\theta,\omega)} = \{\lambda \in \mathcal{C}, \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$
(ii) $\|R(\lambda, A)\|_{\mathcal{L}(\mathcal{X})} \leq \frac{M}{|\lambda-\omega|}, \ \lambda \in \sum_{(\theta,\omega)}$

Definition 2.4. *Let A be a linear closed operator with domain $\mathcal{D}(A)$ defined on $\mathcal{X}$. We call A as the generator of a solution operator if there exists $\omega \geq 0$ and strongly continuous functions $S_\alpha : \mathbf{R}^+ \to \mathcal{L}(\mathcal{X})$ such that $\{\lambda^\alpha; Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \ Re\lambda > \omega, \ x \in \mathcal{X}.$$

S_α is called the solution operator generated by A.

Lemma 2.1. *If the functions $f : [0,b]\times\mathcal{C} \to \mathcal{X}$, $g : [0,b]\times\mathcal{C} \to \mathcal{L}_2^0(Q^{1/2}\mathcal{Y}, \mathcal{X})$ and $h : [0,b] \times \mathcal{C} \times Z \to \mathcal{X}$ satisfy the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then a continuously differentiable function $x(t)$ is a mild solution of (1) if x satisfies the following fractional integral equation*

$$\begin{aligned} x(t) &= \phi(t), \quad t \in [-r, 0], \\ x(t) &= S_\alpha(t)[\phi(0) - f(0,\phi)] + f(t, x_t) + \int_0^t S_\alpha(t-s)\left[\int_0^s g(\tau, x_\tau) dZ_H(\tau)\right] ds \\ &+ \int_0^t \int_Z S_\alpha(t-s) h(s, x_s, \eta) \tilde{N}(ds, d\eta), \end{aligned}$$

where $S_\alpha(t)$ is the solution operator, generated by A is given by

$S_\alpha(t) = E_{\alpha,1}(At^\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_{\hat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda$, $\hat{B}_r$ denotes the Brownwich path [27].

Definition 2.5. *A Cadlag stochastic process $x : [-r,b] \to \mathcal{X}$ is called a mild solution of (1) if*
(i) $x(t)$ is $\Im_t$ − adapted,
(ii) $\int_0^b \mathbf{E}\|x(s)\|^2 ds < \infty$, almost surely
(iii) for each $t \in [0,b]$, $x(t)$ satisfies the following integral equation

$$\begin{aligned} x(t) &= \varphi(t), \quad t \in [-r,0], \\ x(t) &= S_\alpha(t)[\phi(0) - f(0,\phi)] + f(t,x_t) + \int_0^t S_\alpha(t-s)\left[\int_0^s g(\tau,x_\tau)dZ_H(\tau)\right] ds \\ &+ \int_0^t \int_Z S_\alpha(t-s)h(s,x_s,\eta)\tilde{N}(ds,d\eta), \qquad t \in [0,b]. \end{aligned} \tag{2.2}$$

Definition 2.6. *Let us define $M_2([-r,b],\mathcal{X})$ be the space of all $\mathcal{X}$-valued $\Im_t$ adapted processes $\left\{x(t), -r \le t \le b\right\}$ such that*
(i) $x_0 = \phi$ and $x(t)$ is Cadlag on $[0,b]$,
(ii) for all $x \in M_2([-r,b],\mathcal{X})$

$$\mathbf{E}\|x\|^2_{M_2} = \mathbf{E}\|\phi\|^2 + \mathbf{E}\int_0^b \|x(t)\|^2 dt < \infty. \tag{2.3}$$

Lemma 2.2. *[26] The space $M_2([-r,b],\mathcal{X})$ is a Banach space with the norm defined by (4).*

3. Existence of Mild Solutions

In this section, we shall derive the existence and uniqueness of mild solution for system (1). we will work under the following hypotheses:

(A1) There exists a constant $M > 0$ such that $\|S_\alpha(t)\|^2 \le M$, for all $t \in [0,b]$.

(A2) The mappings $g(.), h(.)$ satisfy the following conditions, for all $t \in [0,b], x_1, x_2 \in \mathcal{C}$

$\int_0^s \|g(\tau, x_1) - g(\tau, x_2)\|^2 d\tau \le \kappa(\|x_1 - x_2\|^2),$

$\int_0^t \int_Z \|h(s, x_1, \eta) - h(s, x_2, \eta)\|^2 \tilde{\lambda}(du) ds \vee$

$\left(\int_0^t \int_Z \|h(s, x_1, \eta) - h(t, x_2, \eta)\|^4 \tilde{\lambda}(d\eta) ds\right)^{1/2} \le \int_0^t \kappa(\|x_1 - x_2\|^2) ds,$

$\left(\int_0^t \int_Z \|h(s, x_1, \eta)\|^4 \tilde{\lambda}(d\eta) ds\right)^{1/2} \le \int_0^t \kappa(\|x_1\|^2) ds,$

where $\kappa(.)$ is a concave nondecreasing function from $\mathbf{R}_+$ to $\mathbf{R}_+$ such that $\kappa(0) = 0$, $\kappa(\vartheta) > 0$ for $\vartheta > 0$ and $\int_{0+} \frac{d\vartheta}{\kappa(\vartheta)} = +\infty$.

(A3) For all $t \in [0,b]$, there exists a constant $M_0 > 0$ such that

$$\int_Z \|h(t, 0, u)\|^2 \tilde{\lambda}(d\eta) ds \vee \int_0^s \|g(\tau, 0)\|^2 d\tau \le M_0$$

(A4) f satisfies the Lipschitz condition, that is, there exist a constant $M_f > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\|^2 \le M_f \|x_1 - x_2\|^2 \quad and \quad f(t, 0) = 0, \quad t \ge 0, \quad x_1, x_2 \in \mathcal{C}$$

Let us introduce the sequence of successive approximation defined as follows

$$\begin{aligned}
x^0(t) &= S_\alpha(t)\phi(0), \quad t \in [0,b], \\
x^n(t) &= \phi(t), t \in [-r, 0], n = 1, 2, ... \\
x^n(t) &= S_\alpha(t)[\phi(0) - f(0, \phi)] + f(t, x_t^n) + \int_0^t S_\alpha(t-s) \\
&\quad \left[\int_0^s g(\tau, x_t^{n-1}) dZ_H(\tau)\right] ds \\
&\quad + \int_0^t \int_Z S_\alpha(t-s) h(s, x_s^{n-1}, \eta) \tilde{N}(ds, d\eta), \\
&\quad t \in [0,b], \quad n = 1, 2, ...
\end{aligned} \tag{3.1}$$

Lemma 3.1. *Assume that hypotheses* $(\mathbf{A1})-(\mathbf{A4})$ *and* $M_f < \frac{1}{8}$ *hold, then for all* $t \in [-r, b]$, $n \ge 0$, *there exists a constant* k_1 *such that* $\mathbf{E}\|x^n\|^2_{M_2} \le k_1$

Proof. It is obvious that $x^0 \in M_2([-r,b],\mathcal{X})$. By induction, $x^n(t) \in M_2([-r,b],\mathcal{X})$. From (5), using the Holder's inequality, the Doop martingale inequality, and Burkholder-Davis-Gundy inequality for pure jump stochastic integral in $\mathcal{X}$, we have

$$\begin{aligned}
\mathbf{E}\sup_{0\le s\le t}\|x^n(s)\|^2 &\le 8M(\mathbf{E}\|\phi\|^2 + M_f\mathbf{E}\|\phi\|^2) + 4M_f\mathbf{E}\|x_s^n\|^2 \\
&+ 8MTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}\int_0^t\left[\int_0^s \mathbf{E}\|g(\tau,x_\tau^{n-1})-g(\tau,0)\|^2 d\tau\right]ds} \\
&+ 8MTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}\int_0^t\left[\int_0^s \mathbf{E}\|g(\tau,0)\|^2 d\tau\right]ds} \\
&+ 16Mb\bigg[\int_0^t\int_Z \mathbf{E}\|h(s,x_s^{n-1},\eta)-h(s,0,\eta)\|^2\tilde{\lambda}(d\eta)ds \\
&+ \int_0^t\int_Z \mathbf{E}\|h(s,0,\eta)\|^2\tilde{\lambda}(d\eta)ds\bigg] \\
&+ 8Mb\left(\int_0^t\int_Z \mathbf{E}\|h(s,x_s^{n-1},\eta)\|^4\tilde{\lambda}(d\eta)ds\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\le 8M(\mathbf{E}\|\phi\|^2 + M_f\mathbf{E}\|\phi\|^2) + 4M_f\mathbf{E}\|x_s^n\|^2 \\
&+ 8MTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds + 8MbM_0Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}}} \\
&+ 16Mb\int_0^t \kappa\left(\mathbf{E}\|x_s^{n-1}\|^2\right)ds + 16Mb^2M_0 + 8Mb\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds \\
&\le 8M\left[\mathbf{E}\|\phi\|^2 + M_f\mathbf{E}\|\phi\|^2\right] + 8MM_0bTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}} \\
&+ 16Mb^2K_0 + 4M_f\mathbf{E}\|x_s^n\|^2 + 8MTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds} \\
&+ 24Mb\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds \\
&\le Q_1 + 4M_f\mathbf{E}\|x_s^n\|^2 + 8MTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds} \\
&+ 24Mb\int_0^t \kappa(\mathbf{E}\|x_s^{n-1}\|^2)ds
\end{aligned}$$

where $Q_1 = 8M\left[\mathbf{E}\|\phi\|^2 + M_f\mathbf{E}\|\phi\|^2\right] + 8MM_0bTr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+16Mb^2M_0}$. Because $\kappa(.)$ is concave and $\kappa(0)=0$, we find a pair of positive constants a_1 and a_2 so that

$$\kappa(\vartheta) \le a_1 + a_2\vartheta, \quad \vartheta \ge 0$$

Then,

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le t}\|x^n(s)\|^2 &\le Q_1 + 8Ma_1(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)b}\\ &+ 4M_f\mathbf{E}\|x_s^n\|^2 + 8Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x_s^{n-1}\|^2 ds,\\ &\le Q_2 + 4M_f\mathbf{E}\|x_s^n\|^2 + 8Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x_s^{n-1}\|^2 ds.\end{aligned}$$

where $Q_2 = Q_1 + 8Ma_1(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)b}$ and noting that

$$\mathbf{E}\|x_t\|_{\mathcal{C}}^2 = \sup_{-r\le\theta\le 0}\mathbf{E}\|x(t+\theta)\|^2 \le 2\sup_{-r\le\tau\le 0}\mathbf{E}\|x(\tau)\|^2 + 2\sup_{0\le\tau\le b}\mathbf{E}\|x(\tau)\|^2$$

and

$$\mathbf{E}\|x_s^{n-1}\|^2 \le 2\mathbf{E}\|\phi\|^2 + 2\mathbf{E}\sup_{0\le s\le b}\|x^{n-1}(s)\|^2.$$

$$\begin{aligned}\mathbf{E}\sup\nolimits_{0\le s\le t}\|x^n(s)\|^2 &\le Q_2 + 8M_f\mathbf{E}\|\phi\|^2 + 8M_f\mathbf{E}\sup\nolimits_{0\le s\le t}\|x^n(s)\|^2\\ &+ 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\\ &\times b\mathbf{E}\|\phi\|^2 + 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x^{n-1}(s)\|^2 ds\\ \mathbf{E}\sup\nolimits_{0\le s\le t}\|x^n(s)\|^2(1-8M_f) &\le Q_3 + 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x^{n-1}(s)\|^2 ds,\\ \mathbf{E}\sup\nolimits_{0\le s\le t}\|x^n(s)\|^2 &\le \frac{1}{(1-8M_f)}\\ &\left\{Q_3 + 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x^{n-1}(s)\|^2 ds\right\}\end{aligned}$$

where $Q_3 = Q_2 + 8M_f\mathbf{E}\|\phi\|^2 + 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)b\mathbf{E}\|\phi\|^2}$.

On the other hand, for any $k \ge 1$,

$$\begin{aligned}\max_{1\le n\le k}\mathbf{E}\sup_{0\le s\le t}\|x^{n-1}(s)\|^2 &\le \mathbf{E}\|x^0(s)\|^2 + \max_{1\le n\le k}\mathbf{E}\sup_{0\le s\le t}\|x^n(s)\|^2,\\ \max_{1\le n\le k}\mathbf{E}\sup_{0\le s\le t}\|x^n(s)\|^2 &\le \frac{1}{(1-8M_f)}\Big[Q_3 + 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \mathbf{E}\|x^0(s)\|^2 ds\\ &+ 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \max\nolimits_{1\le n\le k}\mathbf{E}\sup\nolimits_{0\le r\le s}\|x^n(r)\|^2 ds\Big],\\ &\le \frac{1}{(1-8M_f)}\Big[Q_3 + 16M^2a_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)\mathbf{E}\|\phi\|^2 b}\\ &+ 16Ma_2(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)}\int_0^t \max\nolimits_{1\le n\le k}\mathbf{E}\sup\nolimits_{0\le r\le s}\|x^n(r)\|^2 ds\Big],\\ &\le Q_4 + Q_5\int_0^t \mathbf{E}\|x^n(s)\|^2 ds,\end{aligned}$$

where

$$Q_4 = \frac{1}{(1-8M_f)}[Q_3{+}16M^2a_2(Tr(Q)C_{\mathrm{Ht}_1^{2\mathrm{H}-1}+3b)}\mathbf{E}\|\phi\|^2b], Q_5{=}16Ma_2(Tr(Q)C_{\frac{\mathrm{Ht}_1^{2\mathrm{H}-1}+3b)}{(1-8M_f)}}$$

$$\max_{1\le n\le k}\mathbf{E}\sup_{0\le s\le t}\|x^{n-1}(s)\|^2 \ \le \ Q_4+Q_5\int_0^t \max_{1\le n\le k}\mathbf{E}\sup_{0\le r\le s}\|x^n(r)\|^2ds.$$

By Gronwall inequality yields

$$\mathbf{E}\sup_{0\le s\le t}\|x^{n-1}(s)\|^2 \le Q_4e^{Q_5b}.$$

Moreover, $\mathbf{E}\,\|x^n\|^2_{M_2} = \mathbf{E}\,\|x_0^n\|^2+\int_0^b \mathbf{E}\,\|x^n(s)\|^2\,ds \le \mathbf{E}\,\|\phi\|^2+bQ_4e^{Q_5b} < \infty$, which implies $x^n(\cdot)\in M_2([-r,b],\mathcal{X})$ and $\mathbf{E}\,\|x^n\|^2_{M_2} = \mathbf{E}\,\|\phi\|^2+bQ_4e^{Q_5b}$.
□

Lemma 3.2. *If the assumptions of Lemma 3.1 are satisfied with $M_f < \frac{1}{3}$, then there exist positive constants k_2 and k_3 such that for all $t\in[0,b]$, $m,n\ge 1$*

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le t}\|x^{m+n}(s)-x^n(s)\|^2 &\le k_2\int_0^t \kappa(\mathbf{E}\sup_{0\le r\le s}\|x^{m+n-1}(r)-x^{n-1}(r)\|^2)ds,\\ \mathbf{E}\sup_{0\le s\le t}\|x^{m+n}(s)-x^n(s)\|^2 &\le k_3t\end{aligned}$$
(3.2)

Proof. By the definition of x^n, let us derive that for any $m,n\ge 1$ and $t\in J$

$$\begin{aligned}&\mathbf{E}\sup\nolimits_{0\le s\le t}\|x^{m+n}(s)-x^n(s)\|^2\\ &\le 3\mathbf{E}\sup\nolimits_{0\le s\le t}\|f(s,x_s^{m+n})-f(s,x_s^n)\|^2\\ &+3\mathbf{E}\sup\nolimits_{0\le s\le t}\|\textstyle\int_0^t S_\alpha(t-s)\int_0^s(g(\tau,x_\tau^{m+n-1})-g(\tau,x_\tau^{n-1}))dZ_H(\tau)ds\|^2\\ &+3\mathbf{E}\sup\nolimits_{0\le s\le t}\|\textstyle\int_0^t S_\alpha(t-s)\int_Z(h(s,x_s^{m+n-1},\eta)-h(s,x_s^{n-1}),\eta)\tilde N(ds,d\eta)\|^2,\\ &+3M_f\mathbf{E}\sup\nolimits_{0\le s\le t}\|x^{m+n}(s)-x^n(s)\|^2\end{aligned}$$

$$\leq 3(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M \int_0^t \kappa\left(E\sup 0\leq s\leq t\|x_s^{m+n-1}-x_s^{n-1}\|^2\right)ds}$$
$$+6bM\Bigg[\int_0^t\int_Z \mathbf{E}\sup_{0\leq s\leq t}\|h(s,x_s^{m+n-1},\eta)-h(s,x_s^{n-1},\eta)\|^2\tilde{\lambda}(d\eta)ds$$
$$+\left(\int_0^t\int_Z \mathbf{E}\sup_{0\leq s\leq t}\|h(s,x_s^{m+n-1},\eta)-h(s,x_s^{n-1},\eta)\|^4\tilde{\lambda}(d\eta)ds\right)^{1/2}\Bigg]$$
$$\leq 3M_f\mathbf{E}\sup_{0\leq s\leq t}\|x^{m+n}(s)-x^n(s)\|^2$$
$$+3((Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M+4bM)\int_0^t \kappa\left(\sup_{0\leq s\leq t}\|x_s^{m+n-1}-x_s^{n-1}\|^2\right)ds}$$
$$\leq \frac{Q_6}{1-3M_f}\int_0^t \kappa\left(\mathbf{E}\sup_{0\leq r\leq s}\|x^{m+n-1}(r)-x^{n-1}(r)\|^2\right)ds,$$
$$\leq k_2\int_0^t \kappa(2k_1)ds = k_3 t,$$

By inequality (5) and $Q_6 = 3((Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M+4bM)}$ and $k_2 = \frac{Q_6}{1-3M_f}$. □

Theorem 3.1. *Assume that the hypotheses of Lemma 3.1 and 3.2 hold, then system (1) has a unique mild solution $x(t)\in M_2([-r,b],\mathcal{X})$.*

Proof. **Step 1:** Let us show that $x^n(t), t\in[0,b]$ is a Cauchy sequence.

Let $\nu_1(\vartheta) = k_2\kappa(\vartheta)$. Choose $b_1\in[0,b]$ such that $\nu_1(K_3\vartheta)\leq k_3$ for $\vartheta\in[0,b_1]$. We first introduce two sequences of functions $\phi_{n,m}(t)_{m,n\in\mathbf{N}_+}$ and $\phi_n(t)_{n\in\mathbf{N}_+}$ by

$$\begin{aligned}
\phi_1(t) &= k_3 t,\\
\phi_{n+1}(t) &= \int_0^t \nu_1(\phi_n(\vartheta))d\vartheta,\\
\phi_{m,n}(t) &= \mathbf{E}\sup_{0\leq\vartheta\leq t}\|x^{m+n}(\vartheta)-x^n(\vartheta)\|^2.
\end{aligned}$$

Then $\phi_n(t)_{n\in\mathbf{N}_+}$ is monotonically decreasing when $n\to\infty$ and $0\leq\phi_{m,n}(t)\leq\phi_n(t)$ for all $m,n\geq 1,\ t\in[0,b_1]$. In fact, it is obvious that $\phi_{1,m}(t)\leq\phi_1(t)$ and

$$\begin{aligned}
\phi_{2,m}(t) &= \mathbf{E}\sup_{0\leq\vartheta\leq t}\|x^{m+2}(\vartheta)-x^2(\vartheta)\|^2\\
&\leq \int_0^t \nu_1 E\left(\sup_{0\leq\vartheta\leq s}\|x^{m+1}(\vartheta)-x^1(\vartheta)\|^2\right)ds\\
&\leq \int_0^t \nu_1(\phi_1(s))ds\\
&= \int_0^t \nu_1(k_3 s)ds
\end{aligned}$$

$$= \phi_2(t) \leq k_3 t = \phi_1(t),$$

which implies that $\phi_2(t) \leq \phi_2(t) \leq \phi_1(t)$. Now assume the results holds for n, then

$$\begin{aligned}
\phi^{n+1,m}(t) &= \mathbf{E} \sup_{0 \leq \vartheta \leq t} \|x^{m+n+1}(\vartheta) - x^{n+1}(\vartheta)\|^2 \\
&\leq \int_0^t \nu_1(\phi_{m,n}(s)) ds \\
&\leq \int_0^t \nu_1(\phi_n(s)) ds \\
&= \phi_{n+1}(t) \leq \int_0^t \nu_1(\phi_{n-1}(s)) ds \\
&= \phi_n(t).
\end{aligned}$$

This shows that $\phi_n(t)$ is a nonnegative and decreasing continuous function on $[0, b_1]$ by induction on n, so we can define a function $\phi_n(t)$ by $\phi_k(t) \downarrow \phi(t)$, and it is easy to verify that $\phi(0) = 0$ and $\phi(t)$ is a continuous function on $[0, b_1]$. Consequently, $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \int_0^t \nu_1(\phi_{n-1}(s))ds = \int_0^t \nu_1(\phi(s))ds$. From $\phi(0) = 0$, $\int_{0^+} \frac{d\vartheta}{\nu_1(\vartheta)} = +\infty$ together with Bihari inequality, we obtain $\phi(t) \equiv 0$. Thus $0 \leq \phi_{n,n}(t) \leq \phi_n(b_1) \to 0$as $n \to \infty$. This shows that $x^n(t), t \in [0, b_1]$ is a Cauchy sequence in $M_2([-r, b], \mathcal{X})$. The Borel-Cantelli lemma shows that as $n \to \infty, x^n(t) \to x(t)$ holds uniformly for $0 \leq t \leq b$. So, taking limits on both sides of (5), for all $-r \leq t \leq b$, we obtain that $x(t)$ is a solution of (1).

Step 2: Uniqueness Let $x(t), y(t)$ be two solutions of (1). Then the uniqueness is obvious on the interval $[-r, 0]$, and for $0 \leq t \leq b$, it is easy to show that by using Lemma 3.2, we have

$$\mathbf{E} \sup_{0 \leq s \leq t} \|x(s) - y(s)\|^2 \leq k_2 \int_0^t \kappa \left(\sup_{0 \leq r \leq s} \|x(r) - y(r)\|^2 \right) ds$$

The Bihari inequality yields that

$$\mathbf{E}(\sup_{0 \leq s \leq t} \|x(s) - y(s)\|^2) = 0, \quad 0 \leq t \leq b.$$

Therefore, $x(t) = y(t)$ for all $0 \leq t \leq b$. $\square$

4. Optimal Controls

Let $\mathbf{Y}$ be a reflexive Banach space in which controls $\mathbf{u}$ takes values. The multi valued map $\upsilon : [0, b] \to 2^{\mathbf{Y}} \setminus \{\emptyset\}$ has closed, convex and bounded values, $\upsilon(.)$ is graph measurable and $\upsilon(.) \subseteq \varepsilon$, where ε is a bounded set in $\mathbf{Y}$. Introduce the admissible set $\mathcal{U}_{ad} = \Big\{\mathbf{u}(.) \in \mathcal{L}_2(\varepsilon) : \mathbf{u}(t) \in \upsilon(t), a.e.\Big\}$. Now, Consider the fractional stochastic control problem

$$
\begin{aligned}
D_t^\alpha[x(t) - f(t, x_t)] &= A[x(t) - f(t, x_t)] + J_t^{1-\alpha} \\
&\quad \Big[B(t)\mathbf{u}(t) + \int_0^t g(s, x_s) dZ_H(s) ds \\
&\quad + \int_Z h(t, x_t, \eta)\tilde{N}(dt, d\eta)\Big], \epsilon\ J, \quad u \ \epsilon\ \mathcal{U}_{ad}, \\
x(t) &= \phi(t), -r \leq t \leq 0.
\end{aligned} \tag{4.1}
$$

(A5) The operator $B \in \mathcal{L}_2(J, L(\mathbf{Y}, \mathcal{X}))$, $\|B\|_{\mathcal{L}_2}$ stands for the norm of operator B in the Banach space $\mathcal{L}_2(J, \mathcal{L}(\mathbf{Y}, \mathcal{X}))$. It is obvious that $B\mathbf{u} \in \mathcal{L}_2(J, \mathcal{X})$ for all $\mathbf{u} \in \mathcal{U}_{ad}$.

Theorem 4.1. *If the hypotheses* $(\mathbf{A1}) - (\mathbf{A5})$ *are satisfied, for every* $\mathbf{u} \in \mathcal{U}_{ad}$*, then there exists a unique mild solution of system (7) of the form*

$$
\begin{aligned}
x(t) &= S_\alpha(t)[\phi(0) - f(0, \phi)] + f(t, x_t) + \int_0^t S_\alpha(t - s) \\
&\quad \Big[B(s)\mathbf{u}(s) + \int_0^s g(s, x_s) dZ_H(s)\Big] ds \\
&\quad + \int_0^t S_\alpha(t - s) \int_Z h(s, x_s, \eta)\tilde{N}(ds, d\eta), \quad t \in J.
\end{aligned}
$$

Proof. The proof of this theorem is similar to that of Theorem 3.3, and one can easily prove that solution of system (7) by using the method of successive approximation, and hence, it is omitted. □

To prove the existence of optimal control pair of system (7), let us define the performance index

$$
\mathcal{J}(\mathbf{u}) = \int_0^b \mathcal{L}(t, x(t), x_t, \mathbf{u}(t)) dt
$$

Our aim is to find $\mathbf{u}^0 \in \mathcal{U}_{ad}$ such that $\mathcal{J}(\mathbf{u}^0) \leq \mathcal{J}(\mathbf{u})$ for all $\mathbf{u}^0 \in \mathcal{U}_{ad}$, where $x(t)$ denotes the mild solution of (7), we need the following hypotheses:

(**A6**)
(i) The functional $\mathcal{L} : J \times \mathcal{X} \times \mathcal{C} \times \mathbf{Y} \to \mathbf{R} \cup \{\infty\}$ is Borel measurable.
(ii) $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathcal{X} \times \mathcal{C} \times \mathbf{Y}$ for almost all $t \in J$.
(iii) $\mathcal{L}(t, x, x_t, \cdot)$ is concave on $\mathbf{Y}$ for each $x_t \in \mathcal{C}, x \in \mathcal{X}$ and almost all $t \in J$.
(iv) There exist constants $d, e \geq 0$, $j > 0, \rho \geq 0$ and $\rho \in \mathcal{L}_1(J, \mathbf{R})$ such that

$$\mathcal{L}(t, x(t), x_t, \mathbf{u}(t)) \geq \rho(t) + d\|x\|_{\mathcal{X}} + e\|x_t\|_{\mathcal{C}} + \mathcal{J}\|\mathbf{u}\|^2_{\mathbf{Y}}.$$

Theorem 4.2. *Assume that hypotheses* (**A1**) – (**A5**) *and Theorems 3.3 and 4.1 hold and B as a strongly continuous operator, should be defined (7) admits at least one optimal pair.*

Proof. The main task is to minimize the performance index $\mathcal{J}(\mathbf{u})$. In order to prove that, if $inf\,\{\mathcal{J}(\mathbf{u}) : \mathbf{u} \in \mathcal{U}_{ad}\} = \infty$, then the result is obvious. Assume that $inf\,\{\mathcal{J}(\mathbf{u}) : \mathbf{u} \in \mathcal{U}_{ad}\} = \epsilon < \infty$, using the hypothesis (**A6**), we have $\epsilon > -\infty$. By the definition of infimum, there exists a minimizing sequence feasible pair $\{(x^m, \mathbf{u}^m)\} \subset A_{ad} \equiv \Big\{(x, \mathbf{u}) : x$ is a mild solution of the system (7) corresponding to $\mathbf{u} \in \mathcal{U}_{ad}\Big\}$ such that $\mathcal{J}(x^m, \mathbf{u}^m) \to \epsilon$ as $m \to +\infty$. Because $\{\mathbf{u}^m\} \subseteq \mathcal{U}_{ad}$ $m = 1, 2, ...$ and $\{\mathbf{u}^m\}$ is a bounded subset of the separable reflexive Banach space $\mathcal{L}_2(J, \mathbf{Y})$, there exist a subsequence $\{\mathbf{u^m}\}$ and $\mathbf{u}^0 \in \mathcal{L}_2(J, \mathbf{Y})$ such that $\mathbf{u}^m \to \mathbf{u}^0$ weakly in $\mathcal{L}_2(J, \mathbf{Y})$. Because $\mathcal{U}_{ad}$ is closed and convex, owing to the Marzur lemma, $\mathbf{u}^0 \in \mathcal{U}_{ad}$. Suppose x^m is the mild solution of system (7) corresponding to $\mathbf{u}^m$ and x^m satisfying the following integral equations

$$\begin{aligned}
x^m(t) &= \phi(t), \quad t \in [-r, 0] \\
x^m(t) &= S_\alpha(t)[\phi(0) - f(0, \phi)] + f(t, x^m_t) + \int_0^t S_\alpha(t-s)[\int_0^s g(\tau, x^m_\tau) dZ_H(\tau)]ds \\
&+ \int_0^t S_\alpha(t-s) \int_Z h(s, x^m_s, \eta)\tilde{N}(ds, d\eta) + \int_0^t S_\alpha(t-s)B(s)\mathbf{u}^m(s)ds \quad t \in J.
\end{aligned} \tag{4.2}$$

Similarly corresponding to u^0, there exists a mild solution x^0 of (7), that is,

$$\begin{aligned} x^0(t) &= \phi(t), \quad t \in [-r, 0] \\ x^0(t) &= S_\alpha(t)[\phi(0) - f(0,\phi)] + f(t, x_t^0) + \int_0^t S_\alpha(t-s)[\int_0^s g(\tau, x_\tau^0) dZ_H(\tau)]ds \\ &+ \int_0^t S_\alpha(t-s) \int_Z h(s, x_s^0, \eta)\tilde{N}(ds, d\eta) + \int_0^t S_\alpha(t-s)B(s)\mathbf{u}^0(s)ds \quad t \in J. \end{aligned} \tag{4.3}$$

Hence, for $t \in J$, by the hypothesis $(\mathbf{A1}) - (\mathbf{A5})$, and the Holder inequality, after an elementary calculation, we have

$$\begin{aligned} &\mathbf{E}|x^m(t) - x^0(t)|^2 \\ &\leq 4M_f \mathbf{E}|x^m(s) - x^0(t)|^2 + 4(Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M \int_0^t \kappa(\mathbf{E}\|x^m(s) - x_s^0\|^2)ds} \\ &+8Mb\Big[\int_0^t \int_Z \mathbf{E}\|h(s, x_s^m, \eta) - h(s, x_s^0, \eta)\|^2 \tilde{\lambda}(d\eta)ds \\ &+ \Big(\int_0^t \int_Z \|h(s, x_s^m, \eta) - h(s, x_s^0), \eta)\|^4 \tilde{\lambda}(d\eta)ds\Big)^{1/2}\Big] \\ &+4\mathbf{E}\|\int_0^s S_\alpha(t-s)B(s)(\mathbf{u}^m(s) - \mathbf{u}^0(s))ds\|^2 \\ &\leq 4M_f \mathbf{E}\left(\sup_{0\leq s\leq t}\|x^m(s) - x^0(s)\|^2\right) \\ &+4((Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M+2bM)\int_0^t \kappa\big(\mathbf{E}\sup_{0\leq s\leq t}\|x_s^m - x_s^0\|^2\big)ds} \\ &+4bM\mathbf{E}\|B\mathbf{u}^m - B\mathbf{u}^0\|^2_{\mathcal{L}_2(J,\mathbf{Y})}. \end{aligned}$$

By lemma 4.2 in [24], B is strongly continuous and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \mathcal{J}_1 &= 4((Tr(Q)C_{\mathrm{H}t_1^{2\mathrm{H}-1}+3b)M+2bM)\int_0^t \kappa\big(\mathbf{E}\sup_{0\leq s\leq t}\|x_s^m - x_s^0\|^2\big)ds \to 0 \text{ as } m\to\infty}, \\ \mathcal{J}_2 &= 4bME\|B\mathbf{u}_B^m\mathbf{u}^0\|^2_{\mathcal{L}_2(J,\mathbf{Y})} \to 0 \text{ as } m \to \infty. \end{aligned}$$

For each $t \in J$, $x^m(.), x^0(.) \in \mathcal{X}$, we have

$$\mathbf{E}\|x^m(t) - x^0(t)\|^2 \leq \frac{\mathcal{J}_1 + \mathcal{J}_2}{1 - 4M_f}, \quad 4M_f < 1.$$

So, let us infer that $x^m \to x^0$ as $m \to \infty$. Finally using Balder's theorem [25] and hypothesis $(\mathbf{A6})$, we obtain

$$
\begin{aligned}
\epsilon &= \lim_{m\to\infty}\int_0^b \mathcal{L}(t, x^m(t), x_t^m, \mathbf{u}^m(t))dt \\
&\geq \int_0^b \mathcal{L}(t, x^0(t), x_t^0, \mathbf{u}^0(t))dt \\
&= \mathcal{J}(\mathbf{u}^0) \geq \epsilon.
\end{aligned}
$$

Hence, the result is followed that $\mathcal{J}$ attains its minimum at $\mathbf{u}^0 \in \mathcal{U}_{ad}$.

5. Example

Consider the following fractional SPDE's with Rosenblatt process and Poisson jumps

$$
\begin{aligned}
D_t^\alpha\left[y(t,x) - \int_{-r}^0 a_3(s)\sin y(t+s,x)ds\right] &= \frac{\partial^2}{\partial x^2}\left[y(t,x) - \int_{-r}^0 a_3(s)\sin y(t+s,x)ds\right] \\
&+\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\bigg[\int_D k_0(x,\gamma)u(\gamma,t)d\gamma + \int_0^t\int_{-r}^t e^{4(s-t)}y(s,x)dsdZ_{\mathrm{H}}(s) \\
&+\int_Z \eta\left(\int_{-r}^t a_4(s-t)y(s,x)ds\right)\tilde{N}(dt,d\eta)\bigg], \\
&x\in\mathcal{D}=[0,\pi],\quad t\in J:=[0,b],\quad \mathbf{u}\in\mathcal{U}_{ad}, \\
&y(t,0) = y(t,\pi) = 0,\quad t\in J, \\
&y(t,x) = \phi(t,x),\quad -r\leq t\leq 0,\quad x\in\mathcal{D},
\end{aligned}
\tag{5.1}
$$

where D_t^α is Caputo fractional derivative of order $0 \leq \alpha \leq 1$ and $\{Z_{\mathrm{H}}(t) : t \in J\}$ is the Rosenblatt process with parameter $\mathrm{H} \in (\frac{1}{2}, 1)$. Let $\mathbf{Y} = \mathcal{X} = \mathcal{L}_2([0,\pi])$ and define the operator $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ by $Ay = y''$ with the domain $\mathcal{D}(A) = \Big\{y \in \mathcal{X}; y, y'$ are absolutely continuous, $y'' \in \mathcal{X}, y(0) = y(\pi) = 0\Big\}$. Then $A_y = \sum_{n=1}^{\infty} n^2(y, y_n)y_n, y \in \mathcal{D}(A)$, where $y_n(x) = \sqrt{\frac{2}{\pi}}sin(nx), n \in \mathbf{N}$ is the orthogonal set of eigenvectors of A. Furthermore, A generates an analytic compact semigroup of bounded linear operator $S(t), t \geq 0$, on a separable Hilbert space $\mathcal{X}$ which is given by

$$
T(t)y = \sum_{n=1}^{\infty}(y_n, e_n)e_n, \quad y \in \mathcal{X}.
$$

The subordination principle of solution operator, implies that A is the infinitesimal generator of a solution operator $(S_\alpha(t)_{t\geq 0})$. Because $S_\alpha(t)$ is strongly continuous on $[0, \infty)$ by a uniformly bounded theorem, there exist a constant $M > 0$ such that $\|S_\alpha(t)\|^2 \leq M$, for $t \in J$. Define the nonlinear functions $f : J \times \mathcal{C} \to \mathcal{X}, g : J \times \mathcal{C} \to \mathcal{L}_2^0$ and $h : J \times \mathcal{C} \times Z \to \mathcal{X}$ by

$$\begin{aligned} f(t,\phi)(x) &= \int_{-r}^{0} a_3(\theta) sin(\phi(\theta)(x)) d\theta, \quad \theta\epsilon[-r,0], \qquad x \in \mathcal{D} \\ g(t,\phi)(x) &= \int_{-r}^{0} e^{-4\theta}\phi(\theta)(x) d\theta \\ h(t,\phi)(x) &= \int_{-r}^{0} a_4(\theta)\phi(\theta)(x) d\theta \end{aligned}$$

and assuming that $\int_Z \eta^2 \tilde{\lambda}(d\eta) < \infty$. Furthermore, the nonlinear functions f, g and h satisfy the hypotheses $(\mathbf{A1})-(\mathbf{A6})$. Let functions $\mathbf{u} : \tau_y(\mathcal{D}) \to \mathbf{R}$, such that $\mathbf{u} \in \mathcal{L}_2(\tau_y(\mathcal{D}))$ as the controls. This claim is that $t \to \mathbf{u}(.,t)$ going from $[0, b]$ into $\mathbf{Y}$ is measurable. Set $\mathcal{U}(t) = \mathbf{u} \in \mathbf{Y}; \|\mathbf{u}\|^2_{\mathbf{Y}} \leq \mu$, where $\mu \in \mathcal{L}_2(\mathcal{J}, \mathbf{R}^+)$. We restrict the admissible controls $\mathcal{U}_{ad}$ to be all the $\mathbf{u} \in \mathcal{L}_2(\tau_y(\mathcal{D}))$ such that $\|\mathbf{u}(.,t)\|^2 \leq \mu(t)$ a.c. Define $B(t)\mathbf{u}(t)x = \int_{\mathcal{D}} k_0(x,\gamma)\mathbf{u}(\gamma,t)d\gamma$ and consider the following cost function

$$\mathcal{J}(\mathbf{u}) = \int_0^b \int_{\mathcal{D}} (\|y(t,x)\|^2_{\mathcal{X}} + \|\mathbf{u}(t,x)\|^2_{\mathcal{X}}) dxdt + \int_0^b \int_{\mathcal{D}} \int_{-r}^{0} \|y(t+s,x)\|^2_{\mathcal{X}} dsdxdt$$

with respect to system (10). Thus, problem (10) can be written as the form of (7). Hence, all the hypotheses stated in theorem 4.2 are satisfied. Hence, there exists an admissible control $\mathbf{u}^0 \in \mathcal{U}_{ad}$ such that $\mathcal{J}(\mathbf{u}^0) \leq \mathcal{J}(\mathbf{u})$, for all $\mathbf{u} \in \mathcal{U}_{ad}$. □

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