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Trajectory Controllability of Clarke Subdifferential-Type Conformable Fractional Stochastic Differential Inclusions with Non-Instantaneous Impulsive Effects and Deviated Arguments

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Abstract: In this study, the multivalued fixed point theorem, Clarke subdifferential properties, fractional calculus, and stochastic analysis are used to arrive at the system's mild solution (1). Furthermore, the mean square moment for the aforementioned system (1) confirms the conditions for trajectory (T-)controllability. The last part of the paper uses two numerical applications to explain the novel theoretical results that were reached.

Keywords: stochastic differential inclusion; Clarke subdifferential; conformable fractional derivative; trajectory controllability; non-instantaneous impulse; deviated argument

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1. Introduction

Because the fractional operator has a nonlocal quality that allows for the demonstration of long memories or nonlocal effects for the purpose of more precisely confirming physical phenomena, fractional calculus is successful in simulating natural phenomena. Electronic systems, input enhancers, electromagnetic explanatory science, fragmentary multi-poles, and neuron concerns all involve fractional differential equations (FDEs), which represent distinctive areas of theoretical physics and biological sciences (see [1–4], and see the articles in [5–7] for further information). As a result, numerous fractional operators have created numerous structures for differential equations of arbitrary order. To address such a diversity of fractional operators, it has been found that accommodating generalized structures of fractional operators that involve several other operators is the most efficient approach (see [8–12]). Using several distinct definitions of fractional differential operators, Hilfer created the Hilfer fractional derivative (HFD), a new extended formulation of the fractional derivative [13]. The multivalued Hilfer fractional impulsive system of the Sobolev type was explored in [14]. The solvability and controllability properties of Hilfer FDEs have been studied by a large number of authors (refer to [6,15–17]). As an extension of the conventional limit definition of the derivative, the conformable fractional derivative was presented by Khalil et al. in [18] and complies with the conventional properties in the following ways. In addition, it agrees with the traditional definitions of Riemann–Liouville and Caputo on polynomials up to a constant multiple. These include (i) the linearity property, (ii) the product rule, (iii) the quotient rule, (iv) Rolle's theorem, and (v) the mean value theorem.

A generalization of differential equations with diverging arguments is delay differential equations, in which an unknown quantity and its derivatives occur at different values of the respective parameters. They appear in issues with integer and FDEs, applied mathematics, biomechanics, architecture, economics, and various control systems (for additional details, see [19,20] and the references therein). Stochastic differential equations (SDEs), on the other hand, are the best method for representing systems with random effects and external noise [21,22]. From the standpoint of applications, Gaussian noise has its boundaries, whereas Poisson random measures can be used to deal with circumstances in general. For instance, Hausenblas and Marchis [23] explored the river contamination model. For more detail on SDEs, refer to [22,24] and the references therein.

Rapid variations in the state variable are a defining characteristic of many real-life occurrences and processes. These modifications fall into two categories. In the first category, referred to as instantaneous impulses (IIs), changes take place gradually during a time that might be regarded as brief in comparison to the whole duration of the activity. In the second type, sometimes referred to as non-instantaneous impulses (NIIs), changes start impulsively at some locations and continue for a set amount of time. One extremely well-known example of NIIs is the entry of insulin into the bloodstream, which is an abrupt shift, and the following absorption, which is a lengthy process, as it remains active over a finite period. To model this condition, NII differential equations are employed. Examples of these processes can also be found in other academic fields, such as physics, biology, ecology, economics, and population dynamics. For recent studies on NIIs, see [25–28] and the references therein. The focus of Durga and Muthukumar's work [14] is on the best way to handle Sobolev-type Hilfer fractional NII differential inclusion driven by Poisson jumps and the Clarke subdifferential. Ahmed and Ragusa's work [29] established Sobolev-type conformable fractional stochastic evolution inclusions with the Clarke subdifferential and nonlocal conditions.

The idea of controllability, which was first put forth by Kalman [30] in 1960, significantly aided in the development of applied mathematics. See [31,32] and the references therein for more details on various controllability principles for linear and nonlinear systems. The majority of these papers focus on the search for control that guides the system from a certain initial state to a desired final state, but they do not even mention the trajectory's control path. Trajectory controllability, often known as "T-controllability", is a more robust concept of controllability that has recently been developed for nonlinear integrodifferential systems [33–35]. T-controllability (see [36]) is the ability to navigate the system via a given initial state rather than finding the control that directs the system along a predetermined route to the ultimate state to the required final destination. We point out that [37] pioneered the formulation of T-controllability issues for nonlinear integrodifferential equations in finite- and infinite-dimensional spaces. In [38], the authors looked into whether second-order evolution systems in Banach space with diverging arguments and impulses are T-controllable. In the context of the Caputo fractional derivative of order $\alpha \in (1, 2]$, the T-controllability of fractional integrodifferential equations was established in [39]. The approximation and T-controllability of fractional SDEs with non-instantaneous impulses and Poisson jumps were also addressed in [40] using the same techniques.

The significant contributions are listed below:

(i) Studying T-controllability has advantages since it may reduce some expenses associated with guiding the system from the starting state to the end desired state and because it may also protect it.
(ii) For cost-effectiveness and collision avoidance, it could be advantageous, for instance, to launch a rocket into space with an exact course and destination in mind.
(iii) We have extended the problem in Ahmed's [41] and Dimplekumar et al.'s [20] studies, compatible with NII conformable fractional stochastic differential inclusion systems, to conformable fractional stochastic differential inclusions with the Clarke subdifferential type and deviated arguments.

Due to the aforementioned fact, we take into consideration non-instantaneous impulsive (NII) conformable fractional stochastic differential inclusions of the Clarke subdifferential type with deviating arguments and Poisson jumps in the following form:

$$\begin{aligned} \mathcal{D}^{\alpha}\vartheta(\mathrm{t}) &\in \Lambda\vartheta(\mathrm{t}) + \mathfrak{B}\mathfrak{u}(\mathrm{t}) + \partial\Upsilon(\mathrm{t},\vartheta(\mathrm{t})) + \sigma(\mathrm{t},\vartheta(\mathrm{t}),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t}),\mathrm{t}))) \\ &\quad + \int_{\mathcal{Z}} \lambda(\mathrm{t},\vartheta(\mathrm{t}),\delta)\tilde{\mathbb{N}}(d\mathrm{t},d\delta), \quad \mathrm{t} \in \cup_{i=0}^{m}(\mathfrak{a}_i,\mathfrak{b}_{i+1}] \subset l' := (0,\mathrm{T}], \\ \mathfrak{a}_0 &:= 0, \ \mathfrak{b}_{m+1} := \mathrm{T}, \ \mathrm{T} > 0, \\ \vartheta(\mathrm{t}) &= \mathfrak{h}_i(\mathrm{t},\vartheta(\mathfrak{b}_i^-)), \ \mathrm{t} \in (\mathfrak{b}_i,\mathfrak{a}_i], \ i = 1,2,3,\cdots m, \\ \vartheta(0) &= \vartheta_0, \ \ \vartheta_0 \in \mathscr{H}, \end{aligned} \tag{1}$$

where $\mathcal{D}^{\alpha}$ is the conformable fractional derivative of order α, with $0 < \alpha < 1$ and $l := [0,\mathrm{T}]$, $\mathrm{T} > 0$. $\vartheta(\cdot) \in \mathscr{H}$ is the state variable in the Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle$ provided with the norm $\|\cdot\|$. Let $\Lambda : \mathcal{D}(\Lambda) \subset \mathscr{H} \to \mathscr{H}$ be the infinitesimal generator of an analytic semigroup of the bounded linear operator $\mathrm{T}(\mathrm{t})$, $\mathrm{t} \geq 0$ on $\mathscr{H}$. The deviating argument $\mathfrak{e}(\mathrm{t},\cdot)$ is the mapping from $\mathscr{H} \times \mathscr{R}^+ \to \mathscr{R}^+$. $\partial\Upsilon$ signifies the Clarke subdifferential of a locally Lipschitz function $\Upsilon(\mathrm{t},\cdot) : \mathscr{H} \to \mathscr{R}$. $\mathfrak{a}_i$ and $\mathfrak{b}_i$ represent the fixed points that satisfy $\mathfrak{a}_0 < \mathfrak{b}_1 < \mathfrak{a}_1 < \mathfrak{b}_2 < \cdots < \mathfrak{b}_i < \mathfrak{a}_i < \mathfrak{b}_{i+1}$, $i = 0,1,\cdots,m$. Also, $\mathfrak{h}_i : (\mathfrak{b}_i,\mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$ and $\vartheta(\mathfrak{b}_j^-)$ represent the left limit of ϑ at $\mathfrak{b}_i$. Let $(\Omega,\Im,\mathbb{P})$ be the complete probability space, with $\mathbb{P}$ being the probability measure on Ω and the normal filtration $\{\Im_{\mathrm{t}},\mathrm{t} \geq 0\}$. Let $\mathbb{N}(d\mathrm{t},d\delta)$ be the Poisson counting process in the measurable space $(\mathcal{Z},\mathbb{B}(\mathcal{Z}))$ defined on $(\Omega,\Im,\mathscr{P})$. $\tilde{\mathbb{N}}(d\mathrm{t},d\delta) = \mathbb{N}(d\mathrm{t},d\delta) - \mathfrak{h}(d\delta)d\mathrm{t}$ is the compensated martingale measure with a sigma-finite intensity measure $\mathfrak{h}(d\delta)$. $\mathfrak{B}$ is the bounded linear operator from a separable reflexive Banach space $\mathscr{W}$ onto $\mathscr{H}$. The nonlinear functions $\lambda : l \times \mathscr{H} \times \mathcal{Z} \setminus \{0\} \to \mathscr{H}$ and $\sigma : l \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ are continuous. Let $\mathcal{L}^2_{\Im}(l,\mathscr{W})$ be the closed subspace of $\mathscr{W}$ consisting of all measurable and $\Im_{\mathrm{t}}$-adapted, $\mathscr{W}$-valued stochastic processes satisfying $\mathbb{E}\int_0^{\mathrm{t}} \|\mathfrak{u}(\mathrm{t})\|^2_{\mathscr{W}} d\mathrm{t} < \infty$. Let $\mathbb{W}$ be a nonempty closed bounded convex subset of $\mathscr{W}$. Define $\mathscr{W}_{ad} = \{\mathfrak{u}(.) \in \mathcal{L}^2_{\Im}(l,\mathscr{W})\}$ as the admissible control set.

The novelty of our research work is listed as follows:

(i) The conformable fractional stochastic differential inclusions with the Clarke subdifferential system now include T-controllability.

(ii) We have extended the problem in [20,41] to NII conformable fractional stochastic differential inclusions and have used modified techniques to make them compatible with the T-controllability of the Clarke subdifferential system. The system (1) is more advanced than the relative system studied in Refs. [14,33–41].

(iii) Little has been written about the T-controllability of NII conformable fractional stochastic differential inclusions with the Clarke subdifferential, deviating arguments, and Poisson jumps. To close this gap, we have investigated the T-controllability of (1).

The following is a summary of the paper. The concepts and early findings needed to analyze the results are covered in Section 2. In Section 3, the solvability of the suggested system is examined. The generalized Gronwall's inequality is used in Section 4 to derive the T-controllability inclusions for (1) in the mean square. The validity of the result is reported in Section 5.

2. Preliminaries

In this section, basic definitions and lemmas are provided to establish our main results. Throughout the article, $\mathcal{L}^2(\Im,\mathscr{H}) = \mathcal{L}^2(\Omega,\Im_{\mathrm{t}},\mathbb{P},\mathscr{H})$ $(\mathrm{t} > 0)$ denotes the Hilbert space of all square integrable, strongly $\Im_{\mathrm{t}}$-measurable $\mathscr{H}$-valued random variables with $\mathbb{E}\|\vartheta\|^2 < \infty$. Let $\mathcal{L}^2_{\Im}(l,\mathscr{H})$ be the Hilbert space of all $\Im_{\mathrm{t}}$-adapted measurable functions defined on l with the values in $\mathscr{H}$ and its norm, $\|\vartheta\|_{\mathcal{L}^2_{\Im}(l,\mathscr{H})} = \left[\int_0^{\mathrm{T}} \mathbb{E}\|\vartheta(\mathrm{t})\|^2 d\mathrm{t}\right]^{1/2} < \infty$. Let $\mathscr{C}(l,\mathcal{L}^2(\Im,\mathscr{H}))$ be the Banach space of all continuous functions from l into $\mathcal{L}^2(\Im,\mathscr{H})$ equipped with the supremum norm, $\sup_{\mathrm{t}\in l} \mathbb{E}\|\vartheta(\mathrm{t})\|^2 < \infty$. Define

$$\mathfrak{X} := \mathcal{PC}\Big(l', \mathcal{L}^2(\Im, \mathcal{H})\Big) \text{ such that } \vartheta_{|\mathcal{J}_i} \in \mathscr{C}\Big(l_i, \mathcal{L}^2(\Im, \mathscr{H})\Big),\ l_i := (\mathfrak{b}_i, \mathfrak{b}_{i+1}],\ i = 0, 1, 2, \cdots, m.$$

and $\vartheta(\mathfrak{b}_i^+)$ and $\vartheta(\mathfrak{b}_i^-)$ exist and are finite for each $i = 1, 2, \cdots, m$.

Now, $\mathfrak{X}$ is a Banach space furnished with the norm $\|.\|_{\mathcal{PC}}$ given by

$$\|\vartheta\|_{\mathcal{PC}} = \left[\sup_{\mathrm{t}\in l'} \mathbb{E}\|\vartheta(\mathrm{t})\|^2\right]^{1/2}.$$

Let the Banach space be $\mathscr{X}$. $\mathbb{P}(\mathscr{X})$ denotes the set of all non-empty subsets of $\mathscr{X}$

$\mathbb{P}_{cl,bd}(\mathscr{X}) = \{\mathfrak{A} \in \mathbb{P}(\mathscr{X}) : \mathfrak{A} \text{ is closed and bounded}\}$, $\mathbb{P}_{cp}(\mathscr{X}) = \{\mathfrak{A} \in \mathbb{P}(\mathscr{X}) : \mathfrak{A} \text{ is compact}\}$, $\mathbb{P}_{cv}(\mathscr{X}) = \{\mathfrak{A} \in \mathbb{P}(\mathscr{X}) : \mathfrak{A} \text{ is convex}\}$, and $\mathbb{P}_{cd}(\mathscr{X}) = \{\mathfrak{A} \in \mathbb{P}(\mathscr{X}) : \mathfrak{A} \text{ is compact and acyclic}\}$.

The main notion of fractional calculus can be summed up in two different ways. The first method is Riemann–Liouville, which relies on iterating the integral operator n times before replacing it with a single integral through the renowned Cauchy formula, where $n!$ is transformed into the Gamma function, and the fractional integral of noninteger order is defined. Riemann and Caputo fractional derivatives are then defined using integrals. The second method is known as the Grünwald–Letnikov method, and it is based on iterating the derivative n times before fractionalizing using the Gamma function in the coefficients of the binomial distribution. The calculated fractional derivatives in this calculus were challenging and lacked certain fundamental characteristics that typical derivatives have, such as the product rule and the chain rule. A new well-behaved simple fractional derivative called "the conformable fractional derivative" is defined depending just on the basic limit definition of the derivative.

Although Mittag-Leffler functions, which are generalized exponential functions in the well-known fractional calculus, include some functions that do not have certain derivative characteristics, such as those without a Taylor power series representation or those whose Laplace transform cannot be calculated, the theory of this conformable fractional calculus will make it possible to perform these calculations.

Definition 1 ([18]). *Let $0 < \nu \leq 1$. The conformable fractional derivative of order ν of a function $\mathfrak{p}(.)$ for $\mathrm{t} > 0$ is defined as follows:*

$$\frac{d^\nu \mathfrak{p}(\mathrm{t})}{d\mathrm{t}^\nu} = \lim_{v\to 0} \frac{\mathfrak{p}(\mathrm{t} + v\mathrm{t}^{1-\nu}) - \mathfrak{p}(\mathrm{t})}{v}.$$

For the specific condition $\mathrm{t} = 0$, the following definition is derived:

$$\frac{d^\nu \mathfrak{p}(0)}{d\mathrm{t}^\nu} = \lim_{\mathrm{t}\to 0^+} \frac{d^\nu \mathfrak{p}(\mathrm{t})}{d\mathrm{t}^\nu}.$$

The fractional integral $\mathcal{I}^\nu(.)$ associated with the conformable fractional derivative of order ν of a function $s(.)$ is defined by

$$\mathcal{I}^\nu(\mathfrak{p})(\mathrm{t}) = \int_0^{\mathrm{t}} s^{\nu-1}\mathfrak{p}(s)ds.$$

Fractional-order circuit elements have been utilized to simulate various circuit types, circuits, and systems for a number of years. There are numerous types of fractional derivatives. The above-mentioned new, uncomplicated fractional derivative technique has just been introduced under the name conformable fractional derivative. It is simpler to use than other fractional derivatives and has been used to model supercapacitors. Modeling and analysis are essential in order to make full use of the novel circuit elements and analyze the circuits that contain them. Circuit theory and physics are both familiar with the two-capacitor problem; for more information, see ([42–45]).

The development of non-smooth analysis has allowed optimization theory to be expanded to include functions that may not always be differentiable. This extends the generalization of the differentiability theory related to convex functions. The methods developed by Francis Clarke [46] that apply to locally Lipschitz functions on Banach spaces have proven to be the most effective and popular. Despite the fact that this theory was created for and has been utilized in many areas of optimization theory, Simon Fitzpatrick [47] showed that it was also quite helpful in analyzing distance functions that were crucial in approximation theory. The localized Lipschitz functions that are the focus of Clarke's non-smooth analysis do exhibit differentiability qualities, and knowledge of the set of differentiable points is frequently critical. Investigating Clarke's theory and applying it to figure out whether locally Lipschitz functions are T-controllable is the main goal of this manuscript.

Definition 2 ([46]). *Let $\mathscr{X}^*$ be the dual of the Banach space $\mathscr{X}$. Clarke's generalized directional derivative of a locally Lipschitz function $\Upsilon : \mathscr{X} \to \mathscr{R}$ at ϑ in the direction ν, denoted by $\Upsilon^0(\vartheta;\nu)$, is represented by*

$$\Upsilon^0(\vartheta;\nu) = \lim_{\varpi\to\vartheta} \sup_{\lambda\to 0^+} \frac{\Upsilon(\varpi+\lambda\nu)-\Upsilon(\varpi)}{\lambda}.$$

The generalized Clarke subdifferential of Υ at ϑ, denoted by $\partial\Upsilon$, is a subset of $\mathscr{X}^*$ given by

$$\partial\Upsilon(\vartheta) = \{\vartheta^* \in \mathscr{X}^* : \Upsilon^0(\vartheta;\nu) \geq \langle\vartheta^*,\nu\rangle, \forall \nu \in \mathscr{X}\}.$$

Lemma 1 ([40] Generalized Gronwall's inequality). *If $\beta > 0$, $\tilde{a}(t)$ is a non-negative function locally integrable on $0 \leq t \leq \mathtt{T}$, and $\mathfrak{q}(t)$ is a non-negative, non-decreasing continuous function on $0 \leq t \leq \mathtt{T}$, $\mathfrak{q}(t) \leq \mathfrak{c}$, and suppose $\tilde{\mathfrak{u}}(t) \leq \tilde{a}(t) + \mathfrak{q}(t)\int_0^t (t-s)^{\beta-1}\tilde{\mathfrak{u}}(s)ds$ on this interval, then*

$$\tilde{\mathfrak{u}(t)} \leq \tilde{a}(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(\mathfrak{q}(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{\beta-1}\tilde{a}(s)ds, \quad 0 \leq t \leq \mathtt{T}.$$

In particular, when $\tilde{a}(t) = 0$, then $\tilde{\mathfrak{u}}(t) = 0\ \forall\ 0 \leq t < \mathtt{T}$.

Lemma 2 ([48]). *Let the two Banach spaces be $\mathscr{X}$ and $\mathscr{Y}$. If $\mathfrak{F} : \mathscr{X} \to \mathbb{P}_{cp}(\mathscr{Y})$ is a closed compact multivalued function, then $\mathfrak{F}$ is u.s.c.*

Now, we provide some essential definitions for T-controllability.

Let $\mathfrak{T}$ be the collection of all functions $\pi(.)$ defined on $[0,\mathrm{T}]$ with $\pi(0) = \vartheta_0$ and $\pi(\mathrm{T}) = \vartheta_{\mathrm{T}}$, where ϑ_{T} is the reachable state at time T. Moreover, the conformable fractional derivative $\mathcal{D}^\alpha\vartheta(\mathrm{t})$ exists and is continuous a.e. on l'. The set of all achievable trajectories, the (control) model (1), is represented by $\mathfrak{T}$.

Definition 3. *System (1) is said to be T-controllable on l if $\forall\ \vartheta \in \mathfrak{T}$, $\exists$ a control $\mathfrak{u} \in \mathcal{L}^2_{\mathfrak{F}}(l,\mathcal{U})$ where the mild solution $\vartheta(.)$ of (1) satisfies $\vartheta(\mathrm{t}) = \pi(\mathrm{t})$ a.e.*

Theorem 1 ([49]). *Let $\mathfrak{U}$ be an open subset of a Banach space $\mathscr{X}$. Let $\Psi_1 : \overline{\mathfrak{U}} \to \mathscr{X}$ [$\overline{\mathfrak{U}}$, denoting the closure of $\mathfrak{U}$ in $\mathscr{X}$], be single-valued and $\Psi_2 : \overline{\mathfrak{U}} \to \mathbb{P}_{cp,cv}(\mathscr{X})$ be a multivalued operator with $\Psi_1(\overline{\mathfrak{U}}) + \Psi_2(\overline{\mathfrak{U}})$ bounded. If*

(i) Ψ_1 is a contraction with a contraction constant k and

(ii) Ψ_2 is u.s.c and compact,

then either

1. *The operator inclusion $\psi\vartheta \in \Psi_1\vartheta + \Psi_2\vartheta$ has a solution for $\psi = 1$ or*
2. *An element $\mathfrak{a} \in \partial\mathfrak{U}$ such that $\psi\mathfrak{a} \in \Psi_1\mathfrak{a} + \Psi_2\mathfrak{a}$ for some $\psi > 1$, where $\partial\mathfrak{U}$ is the boundary of $\mathfrak{U}$ in $\mathscr{X}$.*

3. Existence Results

With the use of Theorem 1, the solvability results of the system (1) is established.

Definition 4. *An $\Im_t$-adapted stochastic process $\vartheta(t) \in \mathscr{X} : t \in l'$ is said to be a mild solution of (1) when the following conditions hold:*

(i) $\exists$ *an $\Im_t$-adapted measurable function $\omega \in \mathcal{L}^2(\Im, \mathscr{H})$ such that $\omega(t) \in \partial Y(t, \vartheta(t))$ for a.e. $t \in l'$;*

(ii) $\vartheta(t) \in \mathscr{H}$ *has a cadlag path on $t \in l'$ a.s., and the following stochastic integral is satisfied:*

$$\vartheta(t) = \begin{cases} \mathcal{T}\left(\frac{t^\alpha}{\alpha}\right)\vartheta_0 + \int_0^t s^{\alpha-1}\mathcal{T}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}u(s) + \omega(s, \vartheta(s)) + \sigma(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s)))]ds \\ + \int_0^t \int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(s, \vartheta(s), \delta(s))\tilde{N}(ds, d\delta), \quad t \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\big(t, \vartheta(\mathfrak{b}_i^-)\big), \quad i = 1, 2, \cdots m, \\ \mathcal{T}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\big(s, \vartheta(\mathfrak{b}_i^-)\big) + \int_{\mathfrak{a}_i}^t s^{\alpha-1}\mathcal{T}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\big[\mathfrak{B}u(s) + \omega(s, \vartheta(s)) + \sigma\big(s, \vartheta(s), \\ \vartheta(\mathfrak{e}(\vartheta(s), s))\big)\big]ds + \int_{\mathfrak{a}_i}^t \int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\lambda(s, \vartheta(s), \delta(s))\tilde{N}(ds, d\delta), t \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases} \quad (2)$$

In order to prove our results, the following hypotheses are necessary:

(A1) The linear operator $\mathfrak{A} : \mathscr{H} \to \mathscr{H}$ generates a $\mathcal{C}_0$-semigroup $\mathcal{T}(.)$. Thus, $\exists\, \mathbb{M} > 0$ that is constant such that $\|\mathcal{T}(t)\| \leq \mathbb{M}$.

(A2) Let $Y : l \times \mathscr{H} \to \mathscr{R}$ satisfy the following conditions:

- (i) $Y(., \vartheta)$ is measurable for all $\vartheta \in \mathscr{H}$.
- (ii) $Y(t, .)$ is locally Lipschitz continuous for a.e. $t \in l$.
- (iii) There exist a function $\mathfrak{r}_1 \in \mathcal{L}^2(l, \mathscr{R}^+)$ and a constant $\varsigma_\omega \geq 0$ such that

$$\mathbb{E}\|\partial Y(t, \vartheta(t))\|^2 = \sup\{\|\omega(t)\|^2 : \omega(t, \vartheta(t)) \in \partial Y(t, \vartheta(t))\} \leq \mathfrak{r}_1(t) + \varsigma_\omega \mathbb{E}\|\vartheta\|^2, \ \forall \ \vartheta \in \mathscr{H} \text{ and } a.e. t \in l.$$

(A3) The Lipschitz continuity of $\sigma : l \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$: For $\mathfrak{x}_1, \mathfrak{x}_2, y_1, y_2 \in \mathscr{H}$ and $\mathbb{M}_\sigma > 0$.

$$\begin{aligned} \mathbb{E}\|\sigma(t, \mathfrak{x}_1, y_1) - \sigma(t, \mathfrak{x}_2, y_2)\|^2 &\leq \mathbb{M}_\sigma\Big[\mathbb{E}\|\mathfrak{x}_1 - \mathfrak{x}_2\|^2 + \|y_1 - y_2\|^2\Big], \\ \mathbb{E}\|\sigma(\cdot, 0, \vartheta(0))\|^2 &\leq \tilde{\sigma_0}. \end{aligned}$$

(A4) Let $\mathfrak{e} : \mathscr{H} \times \mathscr{R}^+ \to \mathscr{R}^+$ be Lipschitz continuous. For all $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{H}$ and $\mathbb{M}_\mathfrak{e} > 0 \ni$,

$$\mathbb{E}|\mathfrak{e}(\vartheta_1, t) - \mathfrak{e}(\vartheta_2, t)|^2_{\mathscr{R}^+} \leq \mathbb{M}_\mathfrak{e}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2$$

and $\mathfrak{e}(., 0) = 0$.

(A5) $\lambda : l \times \mathscr{H} \times \mathcal{Z} \setminus \{0\} \to \mathscr{H}$ is the Lipschitz constant $\mathbb{M}_\lambda$ a.e. $t \in l$. There exist a function $\mathfrak{r}_2 \in \mathcal{L}^2(l, \mathscr{R}^+)$ and a positive constant ς_λ such that

$$\int_{\mathcal{Z}} \mathbb{E}\|\lambda(t, \vartheta, \delta)h(d\delta)\|^2 \leq \mathfrak{r}_2(t) + \varsigma_\lambda \mathbb{E}\|\vartheta\|^2.$$

(A6) (i) $\mathfrak{h}_i : [\mathfrak{b}_i, \mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$ such that $\mathfrak{h}_i(\cdot, \vartheta)$ is continuous $\forall\, \vartheta \in \mathscr{H}$ and $i = 1, 2, \cdots, m$.
(ii) $\mathfrak{h}_i : [\mathfrak{b}_i, \mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$, $i = 1, 2, \cdots m$, is uniformly continuous on bounded sets, and for $t \in [\mathfrak{b}_i, \mathfrak{a}_i]$, $\mathfrak{h}_i(t, \cdot)$ is a mapping from any bounded subsets of $\mathscr{H}$ into relatively compact subsets of $\mathscr{H}$. Also, $\exists$ positive constants $\mathfrak{f}_i$ with

$$\mathbb{E}\|\mathfrak{h}_i(t, \vartheta_1) - \mathfrak{h}_i(t, \vartheta_2)\|^2 \leq \mathfrak{f}_i\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2, \ t \in [\mathfrak{b}_i, \mathfrak{a}_i], \ \vartheta_1, \vartheta_2 \in \mathscr{H}.$$

Let the multivalued operator $\mathfrak{S} : \mathcal{L}^2_\Im(l, \mathscr{H}) \to \mathbb{P}\big(\mathcal{L}^2_\Im(l, \mathscr{H})\big)$ be defined by

$$\mathfrak{S}(\vartheta) = \Big\{\omega \in \mathcal{L}^2_\Im(l, \mathscr{H}) : \omega(t, \vartheta(t)) \in \partial Y(t, \vartheta(t)), \ a.e. t \in l\Big\}, \ \forall \ \vartheta \in \mathcal{L}^2_\Im(l, \mathscr{H}).$$

Lemma 3. *Suppose that (A1) and (A2) hold. Then, $\mathfrak{S}$ satisfies the following: if $\vartheta_n \to \vartheta$ in $\mathcal{L}^2_{\Im}(l, \mathscr{H})$, $\mathfrak{z}_n \to \mathfrak{z}$ weakly in $\mathcal{L}^2_{\Im}(l, \mathscr{H})$, and $\mathfrak{z}_n \in \mathfrak{S}(\vartheta_n)$, then $\mathfrak{z} \in \mathfrak{S}(\vartheta)$.*

Lemma 4. *If (A1) and (A2) are satisfied for $\vartheta \in \mathcal{L}^2_{\Im}(l, \mathscr{H})$, $\mathfrak{S}(\vartheta)$ is non-empty and convex and has weakly compact values.*

Theorem 2. *For $\forall\ \vartheta_0 \in \mathscr{H}$, the system (1) has a mild solution, assuming that (A1)–(A6) hold, if*

$$\mathfrak{C} = \max_{i=1,2,\cdots,m} \left\{ \frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\mathbb{M}_\lambda], \mathfrak{f}_i, 3\mathbb{M}^2\mathfrak{f}_i + \frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\mathbb{M}_\lambda] \right\} < 1. \tag{3}$$

Proof. Define the multivalued operator $\Phi : \mathscr{X} \to 2^{\mathscr{X}}$ as $\Phi(\vartheta) = \{\mathrm{y} \in \mathscr{X} : \mathrm{y}(\mathfrak{t}) = \eta(\mathfrak{t})\} \ni$:

$$\eta(\mathfrak{t}) = \begin{cases} \mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s)) + \sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s)))]ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \quad \mathfrak{t} \in (0,\mathfrak{b}_1], \\ \mathfrak{h}_i\big(\mathfrak{t},\vartheta(\mathfrak{b}_i^-)\big), \quad i = 1,2,\cdots m, \\ \mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\big(s,\vartheta(\mathfrak{b}_i^-)\big) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\big[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s)) + \sigma\big(s,\vartheta(s), \\ \vartheta(\mathfrak{e}(\vartheta(s),s))\big)\big]ds + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \mathfrak{t} \in [\mathfrak{a}_i,\mathfrak{b}_{i+1}]. \end{cases}$$

We may decompose the operator $\Phi(\vartheta)$ into two components, where $\Phi_1(\vartheta) = \{\mathrm{y} \in \mathscr{X} : \mathrm{y}(\mathfrak{t}) = \eta_1(\mathfrak{t})\} \ni$:

$$\eta_1(\mathfrak{t}) = \begin{cases} \mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s)))ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \quad \mathfrak{t} \in (0,\mathfrak{b}_1], \\ \mathfrak{h}_i\big(\mathfrak{t},\vartheta(\mathfrak{b}_i^-)\big), \quad i = 1,2,\cdots m, \\ \mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\big(s,\vartheta(\mathfrak{b}_i^-)\big) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\sigma\big(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s))\big)ds \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \mathfrak{t} \in [\mathfrak{a}_i,\mathfrak{b}_{i+1}]. \end{cases}$$

and $\Phi_2(\vartheta) = \{\mathrm{y} \in \mathscr{X} : \mathrm{y}(\mathfrak{t}) = \eta_2(\mathfrak{t})\}$ such that

$$\eta_2(\mathfrak{t}) = \begin{cases} \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s))], \quad \mathfrak{t} \in (0,\mathfrak{b}_1], \\ 0, \quad \mathfrak{t} \in (\mathfrak{b}_i,\mathfrak{a}_i] \quad i = 1,2,\cdots m, \\ \int_{\mathfrak{a}_i}^{\mathfrak{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\big[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s))\big]ds, \quad \mathfrak{t} \in [\mathfrak{a}_i,\mathfrak{b}_{i+1}]. \end{cases}$$

Step 1. Claim: Φ_1 is a contraction mapping.

Define $\mathscr{B}_\mathfrak{l} = \{\vartheta \in \mathscr{X} : \mathbb{E}\|\vartheta\|^2 \leq \mathfrak{l}\}$ for $\mathfrak{l} > 0$. Now, for $\mathfrak{t} \in (0,\mathfrak{b}_1]$, $\vartheta_1, \vartheta_2 \in \mathscr{B}_\mathfrak{l}$, using (A3)–(A6), we have

$$
\begin{aligned}
& \mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2_{\mathcal{PC}} \\
= \; & \sup_{\mathrm{t}\in l'} \mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2 \\
\leq \; & \sup_{\mathrm{t}\in l'} \bigg[\mathbb{E}\bigg\| \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)[\sigma(s,\vartheta_1(s),\vartheta_1(\mathfrak{e}(\vartheta_1(s),s))) - \sigma(s,\vartheta_2(s),\vartheta_2(\mathfrak{e}(\vartheta_2(s),s)))]ds \\
+ \; & \int_0^{\mathrm{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)[\lambda(s,\vartheta_1(s),\delta(s)) - \lambda(s,\vartheta_2(s),\delta(s))]\tilde{\mathbb{N}}(ds,d\delta)\bigg\|^2\bigg] \\
\leq \; & 2\sup_{\mathrm{t}\in l'} \bigg[\mathbb{E}\bigg\| \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)\bigg[\sigma(s,\vartheta_1(s),\vartheta_1(\mathfrak{e}(\vartheta_1(s),s))) - \sigma(s,\vartheta_1(s),\vartheta_1(\mathfrak{e}(\vartheta_2(s),s))) \\
- \; & \sigma(s,\vartheta_2(s),\vartheta_2(\mathfrak{e}(\vartheta_2(s),s))) + \sigma(s,\vartheta_1(s),\vartheta_1(\mathfrak{e}(\vartheta_2(s),s)))\bigg]ds\bigg\|^2 \\
+ \; & \mathbb{E}\bigg\| \int_0^{\mathrm{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)[\lambda(s,\vartheta_1(s),\delta(s)) - \lambda(s,\vartheta_2(s),\delta(s))]\tilde{\mathbb{N}}(ds,d\delta)\bigg\|^2\bigg] \\
\leq \; & 2\sup_{\mathrm{t}\in l'} \bigg[2\mathfrak{b}_1\mathbb{M}^2\mathbb{M}_\sigma\bigg(\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\vartheta_1(\mathfrak{e}(\vartheta_1(s),s)) - \vartheta_1(\mathfrak{e}(\vartheta_2(s),s))\|^2 ds + \int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2 \\
\times \; & \|\mathfrak{e}(\vartheta_2(s),s)\|^2 ds\bigg) + \mathfrak{b}_1\mathbb{M}^2\mathbb{M}_\lambda \int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\vartheta_1(s) - \vartheta_2(s)\|^2 ds\bigg] \\
\leq \; & \frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1) + \mathbb{M}_\lambda] \sup_{s\in(0,\mathfrak{b}_1)} \mathbb{E}\|\vartheta_1(s) - \vartheta_2(s)\|^2.
\end{aligned}
$$

For $\mathrm{t} \in (\mathfrak{b}_i, \mathfrak{a}_i]$, we have

$$
\begin{aligned}
\mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2_{\mathcal{PC}} &= \sup_{\mathrm{t}\in l'} \mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2 \\
&\leq \sup_{\mathrm{t}\in l'} \mathbb{E}\big\|\mathfrak{h}_i\big(\mathrm{t},\vartheta_1(\mathfrak{b}_i^-)\big) - \mathfrak{h}_i\big(\mathrm{t},\vartheta_2(\mathfrak{b}_i^-)\big)\big\|^2 \\
&\leq \mathfrak{f}_i \sup_{\mathrm{t}\in(\mathfrak{b}_i,\mathfrak{a}_i]} \mathbb{E}\|\vartheta_1(\mathrm{t}) - \vartheta_2(\mathrm{t})\|^2, \;\; i = 1,2,\cdots,m.
\end{aligned}
$$

For $\mathrm{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i = 1,2,\cdots,m$,

$$
\begin{aligned}
& \mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2_{\mathcal{PC}} \\
= \; & \sup_{\mathrm{t}\in l'} \mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2 \\
\leq \; & \sup_{\mathrm{t}\in l'} \mathbb{E}\bigg\|\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)\big[\mathfrak{h}_i\big(s,\vartheta_1(\mathfrak{b}_i^-)\big) - \mathfrak{h}_i\big(s,\vartheta_2(\mathfrak{b}_i^-)\big)\big] + \int_{\mathfrak{a}_i}^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg) \\
\times \; & [\sigma(s,\vartheta_1(s),\vartheta_1(\mathfrak{e}(\vartheta_1(s),s))) - \sigma(s,\vartheta_2(s),\vartheta_2(\mathfrak{e}(\vartheta_2(s),s)))]ds \\
+ \; & \int_{\mathfrak{a}_i}^{\mathrm{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\bigg(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\bigg)[\lambda(s,\vartheta_1(s),\delta(s)) - \lambda(s,\vartheta_2(s),\delta(s))]\tilde{\mathbb{N}}(ds,d\delta)\bigg\|^2 \\
\leq \; & 3\mathbb{M}^2\mathfrak{f}_i + 3\frac{\mathrm{T}^{2\alpha}\mathbb{M}^2}{2\alpha-1}[2\mathbb{M}_\sigma\mathfrak{l}(2\mathbb{M}_\mathfrak{e}+1)\mathbb{M}_\lambda] \sup_{s\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}]} \mathbb{E}\|\vartheta_1(s) - \vartheta_2(s)\|^2.
\end{aligned}
$$

Therefore, for $\mathrm{t} \in l$,

$$
\mathbb{E}\|(\Phi_1\vartheta_1)(\mathrm{t}) - (\Phi_1\vartheta_2)(\mathrm{t})\|^2_{\mathcal{PC}} \leq \mathfrak{C}_1 \sup_{\mathrm{t}\in l'} \mathbb{E}\|\vartheta_1(\mathrm{t}) - \vartheta_2(\mathrm{t})\|^2.
$$

where $\mathfrak{C}_1 = \max_{i\in\{1,2,\cdots,m\}} \left[3\mathbb{M}^2\mathfrak{f}_i + \frac{3\mathrm{T}^{2\alpha}\mathbb{M}^2}{2\alpha-1}(2\mathbb{M}_\sigma\mathfrak{l}(2\mathbb{M}_\mathfrak{e}+1)+\mathbb{M}_\lambda)\right]$.

$\Longrightarrow$ Φ_1 is a contraction.

Step 2. Claim: Φ_2 is convex for $\vartheta \in \mathscr{X}$.

If $\mathrm{y}_1, \mathrm{y}_2 \in \mathcal{F}_2(\vartheta)$, then there exist $\omega_1, \omega_2 \in \mathfrak{S}(\vartheta)$ such that $\ni \mathrm{t} \in (0, \mathfrak{b}_1]$, and we have

$$\begin{aligned} \mathrm{y}_1(\mathrm{t}) &= \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathrm{u}(s) + \omega_1(s,\vartheta(s))]ds \\ \mathrm{y}_2(\mathrm{t}) &= \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathrm{u}(s) + \omega_2(s,\vartheta(s))]ds. \end{aligned}$$

We may consider $0 \le \kappa \le 1$; then, for $\mathrm{t} \in (0, \mathfrak{b}_1]$, we have

$$(\kappa\mathrm{y}_1 + (1-\kappa)\mathrm{y}_2)(\mathrm{t}) = \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathrm{u}(s) + (\kappa\omega_1 + (1-\kappa)\omega_2)(s,\vartheta(s))]ds.$$

Using Lemma 2, we can obtain the convexity of $\mathfrak{S}(\vartheta)$ and $(\kappa\omega_1 + (1-\kappa)\omega_2) \in \mathfrak{S}(\mathrm{t})$ for $\mathrm{t} \in \mathfrak{S}(\mathrm{t})$. Hence, $(\kappa\mathrm{y}_1 + (1-\kappa)\mathrm{y}_2) \in \mathfrak{S}(\mathrm{t})$ for $\mathrm{t} \in (0, \mathfrak{b}_1]$. The result also holds for $\mathrm{t} \in (\mathfrak{a}_i, \mathfrak{b}_{i+1}]$. Thus, $\Phi_2(\mathrm{t})$ is convex. By Lemma 2, obviously, $\Phi_2(\mathrm{t})$ is non-empty and has weakly compact values for all $\vartheta \in \mathscr{X}$.

Step 3. Claim: $\Phi_2(\vartheta)(\vartheta)$ maps $\mathscr{B}_\mathfrak{l} \to \mathscr{B}_\mathfrak{l}$ in $\mathscr{X}$.

For $\vartheta \in \mathscr{B}_\mathfrak{l}$ and $\mathrm{t} \in (0, \mathfrak{b}_1]$, by using Holder's inequality,

$$\begin{aligned} \mathbb{E}\|\Phi_2(\vartheta)\|^2_{\mathcal{PC}} &= \mathbb{E}\left\|\int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\mathrm{u}(s) + \omega(s,\vartheta(s))]ds\right\|^2_{\mathcal{PC}} \\ &\le \mathbb{M}^2\mathfrak{b}_1\left[\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathcal{L}^2_\Im(l,\mathscr{W})} + \int_0^{\mathrm{t}} s^{2\alpha-2}[\mathfrak{r}_2(s) + \varsigma_\omega\mathfrak{l}]ds\right] \\ &\le \mathbb{M}^2\mathfrak{b}_1\left[\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathcal{L}^2_\Im(l,\mathscr{W})} + \frac{\mathfrak{b}_1^{2\alpha-1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_\Im(l,\mathscr{R}^+)} + \frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\varsigma_\omega\mathfrak{l}\right] \\ &\le \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathcal{L}^2_\Im(l,\mathscr{W})} + \varsigma_\omega\mathfrak{l}\right] + \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_\Im(l,\mathscr{R}^+)} \\ &\le \tilde{\mathfrak{C}}_1. \end{aligned}$$

For $\mathrm{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i = 1, 2, \cdots, m$,

$$\begin{aligned} \mathbb{E}\|\Phi_2(\vartheta)\|^2_{\mathcal{PC}} &= \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\mathrm{u}(s) + \omega(s,\vartheta(s))]ds\right\|^2_{\mathcal{PC}} \\ &\le \frac{\mathbb{M}^2\mathrm{T}^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\mathrm{u}\|^2_{\mathcal{L}^2_\Im(l,\mathscr{W})} + \varsigma_\omega\mathfrak{l}\right] + \frac{\mathbb{M}^2\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_\Im(l,\mathscr{R}^+)} \\ &\le \tilde{\mathfrak{C}}_2. \end{aligned}$$

Hence, $\mathbb{E}\|\Phi_2(\vartheta)\|_{\mathcal{PC}} \le \tilde{\mathfrak{C}}_3$, where $\tilde{\mathfrak{C}}_3 = \max\{\tilde{\mathfrak{C}}_1, \tilde{\mathfrak{C}}_2\}$.

Step 4. Claim: $\{\Phi_2(\vartheta) : \vartheta \in \mathscr{B}_\mathfrak{l}\}$ is equicontinuous.

For every $\vartheta \in \mathscr{B}_\mathfrak{l}$ and $\mathrm{t} \in (0, \mathfrak{b}_1]$, when $\mathrm{t}_1 = 0$, $0 < \mathrm{t}_2 < \epsilon_0$, and ϵ_0 is sufficiently small, it follows that

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}\\
\leq\ &\mathbb{E}\left\|\int_0^{\mathfrak{t}_2} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\right\|^2\\
\leq\ &2\mathbb{M}^2\frac{\mathfrak{t}_2^{2\alpha}}{2\alpha-1}\left[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+2\mathbb{M}^2\frac{\mathfrak{t}_2^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\\
\leq\ &2\mathbb{M}^2\frac{\epsilon_0^{2\alpha}}{2\alpha-1}\left[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+2\mathbb{M}^2\frac{\epsilon_0^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\\
\to\ &0 \text{ as } \epsilon_0\to 0 \text{ for } \vartheta\in\mathscr{B}_{\mathfrak{l}}.
\end{aligned}
$$

In a similar manner, $\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}\to 0$ as $\epsilon_0\to 0$ for $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i=1,2,\cdots,m$. Likewise, for $\mathfrak{t}\in\mathscr{B}_{\mathfrak{l}}$, let $0<\mathfrak{t}_1<\mathfrak{t}_2<\mathfrak{b}_1$; there exists $\omega\in\mathfrak{S}(\vartheta)$ and for all $\vartheta\in(0,\mathfrak{b}_1]$.

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}\\
=\ &\mathbb{E}\left\|\int_0^{\mathfrak{t}_2} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\right.\\
-\ &\left.\int_0^{\mathfrak{t}_1} s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\right\|^2\\
\leq\ &\mathbb{E}\left\|\int_0^{\mathfrak{t}_1-\epsilon} s^{\alpha-1}\left[\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\right.\\
+\ &\int_{\mathfrak{t}_1-\epsilon}^{\mathfrak{t}_1} s^{\alpha-1}\left[\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\\
+\ &\left.\int_{\mathfrak{t}_2}^{\mathfrak{t}_1} s^{\alpha-1}\left[\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds\right\|^2\\
\leq\ &6\left[\left(\frac{(\mathfrak{t}_1-\epsilon)^{2\alpha}}{2\alpha-1}\left[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]\right.\right.\\
+\ &\left.\frac{(\mathfrak{t}_1-\epsilon)^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\right)\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\left\|\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)\right.\\
-\ &\left.\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right\|^2+\left(\frac{\epsilon^{2\alpha}}{2\alpha-1}\left[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+\frac{\epsilon^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\right)\\
\times\ &\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\left\|\mathcal{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathcal{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right\|^2+\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha}}{2\alpha-1}\mathbb{M}^2\left[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]\\
+\ &\left.\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha-1/2}}{2\alpha-1/2}\mathbb{M}^2\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\right].
\end{aligned}
$$

In a similar way, for $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i=1,2,\cdots,m$, $\exists\,\omega\in\mathfrak{S}(\vartheta)\ni$,

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}\\
\leq\ & 6\Bigg[\Bigg(\frac{(\mathfrak{t}_1-\epsilon-\mathfrak{a}_i)^{2\alpha}}{2\alpha-1}\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]\\
+\ & \frac{(\mathfrak{t}_1-\epsilon-\mathfrak{a}_i)^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Bigg)\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Bigg\|\mathcal{T}\Bigg(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Bigg)\\
-\ & \mathcal{T}\Bigg(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Bigg)\Bigg\|^2+\frac{\epsilon^{2\alpha}}{2\alpha-1}\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]+\frac{\epsilon^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Bigg)\\
\times\ & \sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Bigg\|\mathcal{T}\Bigg(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Bigg)-\mathcal{T}\Bigg(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Bigg)\Bigg\|^2+\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha}}{2\alpha-1}\mathbb{M}^2\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]\\
+\ & \frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha-1/2}}{2\alpha-1/2}\mathbb{M}^2\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Bigg].
\end{aligned}
$$

$\mathfrak{t}\in(0,\mathfrak{b}_1]$ and $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$ are independent of ϑ and approach zero as $\mathfrak{t}_2\to\mathfrak{t}_1$ and $\epsilon\to 0$. Thus, $\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2\to 0$ as $\epsilon\to 0$ independent of $\vartheta\in\mathscr{B}_{\mathfrak{l}}$, from which it follows that $\{\Phi_2(\vartheta),\vartheta\in\mathscr{B}_{\mathfrak{l}}\}$ is equicontinuous.

Step 5. Claim: $\Phi_2(\vartheta)$ is completely continuous.

Let ξ be a real number and $\mathfrak{t}\in l$ be fixed with $0<\xi<\mathfrak{t}$. The set $\pi(\mathfrak{t})=\{\Phi_2(\mathfrak{t})\}$ is relatively compact. We may define

$$
(\Phi_2^\xi\vartheta)(\mathfrak{t})=\begin{cases}\int_0^{\mathfrak{t}-\xi}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta^\xi(s))\right]ds, & \mathfrak{t}\in[0,\mathfrak{b}_1],\\ 0, \qquad \mathfrak{t}\in(\mathfrak{b}_i,\mathfrak{a}_i], & i=1,2,\cdots,m,\\ \int_{\mathfrak{a}_i}^{\mathfrak{t}-\xi}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta^\xi(s))\right]ds, & \mathfrak{t}\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}].\end{cases}
$$

Since $\mathcal{T}(\cdot)$ is compact, the set $\pi^\epsilon(\mathfrak{t})=\{\Phi_2^\epsilon(\mathfrak{t})\}$ is relatively compact. Now, for each $0<\xi<\mathfrak{t}$ and $\mathfrak{t}\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i=1,2,\cdots$, we obtain

$$
\begin{aligned}
\mathbb{E}\left\|\Phi_2(\mathfrak{t})-\phi_2^\xi(\mathfrak{t})\right\|^2 \leq\ & 2\mathbb{E}\left\|\int_{\mathfrak{t}-\xi}^{\mathfrak{t}}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\mathfrak{B}\mathfrak{u}(s)ds\right\|^2\\
+\ & 2\mathbb{E}\left\|\int_{\mathfrak{t}-\xi}^{\mathfrak{t}}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\omega(s,\vartheta(s))-\omega(s,\vartheta^\xi(s))\right]ds\right\|^2
\end{aligned}
$$

We see that, when $\xi\to 0$, the above inequality tends to zero. Thus, the set $\pi(\mathfrak{t})$ is relatively compact. Thus, from Step 4 and the Arzela Ascoli theorem, Φ_2 is completely continuous.

Step 6. Claim: $\Phi_2(\vartheta)$ has a closed graph.

Let $\tilde{\vartheta}_n\to\tilde{\vartheta}_*$ in $\mathscr{X}$ and $\tilde{\mathrm{y}}_n\to\tilde{\mathrm{y}}_*$ in $\mathscr{X}$. Our aim is to show that $\tilde{\mathrm{y}}_*\in\Phi_2(\tilde{\vartheta}_*)$. Let $\tilde{\mathrm{y}}_n\in\Phi_2(\tilde{\vartheta}_n)$; then, $\exists\,\omega(s,\tilde{\vartheta}_n)\in\mathfrak{S}(\tilde{\vartheta}_n)$ with

$$
\tilde{\mathrm{y}}_n(\mathfrak{t})=\int_0^{\mathfrak{t}}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\vartheta(s))]ds \tag{4}
$$

With the use of (A2) (iii),

$$
\{(\omega(s,\vartheta_n(s)))\}_{n\geq 1}\subseteq\mathcal{L}^2_{\Im}(l,\mathscr{H})\ \text{ is bounded.} \tag{5}
$$

Let us consider the subsequence $\omega(s,\tilde{\vartheta}_n(s))\to\omega(s,\tilde{\vartheta}_*(s))$ weakly in $\mathcal{L}^2_{\Im}(l,\mathscr{H})$. From the condition of the compactness of $\mathcal{T}(\mathfrak{t})$, using (A2) and Equations (4) and (5),

$$
\tilde{\vartheta}_n(\mathfrak{t})\to\int_0^{\mathfrak{t}}s^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\mathfrak{B}\mathfrak{u}(s)+\omega(s,\tilde{\vartheta}_*(s))\right]ds \tag{6}
$$

We may note that $\tilde{y}_n \to \tilde{y}_*$ in $\mathscr{X}$ and $\omega(s,\tilde{\vartheta}_n) \in \mathfrak{S}(\tilde{\vartheta}_n)$. Lemma 4 and (6) give $\omega(s,\tilde{\vartheta}_*) \in \mathfrak{S}(\tilde{\vartheta}_*)$. Hence, $\tilde{\vartheta}_* \in \Phi_2(\tilde{\vartheta}_*)$. Thus, $\Phi_2(\vartheta)$ has a closed graph. Φ_2 is upper semi-continuous. Thus, $\Phi_2(\vartheta)$ has a closed graph.

Step 7. The operator inclusion $\pi\vartheta \in \Phi_1(\vartheta) + \Phi_2(\vartheta)$ has a solution for $\pi = 1$.

With the use of Theorem 1, it is enough to claim that there is no $\vartheta \in \mathscr{X}$ that satisfies $\pi\vartheta \in \Phi_1(\vartheta) + \Phi_2(\vartheta)$ for $\pi > 1$, and there exists $\omega \in \mathfrak{S}(\vartheta)$ with

$$\vartheta(\mathrm{t}) = \begin{cases} \mathcal{T}\left(\frac{\mathrm{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s)) + \sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s)))]ds \\ + \int_0^{\mathrm{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \quad \mathrm{t} \in (0,\mathfrak{b}_1], \\ \mathfrak{h}_i(\mathrm{t},\vartheta(\mathfrak{b}_i^-)), \quad i = 1,2,\cdots m, \\ \mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i(s,\vartheta(\mathfrak{b}_i^-)) + \int_{\mathfrak{a}_i}^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)[\mathfrak{B}\mathfrak{u}(s) + \omega(s,\vartheta(s)) + \sigma(s,\vartheta(s), \\ \vartheta(\mathfrak{e}(\vartheta(s),s)))]ds + \int_{\mathfrak{a}_i}^{\mathrm{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\lambda(s,\vartheta(s),\delta(s))\tilde{\mathbb{N}}(ds,d\delta), \mathrm{t} \in [\mathfrak{a}_i,\mathfrak{b}_{i+1}]. \end{cases}$$

For $\mathrm{t} \in (0,\mathfrak{b}_1]$, $\exists\, \omega \in \mathfrak{S}(\vartheta)$,

$$\begin{aligned} &\mathbb{E}\|\vartheta(\mathrm{t})\|^2_{\mathcal{PC}} \\ = \;& 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\|\mathfrak{B}\|^2\mathfrak{b}_1\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\mathfrak{u}(s)\|^2 ds + \mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\Big[\mathfrak{r}_1(s) + \varsigma_\omega\mathbb{E}\|\vartheta(s)\|^2\Big]ds \\ + \;& 2\mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s))) - \sigma(s,0,\vartheta(\mathfrak{e}(\vartheta(0),0)))\|^2 ds \\ + \;& 2\mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\sigma(s,0,\vartheta(\mathfrak{e}(\vartheta(0),0)))\|^2 ds + \mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\Big[\mathfrak{r}_2(s) + \varsigma_\omega\mathbb{E}\|\vartheta(s)\|^2\Big]ds\Big] \\ \leq \;& 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\Big(\|\mathfrak{B}\|^2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})} + 2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\sigma_0\Big) + \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\Big[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)} \\ + \;& \|\mathfrak{r}_2\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Big] + \mathfrak{b}_1\mathbb{M}^2[\varsigma_\omega + 2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1) + \varsigma_\lambda]\int_0^{\mathrm{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2 ds\Big] \\ \leq \;& \mathbb{S}_1 + \mathbb{S}_2\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2 ds. \end{aligned}$$

where

$$\begin{aligned} \mathbb{S}_1 &= 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\Big(\|\mathfrak{B}\|^2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})} + 2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\sigma_0\Big) \\ &+ \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\Big[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)} + \|\mathfrak{r}_2\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Big] \\ \mathbb{S}_2 &= 5\mathfrak{b}_1\mathbb{M}^2[\varsigma_\omega + 2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1) + \varsigma_\lambda]. \end{aligned}$$

By the generalized Gronwall's inequality,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2_{\mathcal{PC}} \leq \mathbb{S}_1 e^{\mathbb{S}_2\mathrm{t}} = \mathfrak{d}_1. \tag{7}$$

For $\mathrm{t} \in [\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i = 1,2,\cdots,m$, $\exists\,\omega \in \mathfrak{S}(\vartheta)$ such that

$$\begin{aligned} \mathbb{E}\|\vartheta(\mathrm{t})\|^2 \leq \;& 5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i\mathbb{E}\|\vartheta(\mathrm{t})\|^2 + 5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\Im}(l,\mathscr{W})} + 2\sigma_0\Big] \\ + \;& 5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\Big[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)} + \|\mathfrak{r}_2\|_{\mathcal{L}^2_{\Im}(l,\mathscr{R}^+)}\Big] \\ + \;& 5\mathrm{T}\mathbb{M}^2[\varsigma_\omega + 2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1) + \varsigma_\lambda]\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2 ds. \end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\|\vartheta(\mathrm{t})\|^2 &\leq \frac{1}{1-5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i}\Bigg[\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{W})}+2\sigma_0\Big]+5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\Big[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{R}^+)}\\ &+ \|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{R}^+)}\Big]+5\mathrm{T}\mathbb{M}^2[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda]\int_{\mathfrak{a}_i}^{\mathrm{t}}\mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2ds\Bigg].\end{aligned}$$

Therefore,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2_{\mathcal{PC}} \leq \mathbb{S}_3+\mathbb{S}_4\int_{\mathfrak{a}_i}^{\mathrm{t}}\mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2ds.$$

Here,

$$\begin{aligned}\mathbb{S}_3 &= \max_{i=1,2,\cdots,m}\frac{1}{1-5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i}\Bigg[\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\Big[\|\mathfrak{B}\|^2\|\mathfrak{u}\|^2_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{W})}+2\sigma_0\Big]\\ &+ 5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\Big[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{R}^+)}+\|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{I}}(l,\mathscr{R}^+)}\Big]\Bigg]\\ \mathbb{S}_4 &= 5\mathrm{T}\mathbb{M}^2[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda]\end{aligned}$$

Using Gronwall's inequality,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2_{\mathcal{PC}} \leq \mathbb{S}_3 e^{\mathbb{S}_4\mathrm{t}}=\mathfrak{d}_2. \tag{8}$$

By summarizing (7) and (8),

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2_{\mathcal{PC}} \leq \mathfrak{d} \text{ where } \mathfrak{d}=\max\{\mathfrak{d}_1,\mathfrak{d}_2\}.$$

The set $\chi_\mathfrak{d}=\{\vartheta\in\mathscr{X},\mathbb{E}\|\vartheta\|^2_{\mathcal{PC}}<\mathfrak{d}+1\}$. Clearly, $\chi_\mathfrak{d}$ is an open subset of $\mathscr{X}$. From $\chi_\mathfrak{d}$, there is no $\vartheta\in\mathscr{X}$ that satisfies $\pi\vartheta\in\Phi_1(\vartheta)+\Phi_2(\vartheta)$ for $\pi>1$. Hence, we conclude that the operator inclusion $\vartheta\in\Phi(\vartheta)$ has a mild solution in $\mathcal{J}$. □

4. T-Controllability

This section uses the generalized Gronwall's inequality to establish the T-controllability of the conformable fractional stochastic differential system

Theorem 3. *If (A1)–(A6) are satisfied, the system (1) is T-controllable on* $(0,\mathrm{T}]$.

Proof. Let $\psi(\mathrm{t})$ be a given trajectory on $\mathfrak{T}$. Let us choose the feedback control $\mathfrak{u}(\mathrm{t})$ as

$$\begin{aligned}\mathfrak{u}(\mathrm{t}) &= \mathfrak{B}^{-1}\Big[\mathcal{D}^\alpha\psi(\mathrm{t})-\Lambda\psi(\mathrm{t})-\omega(\mathrm{t},\psi(\mathrm{t}))-\sigma(\mathrm{t},\psi(\mathrm{t}),\psi(\mathfrak{e}(\psi(\mathrm{t}),\mathrm{t})))\\ &- \int_{\mathcal{Z}}\lambda(\mathrm{t},\psi(\mathrm{t}),\delta)\tilde{\mathbb{N}}(d\mathrm{t},d\delta)\Big], \quad \mathrm{t}\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}],\ i=0,1,2,\cdots,m,\\ \mathfrak{u}(\mathrm{t}) &= 0, \quad \mathrm{t}\in(\mathfrak{b}_i,\mathfrak{a}_i],\ i=1,2,\cdots,m.\end{aligned}$$

Thus, (1) implies

$$\begin{aligned}\mathcal{D}^\alpha\vartheta(\mathrm{t}) &= \Lambda\vartheta(\mathrm{t})+\Big[\mathcal{D}^\alpha\psi(\mathrm{t})-\Lambda\psi(\mathrm{t})-\omega(\mathrm{t},\psi(\mathrm{t}))-\sigma(\mathrm{t},\psi(\mathrm{t}),\psi(\mathfrak{e}(\psi(\mathrm{t}),\mathrm{t})))\\ &- \int_{\mathcal{Z}}\lambda(\mathrm{t},\psi(\mathrm{t}),\delta)\tilde{\mathbb{N}}(d\mathrm{t},d\delta)\Big]+\omega(\mathrm{t},\vartheta(\mathrm{t}))+\sigma(\mathrm{t},\vartheta(\mathrm{t}),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t}),\mathrm{t})))\\ &+ \int_{\mathcal{Z}}\lambda(\mathrm{t},\vartheta(\mathrm{t}),\delta)\tilde{\mathbb{N}}(d\mathrm{t},d\delta), \quad \mathrm{t}\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}],\ i=0,1,2,\cdots,m.\end{aligned}$$

Put $\rho(\mathrm{t})=\vartheta(\mathrm{t})-\psi(\mathrm{t})$.

$$\begin{cases} \mathcal{D}^{\alpha}\rho(\mathrm{t}) = \Lambda\rho(\mathrm{t}) + [\omega(\mathrm{t},\vartheta(\mathrm{t})) - \omega(\mathrm{t},\psi(\mathrm{t}))] + [\sigma(\mathrm{t},\vartheta(\mathrm{t}),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t}),\mathrm{t}))) - \sigma(\mathrm{t},\psi(\mathrm{t}),\psi(\mathfrak{e}(\psi(\mathrm{t}),\mathrm{t})))] \\ + \int_{\mathcal{Z}}[\lambda(\mathrm{t},\vartheta(\mathrm{t}),\delta) - \lambda(\mathrm{t},\psi(\mathrm{t}),\delta)]\tilde{\mathbb{N}}(d\mathrm{t},d\delta), \ \ \mathrm{t} \in (\mathfrak{a}_i,\mathfrak{b}_{i+1}], \ \ i = 0,1,2,\cdots,m. \\ \rho(\mathrm{t}) = \mathfrak{h}_i(\mathrm{t},\vartheta(\mathfrak{b}_i^-)) - \mathfrak{h}_i(\mathrm{t},\psi(\mathfrak{b}_i^-)), \quad \mathrm{t} \in (\mathfrak{b}_i,\mathfrak{a}_i], \ \ i = 1,2,\cdots,m, \\ \rho(0) = \vartheta(0) - \psi(0) = 0. \end{cases} \tag{9}$$

Thus, the mild solution of (9) is

$$\begin{aligned} \rho(\mathrm{t}) &= \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^{\alpha}-s^{\alpha}}{\alpha}\right)[\omega(s,\vartheta(s)) - \omega(s,\psi(s))]ds \\ &+ \int_0^{\mathrm{t}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\Big[\sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s))) - \sigma(s,\psi(s),\psi(\mathfrak{e}(\psi(s),s)))\Big]ds \\ &+ \int_0^{\mathrm{t}}\int_{\mathcal{Z}} s^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^{\alpha}-s^{\alpha}}{\alpha}\right)[\lambda(s,\vartheta(s),\delta) - \lambda(s,\psi(s),\delta)]\tilde{\mathbb{N}}(ds,d\delta), \qquad \mathrm{t} \in (0,\mathfrak{b}_1] \\ \rho(\mathrm{t}) &= \mathfrak{h}_i(\mathrm{t},\vartheta(\mathfrak{b}_i^-)) - \mathfrak{h}_i(\mathrm{t},\psi(\mathfrak{b}_i^-)), \quad \mathrm{t} \in (\mathfrak{b}_i,\mathfrak{a}_i], \ i = 1,2,\cdots,m. \end{aligned}$$

For $\mathrm{t} \in (0,\mathfrak{b}_1]$,

$$\begin{aligned} & \mathbb{E}\|\rho(\mathrm{t})\|^2 \\ \leq\ & 3\mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\omega(s,\vartheta(s)) - \omega(s,\psi(s))\|^2 ds + 3\mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\Big\|\sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\vartheta(s),s))) \\ -\ & \sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\psi(s),s))) - \sigma(s,\psi(s),\psi(\mathfrak{e}(\psi(s),s))) + \sigma(s,\vartheta(s),\vartheta(\mathfrak{e}(\psi(s),s)))\Big\|^2 ds \\ +\ & 3\mathfrak{b}_1\mathbb{M}^2\int_0^{\mathrm{t}}\int_{\mathcal{Z}} s^{2\alpha-2}\mathbb{E}\|\lambda(s,\vartheta(s),\delta) - \lambda(s,\psi(s),\delta)\|^2\mathrm{h}(d\delta)ds \end{aligned}$$

$$\begin{aligned} \leq\ & 3\mathfrak{b}_1\mathbb{M}^2\Big[\int_0^{\mathrm{t}} s^{2\alpha-2}\Big[\mathbb{E}\|\partial\Upsilon(s,\vartheta(s)) - \partial\Upsilon(s,\psi(s))\|^2\Big]ds + 2\mathbb{M}_{\sigma}\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\Big\|\vartheta(\mathfrak{e}(\vartheta(s),s) \\ -\ & \mathfrak{e}(\psi(s),s))\Big\|^2 + \mathbb{E}\|\vartheta(s) - \psi(s)\|^2 + \mathbb{E}\|(\vartheta-\psi)(\mathfrak{e}(\psi(s),s) - \mathfrak{e}(\psi(0),0))\|^2 ds \\ +\ & \mathbb{M}_{\lambda}\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\vartheta(s) - \psi(s)\|^2 ds\Big] \\ \leq\ & 3\mathfrak{b}_1\mathbb{M}^2[\mathbb{M}_{\omega} + 2\mathbb{M}_{\sigma}\mathfrak{l}(2\mathbb{M}_{\mathfrak{e}}+1) + \mathbb{M}_{\lambda}]\int_0^{\mathrm{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\rho(s)\|^2 ds \\ \leq\ & \mathbb{S}_5\int_0^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\rho(s)\|^2 ds, \end{aligned}$$

where

$$\mathbb{S}_5 = 3\mathfrak{b}_1\mathbb{M}^2[\mathbb{M}_{\omega} + 2\mathbb{M}_{\sigma}\mathfrak{l}(2\mathbb{M}_{\mathfrak{e}}+1) + \mathbb{M}_{\lambda}].$$

For $\mathrm{t} \in (\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i = 1,2,3,\cdots,m$, $\exists\ \omega \in \mathfrak{S}(\vartheta)$, and we have

$$\begin{aligned} \mathbb{E}\|\rho(\mathrm{t})\|^2 &\leq 4\mathbb{M}^2\mathrm{T}\mathfrak{f}_i\mathbb{E}\|\rho(\mathrm{t})\|^2 + 4\mathrm{T}\mathbb{M}^2[\mathbb{M}_{\omega} + 2\mathbb{M}_{\sigma}\mathfrak{l}(2\mathbb{M}_{\mathfrak{e}}+1) + \mathbb{M}_{\lambda}]\int_{\mathfrak{a}_i}^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\rho(s)\|^2 ds \\ &\leq \frac{1}{1-4\mathbb{M}^2\mathrm{T}\mathfrak{f}_i}\Big[4\mathrm{T}\mathbb{M}^2[\mathbb{M}_{\omega} + 2\mathbb{M}_{\sigma}\mathfrak{l}(2\mathbb{M}_{\mathfrak{e}}+1) + \mathbb{M}_{\lambda}]\int_{\mathfrak{a}_i}^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\rho(s)\|^2 ds\Big] \\ &\leq \mathbb{S}_6\int_{\mathfrak{a}_i}^{\mathrm{t}} s^{2\alpha-2}\mathbb{E}\|\rho(s)\|^2 ds, \end{aligned}$$

where

$$\mathbb{S}_6 = \frac{1}{1 - 4\mathbb{M}^2 \mathrm{T}\mathfrak{f}_i} 4\mathrm{T}\mathbb{M}^2[\mathbb{M}_\omega + 2\mathbb{M}_\sigma \mathfrak{l}(2\mathbb{M}_\mathfrak{e} + 1) + \mathbb{M}_\lambda].$$

$\implies \mathbb{E}\|\rho(\mathrm{t})\|^2_{\mathcal{PC}} = 0$ (i.e) $\vartheta(\mathrm{t}) = \psi(\mathrm{t})$
a.e. Hence, the proof is complete. □

5. Illustration

Example 1. *Consider the conformable stochastic differential inclusion with NIIs as follows:*

$$\begin{aligned}
\mathcal{D}^{\frac{1}{3}}\vartheta(\mathrm{t},s) &\in \frac{\partial^2}{\partial s^2}\vartheta(\mathrm{t},s) + \mathfrak{u}(\mathrm{t},s) + \partial \Upsilon\left(\frac{\mathrm{t}^2 + |\vartheta(\mathrm{t},s)|^2}{7}\right) \\
&+ \frac{1}{15}\left[\frac{\mathrm{t}\vartheta(\mathrm{t},s)}{2} + \vartheta\Big(\mathrm{t}, \sqrt{7}\sin \mathrm{t}|\vartheta(\mathrm{t},s)|/3\Big)\right] + \int_{\mathcal{Z}} \frac{\vartheta(\mathrm{t},s)\sin|s\delta|}{36}\tilde{\mathbb{N}}(d\mathrm{t},d\delta), \\
\mathrm{t} &\in (0, 0.25] \cup (0.5, 1] \text{ and } s \in [0,\pi], \qquad (10)\\
\vartheta(\mathrm{t},\omega) &= \frac{(\mathrm{t}-0.25)e^{-\mathrm{t}}}{1+5e^{\mathrm{t}}} + \frac{|\vartheta(0.25,s)|}{6}, \quad \mathrm{t} \in (0.25, 0.5), \\
\vartheta(\mathrm{t},0) &= \vartheta(\mathrm{t},\pi) = 0, \ \ \mathrm{t} \in (0,1) \times [0,\pi] \\
\vartheta(0,s) &= \vartheta_0(s) \ \ on \ \ [0,\pi].
\end{aligned}$$

Assume that $[0,\pi]$ *is a bounded domain in* $\mathscr{R}^n$, $(n \geq 2)$ *with the Lipschitz boundary. Here,* $\mathcal{D}^{\frac{1}{3}}$ *is a conformable fractional derivative of order* $\frac{1}{3}$. *Let* $\mathscr{H} = \mathscr{U} = \mathcal{L}^2([0,\pi])$, $\omega_0(s) \in \mathscr{H}$, *and let* $\Lambda : \mathcal{L}^2([0,\pi]), \omega_0(s) \in \mathscr{H}$, *and let* $\mathfrak{A} : \mathcal{L}^2([0,\pi]) \to \mathcal{L}^2([0,\pi])$. $\mathfrak{A}\omega = \Delta\omega$, $\omega \in D(\mathfrak{A})$; *we define* $D(\Lambda) = \{\mathrm{y} \in \mathscr{H} : \mathrm{y}, \frac{dy}{ds}$ *as absolutely continuous and* $\frac{d^2\mathrm{y}}{ds^2} \in \mathscr{H}$, $\mathrm{y}(0) = y(\pi) = 0\}$. Λ *generates a* C_0*-semigroup* $\{\mathcal{T}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ *that is analytic and self-adjoint. Furthermore,* Λ *has the discrete spectrum, and* $\exists$ *eigenvalues* $-n^2$ *having orthogonal eigenvectors* $\Psi_n(\mathrm{y}) = \sqrt{\frac{2}{\pi}}\sin(n\mathrm{y})$; *then,* $\Lambda\mathrm{y} = \sum_{n=1}^{\infty} -n^2\langle \mathrm{y}, \Psi_n\rangle\Psi_n$, $n \in \mathscr{R}$. *Also,* $\mathcal{T}(\mathrm{t})\mathrm{y} = \sum_{n=1}^{\infty} e^{-n^2\mathrm{t}}\langle \mathrm{y}, \Psi_n\rangle\Psi_n$, $\mathrm{y} \in \mathscr{H}$ *and* $\forall\ \mathrm{t} > 0$. *In addition,* $\|\mathcal{T}(\mathrm{t})\| \leq \mathbb{M} = 1$, *(A1) holds, and* $\mathcal{R}(\lambda,\Lambda) = (\lambda - \Lambda)^{-1}$ *is a compact operator for every* $\lambda \in D(\Lambda)$.

Define the nonlinear functions as follows:

$$\begin{aligned}
\omega(\mathrm{t},\vartheta(\mathrm{t},s)) &\in \partial\Upsilon(\mathrm{t},\vartheta(\mathrm{t},s)) \text{ and } \omega(\mathrm{t},\vartheta(\mathrm{t},s)) = \frac{\mathrm{t}^2 + |\vartheta(\mathrm{t},s)|^2}{7}, \\
\sigma(\mathrm{t},\vartheta(\mathrm{t},s),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t},s))),\mathrm{t}) &= \frac{1}{15}\left[\frac{\mathrm{t}\vartheta(\mathrm{t},s)}{2} + \vartheta\Big(\mathrm{t}, \sqrt{7}\sin\mathrm{t}|\vartheta(\mathrm{t},s)|/3\Big)\right], \\
\int_{\mathcal{Z}} \lambda(\mathrm{t},\vartheta(\mathrm{t},s),\delta)\tilde{\mathbb{N}}(d\mathrm{t},d\delta) &= \int_{\mathcal{Z}} \frac{\vartheta(\mathrm{t},s)\sin|s\delta|}{36}\tilde{\mathbb{N}}(d\mathrm{t},d\delta), \\
\mathfrak{e}(\vartheta(\mathrm{t},s),\mathrm{t}) &= \sqrt{7}\sin\mathrm{t}|\vartheta(\mathrm{t},s)|/3, \\
\mathfrak{h}\big(\mathrm{t},\vartheta(\mathfrak{b}_i^-,s)\big) &= \frac{(\mathrm{t}-0.25)e^{-\mathrm{t}}}{1+5e^{\mathrm{t}}} + \frac{|\vartheta(0.25,s)|}{6}.
\end{aligned}$$

We define the ball $\mathscr{B}_{\tilde{\mathfrak{r}}} = \{\vartheta \in \mathscr{H} : \mathbb{E}\|\vartheta\|^2 \leq \tilde{\mathfrak{r}}\}$ *with* $\tilde{\mathfrak{r}} > 0$. *For* $\vartheta_1, \vartheta_2 \in \mathscr{B}_{\tilde{\mathfrak{r}}}$,

$$\begin{aligned}
\mathbb{E}|\mathfrak{e}(\vartheta_1(\mathrm{t},s),\mathrm{t}) - \mathfrak{e}(\vartheta_2(\mathrm{t},s),\mathrm{t})|^2 &= \mathbb{E}\left|\frac{\sqrt{7}\sin\mathrm{t} \mid \vartheta_1(\mathrm{t},s)}{3} - \frac{\sqrt{7}\sin\mathrm{t} \mid \vartheta_2(\mathrm{t},s)}{3}\right| \\
&\leq \frac{7}{9}\mathbb{E}|\vartheta_1(\mathrm{t},s) - \vartheta_2(\mathrm{t},s)|^2.
\end{aligned}$$

Also, $\mathfrak{e}(\cdot,0) = 0$. $\implies$ $\mathfrak{e}$ *satisfies (A4) with* $\mathbb{M}_\mathfrak{e} = 7/9$.
Now,

$$
\begin{aligned}
& \mathbb{E}\|\sigma(\mathrm{t},\vartheta_1(\mathrm{t},s),\vartheta_1(\mathfrak{e}(\vartheta_1(\mathrm{t},s))),\mathrm{t})-\sigma(\mathrm{t},\vartheta_2(\mathrm{t},s),\vartheta_2(\mathfrak{e}(\vartheta_2(\mathrm{t},s))),\mathrm{t})\|^2 \\
\leq\ & \frac{1}{225}\mathbb{E}\bigg\|\frac{\mathrm{t}\vartheta_1(\mathrm{t},s)}{2}+\vartheta_1\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_1(\mathrm{t},s)|/3\Big)-\frac{\mathrm{t}\vartheta_2(\mathrm{t},s)}{2}+\vartheta_2\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_2(\mathrm{t},s)|/3\Big) \\
+\ & \vartheta_1\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_2(\mathrm{t},s)|/3\Big)-\vartheta_1\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_2(\mathrm{t},s)|/3\Big)\bigg\|^2 \\
\leq\ & \frac{1}{75}\bigg[\mathbb{E}\|\vartheta_1(\mathrm{t},s)-\vartheta_2(\mathrm{t},s)\|^2+\mathbb{E}\Big\|\vartheta_1\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_2(\mathrm{t},s)|/3\Big)-\vartheta_2\Big(\mathrm{t},\sqrt{7}\sin\mathrm{t}|\vartheta_2(\mathrm{t},s)|/3\Big)\Big\|^2\bigg].
\end{aligned}
$$

This implies that σ satisfies (A3) with $\mathbb{M}_\sigma = \frac{1}{75}$. Also,

$$
\mathbb{E}\|\omega(\mathrm{t},\vartheta(\mathrm{t},s))\|^2 \leq \mathbb{E}\bigg\|\frac{\mathrm{t}^2+|\vartheta(\mathrm{t},s)|}{5}\bigg\|^2 \leq \frac{2}{25}\Big[1+\mathbb{E}\|\vartheta(\mathrm{t},s)\|^2\Big]
$$

and

$$
\begin{aligned}
\mathbb{E}\|\omega(\mathrm{t},\vartheta_1(\mathrm{t},s))-\omega(\mathrm{t},\vartheta_2(\mathrm{t},s))\|^2 &\leq \mathbb{E}\bigg\|\frac{\mathrm{t}^2+|\vartheta_1(\mathrm{t},s)|}{7}-\frac{\mathrm{t}^2+|\vartheta_2(\mathrm{t},s)|}{7}\bigg\|^2 \\
&\leq \frac{2\tilde{\mathfrak{r}}}{49}\mathbb{E}\|\vartheta_1(\mathrm{t},s)-\vartheta_2(\mathrm{t},s)\|^2.
\end{aligned}
$$

This implies that ω satisfies (A2) with $\varsigma_\omega = \frac{2}{49}$.

$$
\begin{aligned}
\int_{\mathcal{Z}}\mathbb{E}\|\lambda(\mathrm{t},\vartheta_1(\mathrm{t},s),\delta)-\lambda(\mathrm{t},\vartheta_2(\mathrm{t},s),\delta)\|^2\mathrm{h}(d\delta) &\leq \int_{\mathcal{Z}}\mathbb{E}\bigg\|\frac{\vartheta_1(\mathrm{t},s)\sin|s\delta|}{36}-\frac{\vartheta_2(\mathrm{t},s)\sin|s\delta|}{36}\bigg\|^2\mathrm{h}(d\delta) \\
&\leq \frac{1}{1296}\mathbb{E}\|\vartheta_1(\mathrm{t},s)-\vartheta_2(\mathrm{t},s)\|^2.
\end{aligned}
$$

This implies that λ satisfies (A5) with $\mathbb{M}_\lambda = \frac{1}{1296}$. Moreover,

$$
\begin{aligned}
\mathbb{E}\big\|\mathfrak{h}\big(\mathrm{t},\vartheta_1(\mathfrak{b}_i^-,s)\big)-\mathfrak{h}\big(\mathrm{t},\vartheta_2(\mathfrak{b}_i^-,s)\big)\big\|^2 &= \mathbb{E}\bigg\|\frac{|\vartheta_1(0.25,s)|}{6}-\frac{|\vartheta_2(0.25,s)|}{6}\bigg\|^2 \\
&\leq \frac{1}{36}\mathbb{E}\|\vartheta_1(0.25,s)-\vartheta_2(0.25,s)\|^2.
\end{aligned}
$$

This implies that (A6) holds with $\mathfrak{f}_i = \frac{1}{36}$. Substituting (3), we can obtain $\mathfrak{C} < 1$. Thus, it satisfies the conditions of Theorem 2, and therefore, there is at least one mild solution for the system (10). Theorem 3's presumptions are satisfied, proving that (10) is T-controllable on $(0,1]$.

Example 2. *Consider the stochastic fractional partial differential inclusion with NIIs as follows:*

$$
\begin{aligned}
\mathcal{D}^\alpha\vartheta(\mathrm{t},w) &\in \frac{\partial^2}{\partial w^2}\vartheta(\mathrm{t},w)+\int_0^1\mathfrak{B}\mathrm{u}(s,w)ds+\partial\Upsilon\bigg(\frac{\mathrm{t}^2-|\vartheta(\mathrm{t},w)|}{9}\bigg) \\
&+ \frac{\mathrm{t}^2\vartheta(\mathrm{t},w)+\vartheta(\mathrm{t},\sin\mathrm{t}|\vartheta(\mathrm{t},w)|)}{5\pi}+\int_{\mathcal{Z}}\bigg(\frac{cos|tw|}{5}+\frac{(\mathrm{t}^2+1)|\vartheta(\mathrm{t},w)|}{5+|\vartheta(\mathrm{t},w)|}\bigg)\tilde{\mathbb{N}}(d\mathrm{t},dw), \\
\mathrm{t} &\in (0,0.25]\cup(0.5,1]\ \textit{and}\ w\in[0,\pi], \\
\vartheta(\mathrm{t},w) &= \frac{\sin|w|+\mathrm{t}^2\vartheta^-_{0.25}(\mathrm{t},w)}{12},\ \mathrm{t}\in(0.25,0.5), \\
\vartheta(\mathrm{t},0) &= \vartheta(\mathrm{t},\pi)=0,\ \mathrm{t}\in[0,1], \\
\vartheta(0,w) &= \vartheta_0(w),\ w\in[0,\pi].
\end{aligned}
\tag{11}
$$

Let $\alpha = 0.9$, and $\mathfrak{B} : [0,\pi] \times [0,1] \to \mathbb{R}$ is a continuous function. Let $\mathscr{H} = \mathscr{U} = \mathcal{L}^2([0,\pi])$, $\omega_0(s) \in \mathscr{H}$, and let $\Lambda : \mathcal{L}^2([0,\pi]), \omega_0(s) \in \mathscr{H}$, and let $\mathfrak{A} : \mathcal{L}^2([0,\pi]) \to \mathcal{L}^2([0,\pi])$.
$\mathfrak{A}\omega = \Delta\omega$, $\omega \in D(\mathfrak{A})$, and we define $D(\Lambda) = \{\mathrm{y} \in \mathscr{H} : \mathrm{y}, \frac{dy}{ds}$ as absolutely continuous and $\frac{d^2\mathrm{y}}{ds^2} \in \mathscr{H}$, $\mathrm{y}(0) = y(\pi) = 0\}$. Λ generates a C_0-semigroup $\{\mathcal{T}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ that is analytic and self-adjoint. Furthermore, Λ has the discrete spectrum, and there exists eigenvalues $-n^2$ having orthogonal eigenvectors $\Psi_n(\mathrm{y}) = \sqrt{\frac{2}{\pi}}\sin(n\mathrm{y})$; then, $\Lambda\mathrm{y} = \sum_{n=1}^{\infty} -n^2\langle \mathrm{y}, \Psi_n\rangle\Psi_n$, $n \in \mathscr{R}$. Also, $\mathcal{T}(\mathrm{t})\mathrm{y} = \sum_{n=1}^{\infty} e^{-n^2\mathrm{t}}\langle \mathrm{y}, \Psi_n\rangle\Psi_n$, $\mathrm{y} \in \mathscr{H}$ and $\forall\, \mathrm{t} > 0$. In addition, $\|\mathcal{T}(\mathrm{t})\| \leq \mathbb{M} = 1$, (A1) holds, and $\mathcal{R}(\lambda,\Lambda) = (\lambda - \Lambda)^{-1}$ is a compact operator for every $\lambda \in D(\Lambda)$. Here, the nonlinear functions are

$$
\begin{aligned}
\partial\Upsilon(\mathrm{t},\vartheta(\mathrm{t})) &= \partial\Upsilon\left(\frac{\mathrm{t}^2 - |\vartheta(\mathrm{t},w)|}{9}\right),\\
\sigma(\mathrm{t},\vartheta(\mathrm{t}),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t}),\mathrm{t}))) &= \frac{\mathrm{t}^2\vartheta(\mathrm{t},w) + \vartheta(\mathrm{t},\sin\mathrm{t}|\vartheta(\mathrm{t},w)|)}{5\pi},\\
\lambda(\mathrm{t},\vartheta(\mathrm{t}),\delta) &= \left(\frac{cos|w\eta|}{5} + \frac{(\mathrm{t}^2+1)|\vartheta(\mathrm{t},w)|}{5+|\vartheta(\mathrm{t},w)|}\right),
\end{aligned}
$$

and

$$
\mathfrak{h}_i\big(\mathrm{t},\vartheta(\mathfrak{b}_i^-)\big) = \frac{\sin|w| + \mathrm{t}^2\vartheta^-_{0.25}(\mathrm{t},w)}{12}.
$$

Further,

$$
\begin{aligned}
\|\partial\Upsilon(\mathrm{t},\vartheta(\mathrm{t}))\|^2 &\leq \frac{1}{81}\|1 - |\vartheta(\mathrm{t},w)|\|^2,\\
&\leq \frac{1}{81}\Big[1 + \|\vartheta(\mathrm{t},w)\|^2 - 2\|\vartheta(\mathrm{t},w)\|\Big]\\
&\leq \frac{1}{81}\Big[1 + \|\vartheta\|^2\Big].
\end{aligned}
$$

$$
\begin{aligned}
\|\sigma(\mathrm{t},\vartheta(\mathrm{t}),\vartheta(\mathfrak{e}(\vartheta(\mathrm{t}),\mathrm{t})))\|^2 &\leq \left\|\frac{\mathrm{t}^2\vartheta(\mathrm{t},w) + \vartheta(\mathrm{t},\sin\mathrm{t}|\vartheta(\mathrm{t},w)|)}{5\pi}\right\|^2,\\
&\leq \frac{4}{25}(1 + 4\|\vartheta\|^2).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_{\mathcal{Z}}\|\lambda(\mathrm{t},\vartheta(\mathrm{t}),\delta)\|^2 &\leq \left\|\frac{cos|w\eta|}{5} + \frac{(\mathrm{t}^2+1)|\vartheta(\mathrm{t},w)|}{5+|\vartheta(\mathrm{t},w)|}\right\|^2 d\eta\\
&\leq \frac{2}{25}(1 + 4\|\vartheta\|^2).
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|\mathfrak{h}_i\big(\mathrm{t},\vartheta(\mathfrak{b}_i^-)\big)\big\| &\leq \left\|\frac{\sin|w| + \mathrm{t}^2\vartheta^-_{0.25}(\mathrm{t},w)}{12}\right\|\\
&\leq \frac{1}{2}\left\|(1 + \frac{1}{4})\vartheta^-_{0.25}(\mathrm{t},w)\right\|\\
&\leq 0.1\|\vartheta\|.
\end{aligned}
$$

Here, $\mathfrak{r}_1 = \varsigma_\omega = \frac{1}{81}$, $\mathbb{M}_\sigma = \frac{4}{25}$, $\mathbb{M}_\mathfrak{c} = 0.1$, $\mathfrak{C} \approx 0.0405 < 1$. *Thus, all the hypotheses of Theorem 2 are fulfilled. Hence, the solution set of (11) is nonempty and compact. And easily, one can verify that the hypotheses of Theorem 3 are fulfilled, so (11) is T-controllable on* $(0, 1]$.

6. Conclusions

This paper presents a mild solution for the given situation using the multivalued fixed point theorem, Clarke subdifferential properties, fractional calculus, and stochastic analysis. Additionally, for the aforementioned system (1), the T-controllability conditions are confirmed in the mean square moment. The obtained innovative theoretical conclusions are illustrated in the later portion of the paper with an application. The presented results can be further expanded to include the averaging principle and the results of well-posedness for conformable fractional stochastic differential equations with Lévy noise and a time and state delay on an infinite horizon.

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