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Optimal control of conformable fractional neutral stochastic integrodifferential systems with infinite delay

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ABSTRACT

The main concern of the manuscript deals with the optimal control problem of conformable fractional neutral stochastic integrodifferential systems with infinite delay. This study is motivated by SAR and RAR systems advantage of providing control over many factors such as power, frequency, phase, polarization, incidence angle, spatial resolution and swath width, all of which are important when designing and operating a radar system. Initially, we investigate the existence of mild solutions for the conformable fractional stochastic integrodifferential equations with infinite delay using stochastic analysis techniques and Banach fixed point theorem. In the later part we establish the existence of mild solutions of the conformable fractional neutral stochastic integrodifferential system with infinite time delay. Furthermore, the existence of optimal control of the corresponding Lagrange optimal control problem is investigated. An example is provided to illustrate the applications of the obtained results. We explain the limitation we have with the existence software, and developed a numerical scheme to justify the theory.

1. Introduction

Numerous real-world phenomena, such as stock prices, heat conduction in materials with memory, rising population, and so on, have fluctuated due to random influences or noise, requiring the accessibility of randomness in mathematical descriptions of these phenomena. Stochastic differential equations (SDEs) are differential equations that involve randomness. SDEs are used in a variety of domains including economics, finance and engineering due to its abstract formulation of many problems. For the study of SDEs and SDEs in the fractional sense, the reader may refer to books [1–5] and articles [6–9]. Recently, much attention has been paid to the qualitative properties of mild solutions to various stochastic integrodifferential equations and the fixed point technique, see [10–12] and the references therein.

Many fields of science, including engineering, mathematics, and biomedicine, use optimal control problems. An optimal control problem becomes a stochastic optimal control problem when the performance index and system dynamics are described by stochastic differential equations. Recently, Sathiyaraj et al. [13] demonstrated controllability and optimal control for a class of time-delayed fractional stochastic integrodifferential systems with Poisson jumps. Several studies have recently been conducted to determine the

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existence of optimal control conditions for a variety of dynamic systems. In the case of fractional optimal control problems, however, there are only a few works in the literature. The reader may refer to [14–18].

Using the fractional variational principle and the Lagrange multiplier technique, Agrawal [15] presented general information for fractional optimal control problems where fractional derivatives were considered in the Riemann–Liouville sense but not the conformable sense. Liu et al. [19] investigated the solvability and optimal controls for a few fractional evolution equations with impulse effects using fractional calculus, the Gronwall inequality, and the Leray Schauder fixed point theorem. Using resolvent operators, Tamilagan et al. [20] investigated the solvability and optimal control for fractional stochastic differential equations driven by Poisson jumps. A fractional neutral stochastic differential system with a Caputo fractional derivative using successive approximation was taken into consideration by Ramkumar et al. [21]. For papers describing the solvability and optimal control for the fractional SDEs, see [22–24].

FDEs are significant and helpful, which makes them and the approaches used to solve them popular. Numerical approaches are also essential for solving FDEs in practice because analytical solutions, such as those utilizing matrix Mittag-Leffler functions or Laplace transforms, are sometimes impractical or unfeasible for increasingly complex FDEs. Numerous such techniques, including the q-homotopy analysis transform method, the B-spline collocation method, the predictor–corrector method, the space-spectral time-fractional Adams–Bashforth–Moulton technique, the fractional Taylor operational matrix technique, the Bernoulli polynomials technique, the differential transform technique, and others, have been established in the literature. Here, we justify the idea using a numerical scheme. The proposed method's very effective outcomes and applicability can make significant contributions to the field of numerical methods.

Many researchers used an integral form for fractional derivative definition. The most popular definitions of fractional derivatives are Riemann–Liouville and Caputo definitions. Lately, Khalil at al.give a new definition of fractional derivative and fractional integral [25]. This new definition benefit from a limit form as in usual derivatives. This new theory is improved by Abdeljawad [26]. Recently, Chalishajar et al. [[27], RICO] discussed the optimal controllability of stochastic systems with deviated argument in infinite time horizon using new definition of a phase space. This work is the extension of authors own work using conformal mapping with numerical simulation. This theoretical and numerical approach makes this work unique. For more articles regarding conformable fractional derivative, one may refer [28,29].

Researchers are increasingly interested in numerical modeling of optimal control of nonlinear time-delay fractional differential equations. Optimal management of nonlinear time-delay fractional differential equations was achieved by Chen et al. [30] using the Dickson polynomial. They approximated the system's states and control variables using a set of Dickson polynomials as basis functions and then utilized a collocation method to translate the issue into a system of nonlinear algebraic equations. Chu et al. [31] presented a dynamical model of SARS-CoV-2 in fractional derivative utilizing fourth-wave coronavirus cases. They fit the data to both the fractional and piecewise stochastic differential equations and show that they both match the data well. Sher et al. [32] investigated the existence and uniqueness of solutions for a class of evolution fractional order differential equations (FODEs) with proportional delay utilizing Caputo derivative under local conditions using topological degree theory (TDT). In addition, Chu et al. [33] employ a combination of Shehu decomposition and variational iteration transform methods to solve fractional third-order dispersive partial differential equations. The graphs and table depict the behavior of the solution for various fractional order values. In [34], Ahmad et al. used an effective local meshless method to discretize the numerical treatment of three-term temporal fractional-order multi-dimensional diffusion equations. The proposed meshless approach based on the multiquadric radial basis function through the time-fractional component discretizes the space derivative of the models. Hajiseyedazizi et al. [35] just published a paper on multi-step approaches for singular fractional q-integro-differential equations with some boundary conditions Ω, which singular at some point $0 \leq t \leq 1$. Chen et al. [36] presented a unique approach to solving optimal control problems using fractional differential equations and time delay. To be more specific, a set of global radial basis functions is utilized to approximate the problem's states and control variables.

1.1. A deterministic and stochastic method to conformal array fabrication for SAR applications

Space-borne Synthetic Aperture Radars (SARs) often use conformal phased array antennas with a large number of elements, which are required to generate multiple shaped beams, including nulls and side-lobe regions with intricate geometries. Only the pattern amplitudes are of importance in such applications, necessitating the learning of the optimal control on the cost basis.

SARs have been employed in both military and civilian settings. They have been shown to be particularly useful in applications such as sea and ice or oil pollution monitoring, oceanography, snow monitoring, earth terrain categorization, and so on. Because of the vast range of applications and benefits of Real Aperture Radars (RARs), a large number of airborne and spaceborne SAR systems have been developed over the last few decades.

The main advantage they offer is the ability to achieve high azimuth resolution. In fact, to improve RAR system resolution, one should either utilize a very long antenna or reduce the signal wavelength. However, in spaceborne and airborne applications, the antenna size and weight are constrained by the platform's structure. Furthermore, short wavelengths are subject to significant attenuation in the environment. SARs, on the other hand, employ the forward motion of the actual antenna on the vehicle to "synthesize" an extremely long antenna, allowing a suitable antenna size to be maintained. Thus the study of optimal control is helpful to day and night monitoring and wide area coverage (even in unfavorable weather conditions).

The antenna in imaging radar systems moves at a constant speed along the track direction. The ability of the system to distinguish between two targets on the ground is characterized as ground resolution. A long antenna in RAR systems generates the beam that illuminates the ground below. The radar along track resolution, also known as the azimuth resolution, is the shortest distance on the ground parallel to the flight path at which two targets can be viewed individually. The azimuth resolution is defined as $\rho_a = R\lambda/l$, where R is the slant range from the antenna to the mean point of the swath, λ is the wavelength, and l is the physical aperture length. The pulse length, τ_p, on the other hand, determines the across-track resolution, or range resolution. It is calculated as $\rho_g = c\tau_p/(2sin\theta)$, where c is the speed of light and θ is the look angle. To improve azimuth resolution, either a longer antenna or a shorter wavelength must be employed in conjunction with optimal conformable system control.

SAR processing results in a high resolution image. In SAR systems, the azimuth resolution is $\rho_a \geq l/2$. Furthermore, SAR systems provide control over numerous aspects such as power, frequency, phase, polarization, incidence angle, spatial resolution, and swath width, all of which are significant for developing and running a system for quantitative information extraction. But, when antenna arrays of many elements are involved, using simple and light feeding networks is highly important to the aim of reducing the cost and the payload of a satellite. Thus, it is important to reduce the dynamic range ratio (DRR) of the excitation, defined as the ratio between the maximum and the minimum amplitude of the excitation using optimal control of the conformal stochastic system.

Motivated by the above facts, let us take into account the Conformable fractional stochastic integrodifferential system with infinite delay of the form:

$$\begin{aligned} \mathcal{D}^{\alpha}\mathfrak{x}(\mathfrak{t}) &= \mathfrak{A}\mathfrak{x}(\mathfrak{t}) + \mathbb{B}(\mathfrak{t})\mathfrak{z}(\mathfrak{t}) + \mathfrak{f}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}, \int_0^{\mathfrak{t}} \mathfrak{g}\left(\mathfrak{t}, s, \mathfrak{x}_s\right) ds\right) + \mathfrak{h}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}, \int_0^{\mathfrak{t}} \tilde{\mathfrak{g}}\left(\mathfrak{t}, s, \mathfrak{x}_s\right) ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}, \\ \mathfrak{x}(\mathfrak{t}) &= \zeta(\mathfrak{t}), \quad \in \mathcal{L}^2(\Omega, \mathfrak{G}_{\mathcal{J}}), \mathfrak{t} \in (-\infty, 0], \end{aligned} \tag{1.1}$$

- $\mathcal{D}^{\alpha}$ is the conformable fractional derivative w.r.t. $\mathfrak{t} \in \mathcal{E}' = (0, \mathfrak{v}]$ and $0 < \alpha < 1$.
- The infinitesimal generator of $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a strongly continuous semigroup $\{\mathfrak{S}(\mathfrak{t})\}\mathfrak{t} \geq 0$ on a Hilbert space $\mathcal{Y}$ with $\langle .,. \rangle\mathcal{Y}$ and norm $\|.\|\mathcal{Y}$.
- Allowing $\mathcal{E} = [0, \mathfrak{v}]$. Then, values are received by the control function $\mathfrak{z}$ from the reflexive Hilbert space, $\mathcal{K}$.
- The appropriate functions are $\mathfrak{f} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{Y}$, $\mathfrak{h} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{L}_2^0$ and $\mathfrak{g}, \tilde{\mathfrak{g}} : \mathcal{E} \times \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \to \mathcal{Y}$.
- Let $\mathcal{Z}$ be a different real separable Hilbert space with $\langle .,. \rangle_{\mathcal{Z}}$ and $\|.\|_{\mathcal{Z}}$ as its norm.
- We may assume that $\{\omega(\mathfrak{t}), \mathfrak{t} \geq 0\}$ is a $\mathcal{Z}-$ valued Brownian motion with finite-trace nuclear covariance operator $\mathcal{Q} \geq 0$
- The abstract phase space $\mathfrak{G}_{\mathcal{J}}$ has the element $\mathfrak{x}_{\mathfrak{t}} : (-\infty, 0] \to \mathcal{Y}$, which is described by $\mathfrak{x}_{\mathfrak{t}}(s) = \mathfrak{x}(\mathfrak{t} + s)$.
- The initial condition $\zeta = \{\zeta(\mathfrak{t}) : \mathfrak{t} \in (-\infty, 0]\}$ is independent of the Wiener process $\{\omega(\mathfrak{t})\}$ with the finite second moment which is $\mathfrak{G}_{\mathcal{J}}-$ valued random variable and $\mathfrak{I}_0-$measurable.

The outline of this paper is as follows: Section 2 is devoted to the notions and preliminaries required to solve the aforementioned system (1.1). In Section 3, for the modeled system, the existence results of mild solutions are investigated. In Section 4, a Conformable fractional stochastic neutral integrodifferential system is framed and the existence results of mild solutions are presented. In Section 5, the optimal results of the system (4.1) are investigated. To validate the obtained results, an example is provided in Section 6 depicts the cost functional considered attains its minimum. In Section 7 numerical simulation is presented to show the application of the equation. At last Section 8 Conclusion is presented.

The primary contribution along with the innovation of the manuscript are listed as follows:

- A conformable fractional stochastic integrodifferential system with infinite delay and the neutral condition is modeled and the existence result is established which is not discussed so far
- The optimal controllability with the numerical simulation of the conformable fractional stochastic integrodifferential system with infinite delay is untreated in the literature.
- Theoretical proofs are justified by numerical simulation uniquely.

2. Preliminaries

Assume that $(\Omega, \mathfrak{I}, \mathcal{P})$ is a complete probability space, that $\mathfrak{I}_{\mathfrak{t}}, \mathfrak{t} \in \mathcal{E}$ represents the normal filtration as being right continuous, and that $\{\mathfrak{I}_0\}$ contains $\mathcal{P}-$null sets. With the covariance operator $\mathcal{Q}$, ω is a $\mathcal{Q}$-Wiener process on $(\Omega, \mathfrak{I}_{\mathfrak{v}}, \mathcal{P}) \ni Tr(\mathcal{Q}) < \infty$. Consider the sequence of bounded non-negative real number $\{h_k\}_{k\geq 0}$ and the complete orthonormal basis $\{\xi_k\}_{k\geq 1}$ in $\mathcal{Z} \ni \mathcal{Q}\xi_k = h_k\xi_k$. The sequence of independent Brownian motion $\{\mathfrak{w}_k\}_{k\geq 1}$ follows $\langle \omega(\mathfrak{t}), \xi \rangle_{\mathcal{Z}} = \sum_{k=1}^{\infty} \sqrt{h_k}\langle \xi_k, \xi \rangle \mathfrak{w}_k(\mathfrak{t})$, $\xi \in \mathcal{Z}$, $\mathfrak{t} \in \mathcal{E}$. Take into account $\mathcal{L}_2^0 = \mathcal{L}_2\left(\mathcal{Q}^{1/2}\mathcal{Z}; \mathcal{Y}\right)$ as the space containing all Hilbert–Schmidt operators from $\mathcal{Q}^{1/2}\mathcal{Z}$ to $\mathcal{Y}$ with $\|\phi\|_{\mathcal{Q}}^2 = Tr(\phi\mathcal{Q}\phi^*)$, where ϕ^* is the adjoint operator of ϕ. The expression $\mathcal{L}_2\left(\Omega, \mathfrak{I}, \mathcal{P}; \mathcal{Y}\right) \equiv \mathcal{L}_2(\Omega; \mathcal{Y})$ is used to describe the set of all strongly measurable square integrable $\mathcal{Y}-$ valued random operators, is a Banach space equipped with the norm $\|\mathfrak{x}(.)\|_{\mathcal{L}_2} = \left(\mathbb{E}\left\|\mathfrak{x}(., \nu_0)\right\|_{\mathcal{Y}}^2\right)^{1/2}$, where $\mathbb{E}\|h_0\| = \int_{\Omega} h_0(\nu_0) d\mathcal{P}$ defines the expectation $\mathbb{E}$. Let $C\left(\mathcal{E}, \mathcal{L}_2(\Omega; \mathcal{Y})\right)$ be the Banach space consisting of all continuous functions $\mathcal{E}$ into $\mathcal{L}_2(\Omega; \mathcal{Y})$ satisfying $\sup_{\mathfrak{t}\in\mathcal{E}} \mathbb{E}\|\mathfrak{x}(\mathfrak{t})\|^2 < \infty$.

Definition 2.1 ([25]). For a function $\mathfrak{p}(.)$ with $\mathfrak{t} > 0$, the conformable fractional derivative of order ν is defined as follows.

$$\frac{d^{\nu}\mathfrak{p}(\mathfrak{t})}{d\mathfrak{t}^{\nu}} = \lim_{\upsilon\to 0} \frac{\mathfrak{p}(\mathfrak{t} + \upsilon\mathfrak{t}^{1-\nu}) - \mathfrak{p}(\mathfrak{t})}{\upsilon}, \quad 0 < \nu < 1.$$

For the specific condition $\mathfrak{t} = 0$, the following definition is derived:

$$\frac{d^{\nu}\mathfrak{p}(0)}{d\mathfrak{t}^{\nu}} = \lim_{\mathfrak{t}\to 0^+} \frac{d^{\nu}\mathfrak{p}(\mathfrak{t})}{d\mathfrak{t}^{\nu}}.$$

The conformable fractional derivative of order ν of a function $\mathfrak{s}(.)$ is related with a fractional integral $\mathcal{I}^{\nu}(.)$ defined by

$$\mathcal{I}^{\nu}(\mathfrak{p})(\mathfrak{t}) = \int_0^{\mathfrak{t}} \mathfrak{s}^{\nu-1}\mathfrak{p}(\mathfrak{s})d\mathfrak{s}.$$

Remark 2.1. By considering a simple differential equation $\mathfrak{x}^{1/2} + \mathfrak{x} = 0$, one can solve using either Caputo or Riemann–Liouville definition, then one can use Laplace transform or fractional power series technique. Nevertheless using Conformable fractional derivative and the fact $\mathcal{T}_{\alpha}\left(e^{\frac{1}{\alpha}}\mathfrak{t}^{\alpha}\right) = e^{\frac{1}{\alpha}}\mathfrak{t}^{\alpha}$, we can easily see that $\mathfrak{x} = ce^{-2\sqrt{\mathfrak{t}}}$.

Following is the abstract phase space $\mathfrak{G}_{\mathcal{J}}$ [37]:

$$\mathfrak{G}_{\mathcal{J}} = \begin{cases} \zeta : (-\infty, 0] \to \mathcal{Y}, \forall\ c > 0,\ (\mathbb{E}\|\zeta(\eta)\|^2)^{1/2} \text{ is a bounded and measurable function on } [-c, 0] \\ \text{with } \int_{-\infty}^{0} \mathcal{J}(s) \sup\limits_{s\le\eta\le 0} \left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2} ds < +\infty, \end{cases}$$

where $\mathcal{J} : (-\infty, 0] \to (0, +\infty)$ be continuous with $\mathfrak{l} = \int_{-\infty}^{0} \mathcal{J}(\mathfrak{t})d\mathfrak{t} < +\infty$ and

$$\|\zeta\|_{\mathfrak{G}_{\mathcal{J}}} = \int_{-\infty}^{0} \mathcal{J}(s) \sup_{s\le\eta\le 0} \left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2} ds, \quad \forall\ \ \zeta \in \mathfrak{G}_{\mathcal{J}}.$$

Clearly $\left(\mathfrak{G}_{\mathcal{J}}, \|.\|_{\mathfrak{G}_{\mathcal{J}}}\right)$ is a Banach space.

The $C((-\infty, \nu], \mathcal{Y})$ be the space of all continuous $\mathcal{Y}-$ valued stochastic processes $\{\xi(\mathfrak{t}) : \mathfrak{t} \in (-\infty, \nu]\}$. Also, $\mathfrak{G}'_{\mathcal{J}} = \{\mathfrak{x} : \mathfrak{x} \in C(-\infty, \nu], \mathcal{Y})\}$ endowed with the seminorm $\mathfrak{G}'_{\mathcal{J}}$ defined as

$$\|\mathfrak{x}\|_{\mathfrak{v}} = \|\zeta\|_{\mathfrak{G}_{\mathcal{J}}} + \sup_{s\in[0,\mathfrak{v}]} \left(\mathbb{E}\|\mathfrak{x}(s)\|^2\right)^{1/2},\ \mathfrak{x} \in \mathfrak{G}'_{\mathcal{J}}.$$

Lemma 2.1 (*[37]*). *If $\mathfrak{x}_0 = \zeta \in \mathfrak{G}_{\mathcal{J}}, \mathfrak{x} \in \mathfrak{G}'_{\mathcal{J}}$, then for $\mathfrak{t} \in \mathcal{E}$, $\mathfrak{x}_{\mathfrak{t}} \in \mathfrak{G}_{\mathcal{J}}$. Moreover,*

$$\mathfrak{l}\left(\mathbb{E}\|\mathfrak{x}(\mathfrak{t})\|^2\right)^{1/2} \le \|\mathfrak{x}_{\mathfrak{t}}\|_{\mathfrak{G}_{\mathcal{J}}} \le \|\mathfrak{x}_0\|_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l} \sup_{s\in[0,\mathfrak{t}]} \left(\mathbb{E}\|\mathfrak{x}(s)\|^2\right)^{1/2},$$

where $\mathfrak{l} = \int_{-\infty}^{0} \mathcal{J}(s)ds < +\infty$.

Definition 2.2. If the mild solution $\mathfrak{x}(.)$ satisfies $\delta(\mathfrak{t}) = \mathfrak{x}(\mathfrak{t})$ a.e., then the system (1.1) is trajectory controllable on $[\mathfrak{t}_0, \mathrm{T}]$.

Lemma 2.2 (*Generalized Gronwall's inequality [3,37]*).*: If $\beta > 0$, $\tilde{\mathfrak{a}}(\mathfrak{t})$ is a non-negative function locally integrable on $0 \le \mathfrak{t} < \mathrm{T}$, for some $(\mathrm{T} < +\infty)$, and $\mathfrak{g}(\mathfrak{t})$ is a non-negative, non-decreasing continuous function on $0 \le \mathfrak{t} \le \mathrm{T}$, $\mathfrak{g}(\mathfrak{t}) \le c$ being a constant and suppose $\tilde{\mathfrak{u}}(\mathfrak{t})$ is non-negative and locally integrable on $0 \le \mathfrak{t} < \mathrm{T}$ with $\tilde{\mathfrak{u}}(\mathfrak{t}) \le \tilde{\mathfrak{a}}(\mathfrak{t}) + \mathfrak{g}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta-1}\tilde{\mathfrak{u}}(s)ds$, on the interval. Then*

$$\tilde{\mathfrak{u}}(\mathfrak{t}) \le \tilde{\mathfrak{a}}(\mathfrak{t}) + \int_0^{\mathfrak{t}} \sum_{n=1}^{\infty} \frac{(\mathfrak{g}(\mathfrak{t})\Gamma(\beta))^n}{\Gamma(n\beta)} (\mathfrak{t} - s)^{\beta-1}\tilde{\mathfrak{a}}(s)ds, \quad 0 \le \mathfrak{t} < \mathrm{T}.$$

In particular, when $\tilde{\mathfrak{a}}(\mathfrak{t}) = 0$, then $\tilde{\mathfrak{u}}(\mathfrak{t}) = 0$ for all $0 \le \mathfrak{t} < \mathrm{T}$.

Definition 2.3 (*[38]*). A mild solution of (1.1) is $\mathfrak{F}_{\mathfrak{t}}-$adapted stochastic process $\mathfrak{x} : (-\infty, \mathfrak{v}] \to \mathcal{Y}$ with $\zeta \in \mathcal{L}^2(\Omega, \mathfrak{G}_{\mathcal{J}})$ on $(-\infty, 0]$, $\mathfrak{x}_0 \in \mathcal{L}_2^0(\Omega, \mathcal{Y})$ and the integral equation below is satisfied:

$$\begin{aligned} \mathfrak{x}(\mathfrak{t}) \ =& \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha} - \frac{s^{\alpha}}{\alpha}\right)\mathfrak{z}(s)ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha} - \frac{s^{\alpha}}{\alpha}\right)\mathfrak{f}\left(s, \mathfrak{x}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{x}_{\rho}\right)d\rho\right)ds \\ &+ \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha} - \frac{s^{\alpha}}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{x}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{x}_{\rho}\right)d\rho\right)d\omega(s) + \mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}\right)\zeta(0). \end{aligned} \tag{2.1}$$

3. Main results

The following are the assumptions to discuss the existence and uniqueness of mild solution as well as the optimal control of the evolution equation:

(A1) The linear operator $\mathfrak{A} : \mathcal{Y} \to \mathcal{Y}$ in (1.1) generates C_0-semigroup $\mathcal{T}(.)$. Thus there exists $\mathcal{M} > 0$ being constant such that $\|\mathcal{T}(\mathfrak{t})\| \le \mathcal{M}\ \forall\ \mathfrak{t} \in \mathcal{E}'$.

(A2) For $\mathfrak{t} \in \mathcal{E}$, the function $\mathfrak{f} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{Y}$ is continuous. $\mathfrak{u}_1, \tilde{\mathfrak{u}}_1 \in \mathfrak{G}_{\mathcal{J}}$, $\mathfrak{u}_2, \tilde{\mathfrak{u}}_2 \in \mathcal{Y}$ and $\exists$ positive constants $N_{\mathfrak{f}}$, $\hat{N}_{\mathfrak{f}}$

$$\begin{aligned} \mathbb{E}\left\|\mathfrak{f}\left(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\right) - \mathfrak{f}\left(\mathfrak{t}, \tilde{\mathfrak{u}}_1, \tilde{\mathfrak{u}}_2\right)\right\|^2 \ &\le\ N_{\mathfrak{f}}\left(\left\|\mathfrak{u}_1 - \tilde{\mathfrak{u}}_1\right\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\left\|\mathfrak{u}_2 - \tilde{\mathfrak{u}}_2\right\|^2\right), \\ \mathbb{E}\left\|\mathfrak{f}\left(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\right)\right\|^2 \ &\le\ \hat{N}_{\mathfrak{f}}\left(1 + \|\mathfrak{u}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2\|^2\right). \end{aligned}$$

(A3) For $\mathfrak{t} \in \mathcal{E}$, the function $\mathfrak{h} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{L}_2^0$ is continuous. $\mathfrak{u}_1, \tilde{\mathfrak{u}}_1 \in \mathfrak{G}_{\mathcal{J}}$, $\mathfrak{u}_2, \tilde{\mathfrak{u}}_2 \in \mathcal{Y}$ and $\exists$ positive constants $N_{\mathfrak{h}}$, $\hat{N}_{\mathfrak{h}}$

$$\mathbb{E}\left\|\mathfrak{h}\left(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\right) - \mathfrak{h}\left(\mathfrak{t}, \tilde{\mathfrak{u}}_1, \tilde{\mathfrak{u}}_2\right)\right\|^2 \le N_{\mathfrak{h}}\left(\|\mathfrak{u}_1 - \tilde{\mathfrak{u}}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2 - \tilde{\mathfrak{u}}_2\|^2\right),$$
$$\mathbb{E}\left\|\mathfrak{h}\left(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\right)\right\|^2 \le \hat{N}_{\mathfrak{h}}\left(1 + \|\mathfrak{u}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2\|^2\right).$$

(A4) For each $(\mathfrak{t}, s) \in \mathcal{E}^2$, the functions $\mathfrak{g}, \tilde{\mathfrak{g}} : \mathcal{E}^2 \times \mathfrak{G}_{\mathcal{J}} \to \mathcal{Y}$ are continuous. For all $\mathfrak{u}, \tilde{\mathfrak{u}} \in \mathfrak{G}_{\mathcal{J}}$, $\exists$ positive constants $\mathfrak{m}_1, \mathfrak{m}_2, \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$

$$\mathbb{E}\|\mathfrak{g}(\mathfrak{t}, s, \mathfrak{u}) - \mathfrak{g}(\mathfrak{t}, s, \tilde{\mathfrak{u}})\|^2 \le \mathfrak{m}_1\|\mathfrak{u} - \tilde{\mathfrak{u}}\|^2_{\mathfrak{G}_{\mathcal{J}}},$$
$$\mathbb{E}\|\tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u}) - \tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le \mathfrak{m}_2\|\mathfrak{u} - \tilde{\mathfrak{u}}\|^2_{\mathfrak{G}_{\mathcal{J}}},$$
$$\mathbb{E}\|\mathfrak{g}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le \tilde{\mathfrak{m}}_1\left(1 + \|\mathfrak{u}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right),$$
$$\mathbb{E}\|\tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le \tilde{\mathfrak{m}}_2\left(1 + \|\mathfrak{u}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right).$$

(A5) Operator $\mathbb{B}(.)$ defined on $\mathcal{L}_\infty(\mathcal{E}, \mathcal{L}(\mathcal{K}, \mathcal{Y}))$, endowed with the sup norm $\|\mathbb{B}\|^2_\infty$.

(A6) Let $M(\mathcal{K})$ be a class of nonempty, closed, convex subsets of $\mathcal{K}$. The multivalued maps $\mathfrak{A} : \mathcal{E} \to M(\mathcal{K})$ are measurable and $\mathfrak{A}(.)$ are contained in Θ, Θ being bounded subset of $\mathcal{K}$.

Admissible set $\mathfrak{A}_{ad} := \{\nu(.) : \mathcal{E} \times \Omega \to \mathcal{Y}\} \ni \nu$ is a $\mathfrak{I}$−adapted stochastic process and $\mathbb{E}\int_0^{\mathfrak{v}} \|\nu(\mathfrak{z})\|^2 d\mathfrak{z} < \infty$. Obviously, $\mathfrak{A}_{ad} \neq \emptyset$ and for $(1 < q < +\infty)$, $\mathfrak{A}_{ad} \subset \mathcal{L}^q(\mathcal{E}, \mathcal{K})$ is convex, bounded and closed. This implies that $\mathbb{B}\mathfrak{z} \in \mathcal{L}^q(\mathcal{E}, \mathcal{Y})$ for $\mathfrak{z} \in \mathfrak{A}_{ad}$.

Theorem 3.1. *With the assumptions (A1)–(A5), a unique mild solution (1.1) exists if*

$$3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\left(1 + \tilde{\mathfrak{m}}_1\mathfrak{v}^2\right) + 3\mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\left(1 + \tilde{\mathfrak{m}}_2\mathfrak{v}^2\right) < 1.$$

Proof. Let the operator $\psi : \mathfrak{G}'_{\mathcal{J}} \to \mathfrak{G}'_{\mathcal{J}}$ be

$$\psi\mathfrak{x}(\mathfrak{t}) = \begin{cases} \zeta(\mathfrak{t}), & \mathfrak{t} \in (-\infty, 0] \\ \displaystyle\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s, \mathfrak{x}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{x}_\rho\right)d\rho\right)ds & \\ \displaystyle+ \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{x}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{x}_\rho\right)d\rho\right)d\omega(s), & \mathfrak{t} \in \mathcal{E}. \end{cases}$$

For $\zeta \in \mathfrak{G}_{\mathcal{J}}$, we may define $\overline{\zeta}$ as

$$\overline{\zeta}(\mathfrak{t}) = \begin{cases} \zeta(\mathfrak{t}), & \mathfrak{t} \in (-\infty, 0] \\ \zeta(0), & \mathfrak{t} \in \mathcal{E}. \end{cases}$$

then $\overline{\zeta} \in \mathfrak{G}'_{\mathcal{J}}$. Let $\mathfrak{x}(\mathfrak{t}) = \mathfrak{w}(\mathfrak{t}) + \overline{\zeta}(\mathfrak{t})$, $-\infty < \mathfrak{t} \le \mathfrak{v}$. Clearly $\mathfrak{x}$ is satisfied if and only if $\mathfrak{w}$ fulfills $\mathfrak{w}_0 = 0$.

$$\begin{aligned} \mathfrak{w}(\mathfrak{t}) = & \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)ds \\ & + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)d\omega(s). \end{aligned}$$

Let $\mathfrak{G}''_{\mathcal{J}} = \{\mathfrak{w} \in \mathfrak{G}'_{\mathcal{J}}; \mathfrak{w}_0 = 0 \in \mathfrak{G}_{\mathcal{J}}\}$. For any $\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}}$.

$$\|\mathfrak{w}\|_{\mathfrak{v}} = \|\mathfrak{w}_0\|_{\mathfrak{G}_{\mathcal{J}}} + \sup_{0\le s\le \mathfrak{v}}\left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right)^{1/2} = \sup_{0\le s\le \mathfrak{v}}\left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right)^{1/2}.$$

This demonstrates that $\left(\mathfrak{G}''_{\mathcal{J}}, \|.\|_{\mathfrak{v}}\right)$ is a Banach space. We assume $\mathcal{B}_{\mathfrak{p}} = \{\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}} : \|\mathfrak{w}\|^2_{\mathfrak{v}} \le \mathfrak{p}\}$; for some $\mathfrak{p} > 0$. then, $\mathcal{B}_{\mathfrak{p}} \subseteq \mathfrak{G}''_{\mathcal{J}}$ is uniformly bounded, $\mathfrak{w} \in \mathcal{B}_{\mathfrak{p}}$, $\forall$ $\mathfrak{p}$. By virtue of Lemma 2.1,

$$\begin{aligned} \|\mathfrak{w}_{\mathfrak{t}} + \overline{\zeta}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}} &\le 2\left(\|\mathfrak{w}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}} + \|\overline{\zeta}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right) \\ &\le 4\left(\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\sup_{s\in[0,\mathfrak{t}]}\left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right) + \|\overline{\zeta}_0\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\sup_{s\in[0,\mathfrak{t}]}\left(\mathbb{E}\|\overline{\zeta}(s)\|^2\right)\right) \\ &\le 4\left(0 + \mathfrak{l}^2\|\mathfrak{w}\|^2_{\mathfrak{v}} + \|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\right) \\ &\le 4\left(\mathfrak{l}^2\mathfrak{p} + \|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\right) = \mathfrak{p}'. \end{aligned} \tag{3.1}$$

Now we may define $\tilde{\Psi}:\mathfrak{S}''_{\mathcal{J}}\to\mathfrak{S}''_{\mathcal{J}}$ as

$$\tilde{\Psi}\mathfrak{w}(\mathfrak{t})=\begin{cases}0, & \mathfrak{t}\in(-\infty,0]\\ \int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds & \\ +\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\big)d\rho\big)ds & \\ +\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)d\omega(s), & \mathfrak{t}\in\mathcal{E},\end{cases}\tag{3.2}$$

implies that $\tilde{\Psi}$ has a unique fixed point. Let us divide the proof into several steps.

Step 1: To claim $\exists\ \mathfrak{p}>0\ni\tilde{\Psi}(\mathcal{B}_\mathfrak{p})\subset\mathcal{B}_\mathfrak{p}$.

On the contrary let us assume that for $\mathfrak{p}>0\ \exists\ \mathfrak{w}^p(.)\in\mathcal{B}_\mathfrak{p}$ and $\tilde{\Psi}(\mathfrak{w}^p)\notin\mathcal{B}_\mathfrak{p}$, (i.e), $\mathbb{E}\|(\tilde{\Psi}\mathfrak{w}^\mathfrak{p})(\mathfrak{t})\|^2>\mathfrak{p}$ for some $\mathfrak{t}\in\mathcal{E}$. From Lemma 2.2 and the assumptions (A1)–(A6), yield

$$\begin{aligned}\mathfrak{p}&\le\mathbb{E}\|\left(\tilde{\Psi}\mathfrak{w}^\mathfrak{p}\right)(\mathfrak{t})\|^2\\&\le 3\left[\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2\right.\\&\quad+\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\big(s,\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\big(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\big)d\rho\big)ds\right\|^2\\&\quad+\left.\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\right)d\rho\right)d\omega(s)\right]\right\|^2\\&\le 3[\mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_3],\end{aligned}$$

where

$$\begin{aligned}\mathfrak{S}_1&=\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2\\&\le\mathcal{M}^2\int_0^{\mathfrak{t}}\mathfrak{s}^{2(\alpha-1)}\mathbb{E}\|\mathfrak{z}(s)\|^2ds\\&\le\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^\mathfrak{q}(\mathcal{E},\mathcal{K})},\\ \mathfrak{S}_2&\le\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\big(s,\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\big(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\big)d\rho\big)ds\right\|^2\\&\le\mathcal{M}^2\int_0^{\mathfrak{t}}\mathfrak{s}^{2(\alpha)-1}\hat{N}_\mathfrak{f}\left(1+\|\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s\|^2+\mathbb{E}\left\|\int_0^s\mathfrak{g}\left(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\right)d\rho\right\|^2\right)ds\\&\le\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}')\right),\end{aligned}$$

$$\begin{aligned}\mathfrak{S}_3&\le\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\big(s,\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\big(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\big)d\rho\big)d\omega(s)\right\|^2\\&\le\mathcal{M}^2Tr(\mathcal{Q})\int_0^{\mathfrak{t}}\mathfrak{s}^{2(\alpha)-1}\hat{N}_\mathfrak{h}\left(1+\|\mathfrak{w}^\mathfrak{p}_s+\overline{\zeta}_s\|^2+\mathbb{E}\left\|\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}^\mathfrak{p}_\rho+\overline{\zeta}_\rho\right)d\rho\right\|^2\right)ds\\&\le\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{h}\left(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}')\right).\end{aligned}$$

Thus combining the above form we obtain

$$\begin{aligned}\mathfrak{p}&\le 3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^\mathfrak{q}(\mathcal{E},\mathcal{K})}+3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}')\right)\\&\quad+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{h}\left(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}')\right).\end{aligned}$$

Dividing throughout by $\mathfrak{p}$ and by taking $\mathfrak{p}\to\infty$,

$$1\le 3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\tilde{\mathfrak{m}}_1\mathfrak{v}^2\right)+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{h}\left(1+\tilde{\mathfrak{m}}_2\mathfrak{v}^2\right);$$

which goes against what we assumed. therefore, for some $\mathfrak{p}>0$, $\tilde{\Psi}(\mathcal{B}_\mathfrak{p})\subset\mathcal{B}_\mathfrak{p}$.

Step 2: To claim $\tilde{\Psi}$ is a contraction on $\mathcal{B}_\mathfrak{p}$.

Let us consider $\mathfrak{w},\hat{\mathfrak{w}}\in\mathcal{B}_\mathfrak{p}$, then

$$
\begin{aligned}
&\mathbb{E}\left\|\tilde{\Psi}\mathfrak{w}(\mathfrak{t})-\tilde{\Psi}\hat{\mathfrak{w}}(\mathfrak{t})\right\|^{2}\\
&\leq\ 2\Bigg[\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha}\right)\Big[\mathfrak{f}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\\
&\quad-\mathfrak{f}\left(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\left(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\Big]ds\Bigg\|^2+\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha}\right)\\
&\quad\times\left[\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)-\mathfrak{h}\left(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\right]d\omega(s)\Bigg\|^2\Bigg]\\
&\leq\ 2\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{f}}\left(\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{m}_1\mathfrak{v}^2\|\mathfrak{w}_\rho-\hat{\mathfrak{w}}_\rho\|^2\right)+2\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\big(\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\
&\quad+\ \mathfrak{m}_2\mathfrak{v}^2\|\mathfrak{w}_\rho-\hat{\mathfrak{w}}_\rho\|^2\big)\\
&\leq\ \left[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\right]\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\
&\leq\ \left[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\right]\\
&\quad\times\left(\mathfrak{l}^2\sup_{s\in\mathcal{E}}\mathbb{E}\left\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\right\|^2+\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\|\hat{\mathfrak{w}}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}\right)\\
&\leq\ \mathfrak{l}^2\left[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\right]\sup_{s\in\mathcal{E}}\mathbb{E}\left\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\right\|^2\\
&\leq\ \mathsf{P}^*\sup_{s\in\mathcal{E}}\mathbb{E}\left\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\right\|^2,
\end{aligned}
$$

here

$$
\mathsf{P}^*=\mathfrak{l}^2\left[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\right].
$$

Thus by taking supreme over $\mathfrak{t}$, we obtain $\|\tilde{\Psi}\mathfrak{w}-\tilde{\Psi}\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}\leq\mathsf{P}^*\|\mathfrak{w}-\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}$.

As a result, $\tilde{\Psi}$ is a contraction on $\mathcal{B}\mathfrak{p}$ and has a unique fixed point $\mathfrak{w}(.)\in\mathcal{B}\mathfrak{p}$, which is the mild solution of (1.1). Hence the facts. □

4. Systems of neutral stochastic differential equations with infinite delay

Neutral differential systems have gained popularity in applied mathematics recently. A number of partial differential systems, such as heat flow in materials, wave propagation, and various natural phenomena, get assistance from neutral systems with or without delay. Let us now consider the following neutral stochastic infinite-delay integrodifferential system of the following form:

$$
\begin{aligned}
\mathcal{D}^{\alpha}\left[\mathfrak{x}(\mathfrak{t})-\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right]&=\mathfrak{A}\left[\mathfrak{x}(\mathfrak{t})-\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right]+\mathfrak{z}(\mathfrak{t})+\mathfrak{f}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right)\\
&\quad+\mathfrak{h}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}},\\
\mathfrak{x}(\mathfrak{t})&=\zeta(\mathfrak{t}),\quad\in\mathcal{L}^2(\Omega,\mathfrak{G}_{\mathcal{J}}),\mathfrak{t}\in(-\infty,0],
\end{aligned}\tag{4.1}
$$

A strongly continuous semigroup $\{\mathfrak{S}(\mathfrak{t})\}_{\mathfrak{t}\geq0}$ is generated by $\mathfrak{A}$ on $\mathcal{Y}$.

Consider the following hypotheses:

(A5) $\mathsf{k}:[0,\mathfrak{v}]\times\mathfrak{G}_{\mathcal{J}}\to\mathcal{Y}$ is a continuous function such that it satisfies the following requirement

$$
\begin{aligned}
\mathbb{E}\,\|\mathsf{k}(\mathfrak{t},\mathfrak{x})-\mathsf{k}(\mathfrak{t},\hat{\mathfrak{x}})\|^2&\leq N_{\mathsf{k}}\|\mathfrak{x}-\hat{\mathfrak{x}}\|^2_{\mathfrak{G}_{\mathcal{J}}},\quad\mathfrak{x},\hat{\mathfrak{x}}\in\mathfrak{G}_{\mathcal{J}},\ \mathfrak{t}\in\mathfrak{G}_{\mathcal{J}},\\
\mathbb{E}\,\|\mathsf{k}(\mathfrak{t},\mathfrak{x})\|^2&\leq\hat{N}_{\mathsf{k}}\left(1+\|\mathfrak{x}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right),\quad\mathfrak{x}\in\mathfrak{G}_{\mathcal{J}},\ \mathfrak{t}\in\mathfrak{G}_{\mathcal{J}}.
\end{aligned}
$$

Theorem 4.1. *Assume that (A1)–(A6) gets satisfied. Then* (4.1) *has a unique mild solution provided,*

$$
5\hat{N}_{\mathsf{k}}+\left[5\mathcal{M}^2\hat{N}_{\mathfrak{f}}(1+\tilde{\mathfrak{m}}_1\mathfrak{v}^2)+5\mathcal{M}^2\hat{N}_{\mathfrak{h}}Tr(Q)(1+\tilde{\mathfrak{m}}_2\mathfrak{v}^2)\right]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}<1.
$$

Proof. Let us define $\eta:\mathfrak{G}'_{\mathcal{J}}\to\mathfrak{G}'_{\mathcal{J}}$ as

$$
\eta\mathfrak{x}(\mathfrak{t})=\begin{cases}\zeta(\mathfrak{t}), & \mathfrak{t}\in(-\infty,0],\\
\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}\right)[\zeta(0)-\mathsf{k}(0,\zeta)]+\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha}\right)\mathfrak{z}(s)+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha}\right)\\
\mathfrak{f}\left(s,\mathfrak{x}_s,\int_0^s\mathfrak{g}\left(s,\rho,\mathfrak{x}_\rho\right)d\rho\right)ds+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^{\alpha}}{\alpha}-\frac{s^{\alpha}}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{x}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{x}_\rho\right)d\rho\right)d\omega(s), & \mathfrak{t}>0.\end{cases}
$$

Consider $\overline{\zeta}$ as

$$\overline{\zeta}(\mathfrak{t}) = \begin{cases} \zeta(\mathfrak{t}), & \mathfrak{t} \in (-\infty, 0] \\ \zeta(0), & \mathfrak{t} \in \mathcal{E} \end{cases} \quad \zeta \in \mathfrak{G}_{\mathcal{J}};$$

then $\overline{\zeta} \in \mathfrak{G}'_{\mathcal{J}}$. Let $\mathfrak{x}(\mathfrak{t}) = \mathfrak{w}(\mathfrak{t}) + \overline{\zeta}(\mathfrak{t})$, $-\infty < \mathfrak{t} \le \mathfrak{v}$. Clearly $\mathfrak{x}$ is satisfied if and only if $\mathfrak{w}$ fulfills $\mathfrak{w}_0 = 0$.

$$\begin{aligned} \mathfrak{w}(\mathfrak{t}) \;=\; & -\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\mathsf{k}(0,\zeta) + \mathsf{k}\left(\mathfrak{t}, \mathfrak{w}_\mathfrak{t} + \overline{\zeta}(\mathfrak{t})\right) + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right) \\ & \times \mathfrak{f}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)ds + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right) \\ & \times \mathfrak{h}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)d\omega(s). \end{aligned}$$

For any $\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}} = \{\mathfrak{w} \in \mathfrak{G}'_{\mathcal{J}}; \mathfrak{w}_0 = 0 \in \mathfrak{G}_{\mathcal{J}}\}$,

$$\|\mathfrak{w}\|_\mathfrak{v} = \|\mathfrak{w}_0\|_{\mathfrak{G}_{\mathcal{J}}} + \sup_{0\le s\le \mathfrak{v}} \left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right)^{1/2} = \sup_{0\le s\le \mathfrak{v}} \left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right)^{1/2}.$$

Eventually, $\left(\mathfrak{G}''_{\mathcal{J}}, \|.\|_\mathfrak{v}\right)$ is a Banach space. We consider $\mathcal{B}_\mathfrak{p} = \{\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}} : \|\mathfrak{w}\|^2_\mathfrak{v} \le \mathfrak{p}\}$; For some $\mathfrak{p} > 0$, then $\mathcal{B}_\mathfrak{p} \subseteq \mathfrak{G}''_{\mathcal{J}}$ is uniformly bounded, for each $\mathfrak{p}$. For $\mathfrak{w} \in \mathcal{B}_\mathfrak{p}$ and by the virtue of Lemma 2.1

$$\begin{aligned} \|\mathfrak{w}_\mathfrak{t} + \overline{\zeta}_\mathfrak{t}\|^2_{\mathfrak{G}_{\mathcal{J}}} & \le 2\left(\|\mathfrak{w}_\mathfrak{t}\|^2_{\mathfrak{G}_{\mathcal{J}}} + \|\overline{\zeta}_\mathfrak{t}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right) \\ & \le 4\left(\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2 \sup_{s\in[0,\mathfrak{t}]}\left(\mathbb{E}\|\mathfrak{w}(s)\|^2\right) + \|\overline{\zeta}_0\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2 \sup_{s\in[0,\mathfrak{t}]}\left(\mathbb{E}\|\overline{\zeta}(s)\|^2\right)\right) \\ & \le 4\left(0 + \mathfrak{l}^2\|\mathfrak{w}\|^2_\mathfrak{v} + \|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\right) \\ & \le 4\left(\mathfrak{l}^2\mathfrak{p} + \|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\right) = \mathfrak{p}'. \end{aligned} \tag{4.2}$$

Now we may define $\tilde{\Psi} : \mathfrak{G}''_{\mathcal{J}} \to \mathfrak{G}''_{\mathcal{J}}$ as:

$$\tilde{\Psi}\mathfrak{w}(\mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \in (-\infty, 0] \\ \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds & \\ + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{f}\big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\big)d\rho\big)ds & \\ + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)d\omega(s), & \mathfrak{t} \in \mathcal{E}, \end{cases} \tag{4.3}$$

implies that $\tilde{\Psi}$ has a unique fixed point. Let us divide the proof into several steps.

Step 1: $\overline{\eta}(\mathcal{B}_\mathfrak{p}) \subset \mathcal{B}_\mathfrak{p}$ for $\mathfrak{p} > 0$.

Assuming the contrary, for each $\mathfrak{p} > 0$ there exists $\mathfrak{w}(.) \in \mathcal{B}_\mathfrak{p}$ and $\overline{\eta}(\mathfrak{w}) \notin \mathcal{B}_\mathfrak{p}$ (i.e), $\mathbb{E}\left\|(\overline{\eta}\mathfrak{w})(\mathfrak{t})\right\|^2 > \mathfrak{p}$ for some $\mathfrak{t} \in \mathcal{E}$.

$$\begin{aligned} \mathfrak{p} \;\le\; & \mathbb{E}\left\|(\overline{\eta}\mathfrak{w})(\mathfrak{t})\right\|^2 \\ \le\; & 5\mathbb{E}\left\|\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\mathsf{k}(0,\zeta)\right\|^2 + 5\mathbb{E}\left\|\mathsf{k}\left(\mathfrak{t}, \mathfrak{w}_\mathfrak{t} + \overline{\zeta}_\mathfrak{t}\right)\right\|^2 + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2 \\ & + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2 \\ & + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\right)d\rho\right)d\omega(s)\right\|^2 \\ \le\; & \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_4 + \mathfrak{S}_5. \end{aligned}$$

where

$$\begin{aligned} \mathfrak{S}_1 &= \mathbb{E}\|\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\mathsf{k}(0,\zeta)\|^2 \le \mathcal{M}^2\hat{N}_\mathsf{k}\left(1 + \|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}}\right), \\ \mathfrak{S}_2 &= \mathbb{E}\left\|\mathsf{k}\left(\mathfrak{t}, \mathfrak{w}_\mathfrak{t} + \overline{\zeta}_\mathfrak{t}\right)\right\|^2 \le \hat{N}_\mathsf{k}\left(1 + \|\mathfrak{w}_\mathfrak{t} + \overline{\zeta}_\mathfrak{t}\|^2\right) \le \hat{N}_\mathsf{k}(1 + \mathfrak{p}_1), \\ \mathfrak{S}_3 &= \mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2 \\ &\le \mathcal{M}^2\|\mathbb{B}\|^2_\infty\mathbb{E}\left[\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\|\mathfrak{z}(s)\|^2ds\right] \le \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^\mathfrak{q}(\mathcal{E},\mathcal{K})}, \end{aligned}$$

$$
\begin{aligned}
\mathfrak{S}_4 &\leq \mathbb{E}\left\| \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2\\
&\leq \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathbb{E}\left\|\int_0^s \mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right\|^2\right)\\
&\leq \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+\overline{\mathfrak{m}}_1\mathfrak{v}^2\left(1+\|\mathfrak{w}_\rho+\overline{\zeta}_\rho\|^2_{\mathfrak{G}_{\mathcal{J}}}\right)\right)\\
&\leq \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}_1)\right),\\
\mathfrak{S}_5 &= \mathbb{E}\left\| \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)d\omega(s)\right\|^2\\
&\leq \mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2\\
&\leq \mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+\overline{\mathfrak{m}}_2\mathfrak{v}^2\left(1+\left\|\mathfrak{w}_\rho+\overline{\zeta}_\rho\right\|^2_{\mathfrak{G}_{\mathcal{J}}}\right)\right)\\
&\leq \mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}_1)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathfrak{p} \;\leq\;& 5\mathcal{M}^2\hat{N}_{\mathfrak{k}}\left(1+\|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}}\right)+5\hat{N}_{\mathfrak{k}}(1+\mathfrak{p}_1)+5\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\\
&+5\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}_1)\right)+5\mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}_1)\right).
\end{aligned}
$$

Dividing by $\mathfrak{p}$ throughout and let $\mathfrak{p}\to\infty$, we obtain

$$
1 \leq 5\hat{N}_{\mathfrak{k}}+5\mathcal{M}^2\hat{N}_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\overline{\mathfrak{m}}_1\mathfrak{v}^2)+5\mathcal{M}^2 Tr(Q)\hat{N}_{\mathfrak{h}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\overline{\mathfrak{m}}_2\mathfrak{v}^2)
$$

It opposes our theory (4.1). Therefor $\overline{\eta}(\mathcal{B}_{\mathfrak{p}})\subset\mathcal{B}_{\mathfrak{p}}$, $\mathfrak{p}>0$.

Step 2: $\overline{\eta}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$.

Using $\mathfrak{w},\hat{\mathfrak{w}}\in\mathcal{B}_{\mathfrak{p}}$,

$\mathbb{E}\left\|\overline{\eta}\mathfrak{w}(\mathfrak{t})-\overline{\eta}\hat{\mathfrak{w}}(\mathfrak{t})\right\|^2$

$$
\begin{aligned}
=\;& \mathbb{E}\Bigg\| \left[\mathfrak{k}\left(\mathfrak{t},\mathfrak{w}_{\mathfrak{t}}+\overline{\zeta}_{\mathfrak{t}}\right)-\mathfrak{k}\left(\mathfrak{t},\hat{\mathfrak{w}}_{\mathfrak{t}}+\overline{\zeta}_{\mathfrak{t}}\right)\right]+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\Big[\mathfrak{f}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\\
&-\mathfrak{f}\left(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\left(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\Big]ds+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\\
&\times\left[\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)-\mathfrak{h}\left(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\right)d\rho\right)\right]ds\Bigg\|^2\\
\leq\;& 3N_{\mathfrak{k}}\|\mathfrak{w}_{\mathfrak{t}}-\hat{\mathfrak{w}}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}+3\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\left(1+\mathfrak{m}_1\mathfrak{v}^2\right)\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+3\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)\\
&\times\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\
\leq\;& \left[3N_{\mathfrak{k}}+3\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\left(1+\mathfrak{m}_1\mathfrak{v}^2\right)+3\mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)\right]\\
&\times\left(\mathfrak{l}^2\sup_{s\in\mathcal{E}}\mathbb{E}\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\|^2+\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\|\hat{\mathfrak{w}}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}\right)\\
\leq\;& \mathfrak{l}^2\left[3N_{\mathfrak{k}}+\left[3\mathcal{M}^2N_{\mathfrak{f}}(1+\mathfrak{m}_1\mathfrak{v}^2)+3\mathcal{M}^2Tr(Q)N_{\mathfrak{h}}\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)\right]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\right]\sup_{s\in\mathcal{E}}\mathbb{E}\left\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\right\|^2\\
\leq\;& \mathcal{Q}^*\sup_{s\in\mathcal{E}}\mathbb{E}\left\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\right\|^2,
\end{aligned}
$$

with

$$
\mathcal{Q}^*=\mathfrak{l}^2\left[3N_{\mathfrak{k}}+\left[3\mathcal{M}^2N_{\mathfrak{f}}(1+\mathfrak{m}_1\mathfrak{v}^2)+3\mathcal{M}^2Tr(Q)N_{\mathfrak{h}}\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)\right]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\right].
$$

By taking supremum over $\mathfrak{t}$,

$$
\|\overline{\eta}\mathfrak{w}-\overline{\eta}\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}\leq\mathcal{Q}^*\|\mathfrak{w}-\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}.
$$

Thus $\overline{\eta}$ is a contraction. As a result, the mild solution of (4.1) has a unique fixed point $\mathfrak{w}(.)\in\mathcal{B}_{\mathfrak{p}}$ for the function $\overline{\eta}$. □

5. Optimal control

Consider the Lagrange problem (LP). Find a control $\mathfrak{z}^0 \in \mathfrak{A}_{ad} \ni \mathscr{L}(\mathfrak{z}^0) \le \mathscr{L}(\mathfrak{z}), \quad \forall \ \mathfrak{z} \in \mathfrak{A}_{ad}$ where

$$\mathscr{L}(\mathfrak{z}) = \mathbb{E}\{\int_0^{\mathfrak{b}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}^{\mathfrak{z}}, \mathfrak{x}^{\mathfrak{z}}(\mathfrak{t}), \mathfrak{z}(\mathfrak{t})\right) dz\},$$

and $\mathfrak{x}^{\mathfrak{z}}$ relates to the control $\mathfrak{z} \in \mathfrak{A}_{ad}$ and is the mild solution of system (4.1). To demonstrate that there is a solution to the problem, we establish the following hypotheses (LP).

(A7) (i) $\mathscr{M} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \times \mathcal{K} \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.
(ii) $\mathscr{M}(\mathfrak{t}, ., ., .)$ on $\mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \times \mathcal{K}$ is a sequentially lower semicontinuous functional $\forall$ $\mathfrak{t} \in \mathcal{E}$.
(iii) $\mathscr{M}(\mathfrak{t}, \mathfrak{x}, \hat{\mathfrak{x}}, .)$ is convex on $\mathcal{K}$ for each $\mathfrak{x} \in \mathfrak{G}_{\mathcal{J}}, \hat{\mathfrak{x}} \in \mathcal{Y}$ and almost all $\mathfrak{t} \in \mathcal{E}$.
(iv) There exist constants $\tilde{\mathfrak{a}}, \tilde{\mathfrak{b}} \ge 0$, $\tilde{\mathfrak{c}} > 0$, $\mathfrak{g}$ is non-negative and $\mathfrak{g} \in \mathcal{L}^1(\mathcal{E}, \mathbb{R})$ such that

$$\mathfrak{g}(\mathfrak{t}) + \tilde{\mathfrak{a}}\|\mathfrak{x}\|^2_{\mathfrak{G}_{\mathcal{J}}} + \tilde{\mathfrak{b}}\|\hat{\mathfrak{x}}\|^2 + \tilde{\mathfrak{c}}\|\mathfrak{z}\|^{\mathfrak{q}}_{\mathcal{K}} \le \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}, \hat{\mathfrak{x}}, \mathfrak{z}\right)$$

Theorem 5.1. *Assume that (A7), Theorem 4.1, and the existence of the strongly continuous operator $\mathbb{B}$ are all true. Then, the Lagrange Problem (LP) accepts at least one optimal control pair, that is, a control $\mathfrak{z}^0 \in \mathfrak{A}_{ad}$ such that $\forall \mathfrak{z} \in \mathfrak{A}_{ad}$,*

$$\mathscr{L}(\mathfrak{z}^0) = \mathbb{E}\left\{\int_0^{\mathfrak{v}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}^0, \mathfrak{x}^0(\mathfrak{t}), \mathfrak{z}^0(\mathfrak{t})\right) d\mathfrak{t}\right\} \le \mathscr{L}(\mathfrak{z}).$$

Proof. For $\inf\{\mathscr{L}(\mathfrak{z}) \mid \mathfrak{z} \in \mathfrak{A}_{ad}\} = +\infty$, nothing needs to be verified. Without loss of generality, we deduce that $\inf\{\mathscr{L}(\mathfrak{z}) \mid \mathfrak{z} \in \mathfrak{A}ad\} = \Theta < +\infty$. By (A7), we have $\Theta > -\infty$. According to the notion of infimum, there is a minimizing sequence of feasible pairs $\{(\mathfrak{x}^n, \mathfrak{z}^n)\} \subset \mathcal{U}_{ad}$, where $\mathcal{U}_{ad} = \{(\mathfrak{x}, \mathfrak{z}) : \mathfrak{x}$ is a mild solution of [(4.1)] relating to $\mathfrak{z} \in \mathfrak{A}_{ad}\}$ such that $\mathscr{L}(\mathfrak{x}^n, \mathfrak{z}^n) \to \Theta$, as $n \to +\infty$. Then $\{\mathfrak{z}^n\} \subseteq \mathfrak{A}_{ad}$, $n = 1, 2, \ldots, \{\mathfrak{z}^n\}$ is a bounded subset of the separable reflexive Hilbert space $\mathcal{L}^{\mathfrak{q}}(\mathcal{E}, \mathcal{K})$ and $\exists$ a subsequence, $\{\mathfrak{z}^n\}$ and $\mathfrak{z}^0 \in \mathcal{L}^{\mathfrak{q}}(\mathcal{E}, \mathcal{K}) \ni \mathfrak{z}^n \xrightarrow{w} \mathfrak{z}^0$ in $\mathcal{L}^{\mathfrak{q}}(\mathcal{E}, \mathcal{K})$. Due to the closed convex nature of $\mathfrak{A}_{ad}$, the Mazur Lemma, which states that $\mathfrak{z}^0 \in \mathfrak{A}_{ad}$., holds true. If $\{\mathfrak{x}^n\}$, which corresponds to $\{\mathfrak{z}^n\}$, is the solution sequence for (4.1), then

$$\mathfrak{x}^n(\mathfrak{t}) = \begin{cases} \zeta(\mathfrak{t}), & \mathfrak{t} \in (-\infty, 0], \\ \mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)[\zeta(0) - \mathsf{k}(0, \zeta)] + \mathsf{k}(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}^n) + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathbb{B}(s)\mathfrak{z}^n(s) + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right) \\ \mathfrak{f}\left(s, \mathfrak{x}_s^n, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{x}_\rho^n\right) d\rho\right) ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s, \mathfrak{x}_s^n, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{x}_\rho^n\right) d\rho\right) d\omega(s), & \mathfrak{t} > 0. \end{cases}$$

By referring Theorem 4.1, it is evident that $\exists\ \tau > 0 \ni \quad n = 0, 1, 2, \ldots$.

$$\|\mathfrak{x}^n\|^2_{\mathfrak{v}} \le \tau,$$

Assume that $\mathfrak{x}^n(\mathfrak{t}) = \mathfrak{w}^n(\mathfrak{t}) + \overline{\zeta}(\mathfrak{t})$; $\mathfrak{w}^n \in \mathfrak{G}''_{\mathcal{J}}$ and $\overline{\zeta} : (-\infty, \mathfrak{v}] \to \mathcal{Y}$ are the functions defined in the earlier part of the proof. For $\mathfrak{t} \in \mathcal{E}$, we get

$$\begin{aligned} &\mathbb{E}\left\|\mathfrak{w}^n(\mathfrak{t}) - \mathfrak{w}^0(\mathfrak{t})\right\|^2 \\ &\le \ 4\mathbb{E}\left\|\mathsf{k}\left(\mathfrak{t}, \mathfrak{w}_{\mathfrak{t}}^n + \overline{\zeta}_{\mathfrak{t}}\right)\right\|^2 + 4\mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\left[\mathbb{B}(s)\mathfrak{z}^n(s) - \mathbb{B}(s)\mathfrak{z}^0(s)\right] ds\right\|^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\left[\mathfrak{f}\left(s, \mathfrak{w}_s^n + \overline{\zeta}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{w}_s^n + \overline{\zeta}_s\right) d\rho\right)\right.\right. \\ &\quad \left.\left. - \mathfrak{f}\left(s, \mathfrak{w}_s^0 + \overline{\zeta}_s, \int_0^s \mathfrak{g}\left(s, \rho, \mathfrak{w}_s^0 + \overline{\zeta}_s\right) d\rho\right)\right] ds\right\|^2 + 4\mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)\right. \\ &\quad \left. \times \left[\mathfrak{h}\left(s, \mathfrak{w}_s^n + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_s^n + \overline{\zeta}_s\right) d\rho\right) - \mathfrak{h}\left(s, \mathfrak{w}_s^0 + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\left(s, \rho, \mathfrak{w}_s^0 + \overline{\zeta}_s\right) d\rho\right)\right] d\omega(s)\right\|^2 \\ &\le \ 4N_{\mathsf{k}}\|\mathfrak{w}_{\mathfrak{t}}^n - \mathfrak{w}_{\mathfrak{t}}^0\|^2_{\mathfrak{G}_{\mathcal{J}}} + 4\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left[\int_0^{\mathfrak{t}} \left\|\mathbb{B}(s)\mathfrak{z}^n(s) - \mathbb{B}(s)\mathfrak{z}^0(s)\right\|^2 ds\right] + \left[4\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{f}}\left(1 + \mathfrak{m}_1\mathfrak{v}^2\right)\right. \\ &\quad \left. + 4\mathcal{M}^2 Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\left(1 + \mathfrak{m}_2\mathfrak{v}^2\right)\right]\left\|\mathfrak{w}_s^n - \mathfrak{w}_s^0\right\|^2. \end{aligned}$$

So *exists* $\mathscr{S}^* > 0 \ni \sup_{s\in\mathcal{E}} \mathbb{E}\left\|\mathfrak{w}^n(s) - \mathfrak{w}^0(s)\right\|^2 \le \mathscr{S}^*\left\|\mathbb{B}\mathfrak{z}^n - \mathbb{B}\mathfrak{z}^0\right\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})}, \quad \forall \ \mathfrak{t} \in \mathcal{E}$.

Hence, $\mathbb{B}$ is strongly continuous. This implies that

$$\left\|\mathbb{B}\mathfrak{z}^n - \mathbb{B}\mathfrak{z}^0\right\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})} \to 0 \ as \ n \to \infty.$$

Then, we have

$$\left\|\mathfrak{w}^n - \mathfrak{w}^0\right\|^2_{\mathfrak{v}} \to 0 \ as \ n \to \infty.$$

which is equivalent to

$$\|\mathfrak{x}^n - \mathfrak{x}^0\|_{\mathfrak{v}}^2 \to 0 \ as \ n \to \infty.$$

Therefore,

$$\mathfrak{x}^n \to \mathfrak{x}^0 \ in \ \mathfrak{G}'_{\mathcal{J}} \ as \ n \to \infty.$$

We deduce that

$$\left(\mathfrak{x}_{\mathfrak{t}} \times \mathfrak{x}, \mathfrak{z}\right) \to \mathbb{E}\left\{\int_0^{\mathfrak{v}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}, \mathfrak{x}(\mathfrak{t}), \mathfrak{z}(\mathfrak{t})\right) d\mathfrak{t}\right\}$$

is sequentially lower semicontinuous in the weak topology of $\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{K}) \subset \mathcal{L}^1(\mathcal{E},\mathcal{K})$ and strong topology of $\mathcal{L}^1(\mathcal{E},\mathfrak{G}\mathcal{J}\times\mathcal{Y})$ from Balder's theorem [39]. So, $\mathscr{L}$ is weakly lower semicontinuous on $\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{Y})$. By (A7), $\mathscr{L} > -\infty$ and $\mathscr{L}$ succeeds its minimum at $\mathfrak{z}^0 \in \mathfrak{A}_{ad}$, i.e.,

$$\Theta = \lim_{n\to\infty} \mathbb{E}\left\{\int_0^{\mathfrak{v}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}^n_{\mathfrak{t}}, \mathfrak{x}^n(\mathfrak{t}), \mathfrak{z}^n(\mathfrak{t})\right) d\mathfrak{t}\right\} \geq \mathbb{E}\left\{\int_0^{\mathfrak{v}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}^0_{\mathfrak{t}}, \mathfrak{x}^0(\mathfrak{t}), \mathfrak{z}^0(\mathfrak{t})\right) d\mathfrak{t}\right\} = \mathscr{L}(\mathfrak{z}^0) \geq \Theta.$$

Hence the assertion. □

6. Illustration

Let us consider a conformable fractional stochastic neutral control system of the form:

$$\begin{cases} \mathcal{D}^{\alpha}\left[\mathfrak{x}(\mathfrak{t},\varepsilon) - \int_{-\infty}^{0} \mathfrak{b}(\mathfrak{t},\varepsilon)\mathfrak{x}(\mathfrak{t},\varepsilon)d\varepsilon\right] = \frac{\partial^2}{\partial\varepsilon^2}\mathfrak{x}(\mathfrak{t},\varepsilon) + \int_{[0,1]} \mathscr{U}(\varepsilon,s)\mathfrak{z}(s,\mathfrak{t})ds \\ +\Xi\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds, \int_0^{\mathfrak{t}}\int_{-\infty}^{0} \Xi_2(s,\varepsilon,\epsilon-s)\,\mathfrak{x}(\epsilon,\varepsilon)d\epsilon ds\right) \\ +\Upsilon\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds, \int_0^{\mathfrak{t}}\int_{-\infty}^{0} \Xi_3(s,\varepsilon,\epsilon-s)\,\mathfrak{x}(\epsilon,\varepsilon)d\epsilon ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}, \quad \mathfrak{t}\in\mathcal{E}' \\ \mathfrak{x}(\mathfrak{t},0) = \mathfrak{x}(\mathfrak{t},\pi) = 0, \mathfrak{t}\geq 0, \\ \mathfrak{x}(\mathfrak{t},\varepsilon) = \phi(\mathfrak{t},\varepsilon), \ \varepsilon\in[0,\pi], -\infty < \mathfrak{t} < 0, \end{cases} \tag{6.1}$$

where $\mathcal{D}^{\alpha}$ is the Conformable derivative, and the neutral function be $\mathfrak{b}(\mathfrak{t},\varepsilon)$. $(\Omega,\mathfrak{F},\mathcal{P})$ is the filtered Probability space with $\omega(\mathfrak{t})$ being a one-dimensional standard Weiner process in $\mathcal{Y}$. The functions $\phi(\mathfrak{t},\varepsilon), \Xi, \Upsilon, \Xi_2$, and Ξ_3 are continuous.

Let us equalize the Considered Hilbert spaces, (i. (e) $\left(\mathcal{K} = \mathcal{Y} = \mathcal{L}^2([0,\pi])\right)$. The operator $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathcal{Y} \to \mathcal{Y}$ is expressed by $\mathfrak{A}\mathfrak{u} = \mathfrak{u}'' \in \mathscr{D}(\mathfrak{A})$, where

$$\mathscr{D}(\mathfrak{A}) = \{\mathfrak{u}\in\mathcal{Y} : \mathfrak{u}, \mathfrak{u}' \text{ are absolutely continuous, } \mathfrak{u}''\in\mathcal{Y}, \mathfrak{u}(0) = \mathfrak{u}(\pi) = 0\}.$$

$\mathfrak{A}$ generates a continuous semigroup $\mathfrak{S}(\mathfrak{t})_{\mathfrak{t}\geq 0}$ being compact, analytic, and self-adjoint, having a discrete spectrum. Let $-\upsilon^2, \upsilon\in\mathbb{N}$ are the eigenvalues and the corresponding normalized eigenvectors are

$$\mathfrak{w}_{\upsilon}(\varepsilon) = \left(\frac{2}{\pi}\right)^{1/2}\sin(\upsilon\varepsilon), \ \upsilon = 1, 2, \ldots .$$

Let us consider the following assumptions:

(i) Provided, $\mathfrak{u}\in\mathscr{D}(\mathfrak{A})$, $\mathfrak{A}\mathfrak{u} = \sum_{\upsilon=1}^{\infty}\langle\mathfrak{u},\mathfrak{w}_{\upsilon}\rangle\mathfrak{w}_{\upsilon}$.

(ii) For $\mathfrak{u}\in\mathcal{Y}$, $\mathfrak{Q} = \sum_{\upsilon=1}^{\infty}\frac{1}{\upsilon}\langle\mathfrak{u},\mathfrak{w}_{\upsilon}\rangle\mathfrak{w}_{\upsilon}$ in the space $\mathscr{D}(\mathfrak{Q}) = \{\mathfrak{u}\in\mathcal{Y}, \sum_{\upsilon=1}^{\infty}\upsilon\langle\mathfrak{u},\mathfrak{w}_{\upsilon}\rangle\mathfrak{w}_{\upsilon}\}$, $\mathfrak{Q} = 1 - \mathfrak{A}$.

Consider $\mathcal{J}(s) = e^{2s}, s < 0$, then $\mathfrak{l} = \int_{-\infty}^{0}\mathcal{J}(s)ds = \frac{1}{2}$. $\mathfrak{G}_{\mathcal{J}}$ be the phase space equipped with the norm

$$\|\zeta\|_{\mathfrak{G}_{\mathcal{J}}} = \int_{-\infty}^{0}\mathcal{J}(s)\sup_{s\leq\eta\leq 0}\left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2}ds.$$

Then, $\left(\mathfrak{G}_{\mathcal{J}}, \|.\|_{\mathfrak{G}_{\mathcal{J}}}\right)$ is a Banach space.

For $(\mathfrak{t},\zeta)\in[0,\mathfrak{v}]\times\mathfrak{G}_{\mathcal{J}}$, where $\zeta(.,\varepsilon) = \phi(.,\varepsilon)\in(-\infty,0]\times[0,\pi]$, we consider

$$\begin{aligned} \mathfrak{x}(\mathfrak{t})(\varepsilon) &= \mathfrak{x}(\mathfrak{t},\varepsilon), \\ \mathfrak{g}(\mathfrak{t},\zeta)(\varepsilon) &= \int_{-\infty}^{0}\Xi_2(\mathfrak{t},\varepsilon,s)\zeta(s)(\varepsilon)ds, \\ \mathfrak{f}\left(\mathfrak{t},\zeta,\int_0^{\mathfrak{t}}\mathfrak{g}(s,\zeta)\right)(\varepsilon) &= \Xi\left(\mathfrak{t},\int_{-\infty}^{\mathfrak{t}}\Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds, \int_0^{\mathfrak{t}}\mathfrak{g}(\mathfrak{t},\zeta)(\varepsilon)ds\right), \\ \tilde{\mathfrak{g}}(\mathfrak{t},\zeta)(\varepsilon) &= \int_{-\infty}^{0}\Xi_3(\mathfrak{t},\varepsilon,s)\zeta(s)(\varepsilon)ds, \\ \mathfrak{h}\left(\mathfrak{t},\zeta,\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}(s,\zeta)\right)(\varepsilon) &= \Upsilon\left(\mathfrak{t},\int_{-\infty}^{\mathfrak{t}}\Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds, \int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}(\mathfrak{t},\zeta)(\varepsilon)ds\right), \\ \mathsf{k}(\mathfrak{t},\zeta)(\varepsilon) &= \int_{-\infty}^{0}\mathfrak{b}(\mathfrak{t},\varepsilon)\zeta(\mathfrak{t},\varepsilon)d\varepsilon. \end{aligned}$$

It is obvious that the functions $\mathfrak{f}, \mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{h}$, and $\mathfrak{k}$ satisfy the assumptions (A1)–(A7).

The function $\mathfrak{z} \in \mathfrak{G}\mathfrak{x}([0,\pi]) \to \mathbb{R}$ is a control where $\mathfrak{z} \in \mathcal{L}^2(\mathfrak{G}\mathfrak{x}([0,\pi]))$, $\mathfrak{t} \to \mathfrak{z}(\mathfrak{t})$ is measurable. Let the set $\mathfrak{A} = \{\mathfrak{z} \in \mathcal{K} : \|\mathfrak{z}\|_{\mathcal{K}} \le \varpi\}$ where $\varpi \in \mathcal{L}^2(\mathcal{E}, \mathbb{R}^+)$. We confine the admissible control $\mathfrak{A}_{ad}$ to be $\mathfrak{z} \in \mathcal{L}^2(\mathfrak{G}\mathfrak{x}([0,\pi])) \ni \|\mathfrak{z}(.,\mathfrak{t})\| \le \varpi(\mathfrak{t})$.

We express $\mathbb{B}(\mathfrak{t})\mathfrak{z}(\mathfrak{t})(\epsilon) = \int_{[0,1]} \mathscr{U}(\epsilon, s)\mathfrak{z}(s,\mathfrak{t})ds$ and suppose that

(i) $\mathscr{U}$ is a continuous function.

(ii) $\mathfrak{z} \in \mathcal{L}^2([0,\pi] \times \mathcal{E})$ and $\mathscr{M} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \times \mathcal{K} \to \mathbb{R} \cap \infty$ being defined by

$$\mathscr{M}\left(\mathfrak{t}, \mathfrak{x}^{\mathfrak{z}}_{\mathfrak{t}}, \mathfrak{x}^{\mathfrak{z}}(\mathfrak{t}), \mathfrak{z}(\mathfrak{t})\right)(\epsilon) = \int_{[0,\pi]}\int_{-\infty}^{0} |\mathfrak{x}^{\mathfrak{z}}(\mathfrak{t}+s,\epsilon)|^2 ds d\epsilon + \int_{[0,\pi]} |\mathfrak{x}^{\mathfrak{z}}(\mathfrak{t},\epsilon)|^2 d\epsilon + \int_{[0,\pi]} |\mathfrak{z}(\epsilon,\mathfrak{t})|^2 d\epsilon.$$

It is explicit that all the requirements of Theorem 4.1 gets satisfied. Thus there occurs an admissible control pair $(\mathfrak{x}, \mathfrak{z})$ such that the associated cost functional

$$\mathscr{L}(\mathfrak{z}) = \mathbb{E}\left\{\int_0^{\mathfrak{v}} \mathscr{M}\left(\mathfrak{t}, \mathfrak{x}^{\mathfrak{z}}_{\mathfrak{t}}, \mathfrak{x}^{\mathfrak{z}}(\mathfrak{t}), \mathfrak{z}(\mathfrak{t})\right) dt\right\}$$

achieves its minimum.

7. Application

In this section we will show numerical simulations of the application Eq. (7.1). In terms of approximating a numerical solution this example has a variety of interesting and challenging components. If it only had the parabolic partial differential equation (PDE) component then we could solve it with a variety of software suites and the methods described in [40,41]. If it was only the neutral delay term then we could use the methods described in [42]. If it was only the Brownian and fBm components then we could use the methods described on in [43]. This equation has all of these components in addition to an integral term.

To the author's knowledge there is no existing software suite built to handle all of these components in the example NSIDE with infinite delay and mixed fBm so we have built one in Matlab. All of the specifics of the simulation are available within the code but the primary techniques are as follows. A forward finite difference was used in the time derivative as to make it an explicit method. A centered difference approximation was used for all spatial (θ) derivatives. The trapezoid rule as implemented by the Matlab function trapz.m was used to approximate the integral term at each discretization point. The Brownian term dB is normally distributed with mean zero and variance $d\iota$ where $d\iota$ is the time discretization step. the fractional Brownian term dZ is normally distributed with mean zero and variance $d\iota^{2h}$ where h is the fBm parameter. The delayed derivative terms were approximated by difference derivatives on the mesh as well.

In the included simulations (Figs. 1–3) the following functions and parameters have been used. We have used $n = 2$ and 20 points in each spatial dimension for a total of 400 spatial points at each timestep. We have used 5000 time-steps so $d\iota = 0.0002$. The fBm parameter $h = 0.7$. The functions are $\phi(\iota,\theta) = \iota^2 + \sum_i^n \theta_i^2$, $\Lambda(s) = cos(s)$, and $c(\iota,\theta) = \frac{1}{\iota^2+\sum_i^n \theta_i^2}$. Fig. 1 shows the function at the beginning at $\iota = 0$ and a third way through the simulation. Fig. 2 shows the simulation at roughly two thirds through the simulation and at the end. Fig. 3 shows the simulation at roughly third eighth through the simulation and at the end.

$$\frac{\partial}{\partial\iota}\left[\vartheta(\iota,\theta) + \frac{\iota^2 + e^\iota|\vartheta(\iota-\varphi,\theta)|^2}{18}\right] \tag{7.1}$$

$$= \frac{\partial^2}{\partial\theta^2}\left[\vartheta(\iota,\theta) + \frac{\iota^2 + e^\iota|\vartheta(\iota-\varphi,\theta)|^2}{18}\right]$$

$$+ \frac{e^\iota\vartheta(\iota-\varphi,\theta)/\sqrt{2} + \vartheta(\iota, \sin\iota\,|\vartheta(\iota,\theta)|)/\sqrt{2}}{9} \tag{7.2}$$

$$+ e^\iota\int_{-\infty}^{t}\sin(\iota-\zeta)d\iota\int_0^t\int_{-\infty}^0\left[\frac{\iota\sin\theta}{8\pi} + \frac{e^\iota|\vartheta(\iota-\varphi,\theta)|}{2+|\vartheta(\iota,\theta)|}\right]d\iota d\theta$$

$$+ \left[e^\iota\int_{-\infty}^{t}\sin(\iota-\zeta)d\iota\int_0^t\int_{-\infty}^0\left[\frac{\iota\sin\theta}{8\pi} + \frac{e^\iota|\vartheta(\iota-\varphi,\theta)|}{2+|\vartheta(\iota,\theta)|}\right]d\iota d\theta\right]\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}} \tag{7.3}$$

$$\iota \in (0,1],\ \theta \in [0,\pi],$$

$$\vartheta(\iota, 0) = 0,$$

$$\vartheta(\iota,\theta) = \varphi(\iota,\theta),\ \theta \in [0,\pi],\ \iota \in (-\infty, 0]. \tag{7.4}$$

8. Conclusion

In this article, we discussed the optimal control of conformable fractional neutral stochastic integrodifferential system. This equation has infinite delays and takes place in a separable Hilbert space. In addition, using Banach fixed point theorem, the existence results and some conclusions concerning the optimal control are obtained. An illustration of the theory that has been presented is provided as the concluding part. We developed a numerical scheme to justify the theory. The code contains all of the details of the simulation based on finite difference method in Matlab. This work is a unique combination of theoretical proof with numerical estimations.

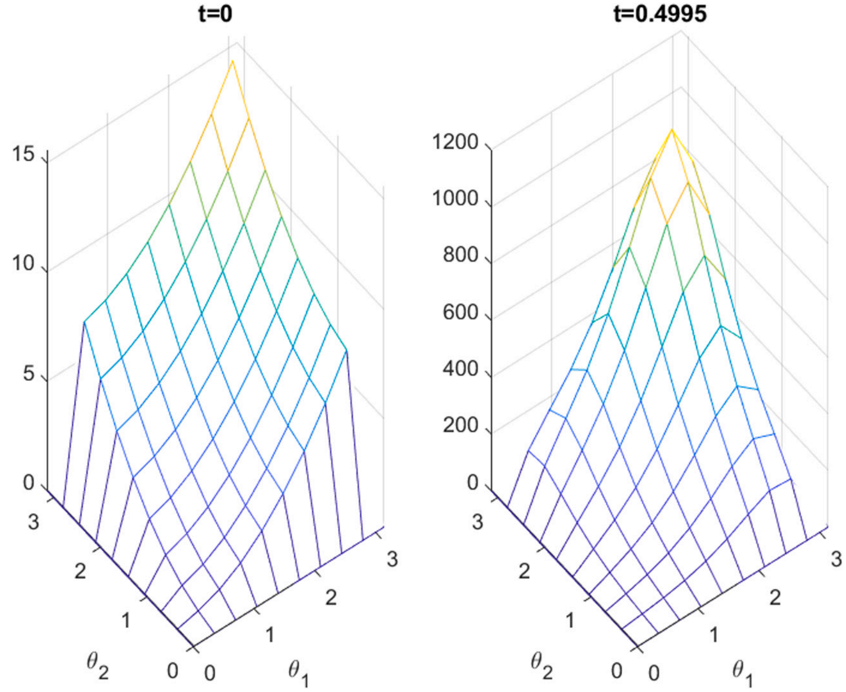

Fig. 1. Function at the beginning at $\iota = 0$ and a third way through the simulation.

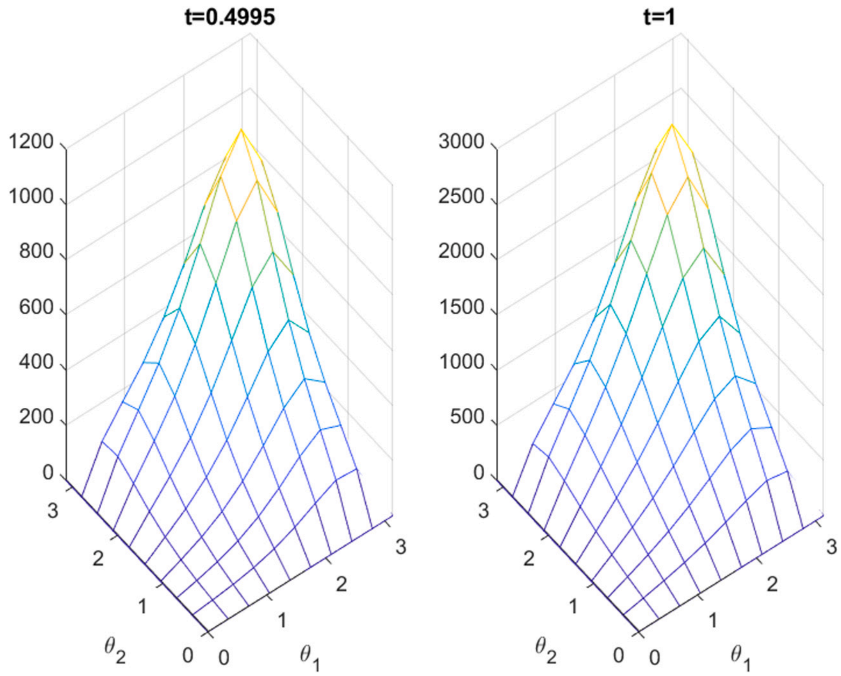

Fig. 2. Simulation at roughly one half

This work can further be extended to trajectory (T-) controllability of conformable fractional stochastic differential equations. T-controllability is the strongest notion of controllability, so one has to prove the result without assuming the compactness of the semigroup or the resolvent operator using measure of noncompactness or some other way. Also, we are planning to weaker the Lipschitz condition on nonlinear operator by the "Integral Contractor method with Regularity" for the advanced stochastic fractional order system, like, ψ-Hilfer system, Hadamard system, Hilfer–Katugampola system, etc.

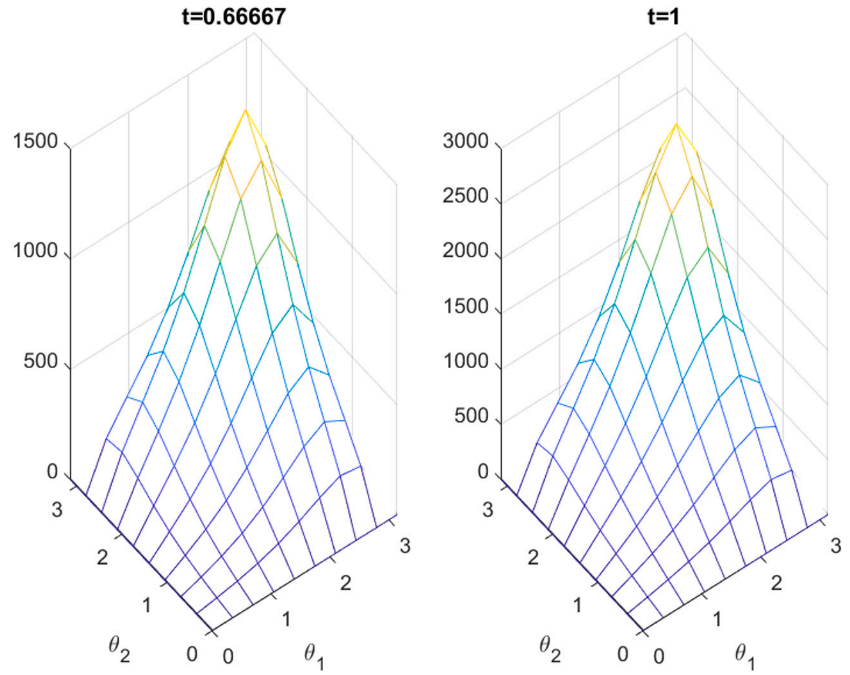

Fig. 3. Simulation roughly two-thirds through the simulation and at the end

We have studied the solution of the model as a piecewise random stochastic with numerical schemes including Newton polynomial. This technique is more accurate and useful for solving the linear and nonlinear partial differential equations. We can generalize the same for the future work. One can use the hybrid control system using the same argument. •We cannot apply this result to the boundary value problem. The existence result has not been reported so far for the system with infinite time/state delay with boundary value. This is the limitation. The studied numerical simulation is valid only for the time delay system, but not for the state delay, and not for the deviating arguments. One needs to generate separate codes to extend the result studied in this manuscript.

Declaration of competing interest

There is no conflict of interest.

Data availability

No data was used for the research described in the article.

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