Trajectory Controllability of stochastic integro-differential equation with state dependent delay driven by mixed Brownian motion suffered by non-instantaneous impulses

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Abstract

We present a framework of studying the mild solution to non-instantaneous impulsive stochastic integro-differential equations (SIDEs) with state-dependent delay and the mixed Brownian motion in Hilbert spaces. The solvability of the proposed stochastic system is obtained using stochastic analysis, fixed point theorems, and the resolvent operators. Furthermore, under some reasonable assumptions, the Trajectory controllability [T-controllability] of the investigated system is established using extended Gronwall's inequality. Finally, an example is provided to demonstrate the theoretical findings.

Keywords: T-controllability, Non-instantaneous impulse, Stochastic Integro-differential equation, Mixed Brownian motion, State dependent delay.

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1 Introduction

Unexpected fluctuations are inherent and prevalent in both natural and man-made systems which lead to study stochastic models rather than deterministic ones. Stochastic evolution equations are natural generalizations of ordinary differential equations incorporating the randomness into the equations.

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Thereby, making the system more realistic, several authors ([1, 2, 3, 4, 5, 6, 7] and the references therein) explore the qualitative properties of solutions for stochastic differential equations (SDEs). It has been established that stochastic systems are effective tools with more reliability for formulating and analyzing phenomena, such as population modeling, stock prices, heat conduction in memory materials, etc. Fractional Brownian motion (fBm) was introduced by Kolmogorov [40], and studied by Mandelbrot & Van Ness [41], it is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $\mathrm{H} \in (0, 1)$. The fBm is a generalization of classical Brownian motion, it admits the stationary increments, self-similarity and has a long-memory when $\mathrm{H} > \frac{1}{2}$ and it reduces to Brownian motion when $\mathrm{H} = \frac{1}{2}$. The fBm make this process a natural candidate as a model for noise in a wide range of phenomena, especially in financial markets, communications networks, traffic networks, and medicine, etc., see [8, 9]. SDEs with fBm have been considered greatly by research community in various aspects, see [10, 11, 12, 21, 28].

Impulsive differential equations play a crucial role in population dynamics, medical science and any other engineering domains. Naturally, all physical systems which evolve with respect to time are suffered by small abrupt changes in the form of impulses. These impulses are divided into two types:

(i) Instantaneous impulse differential equations (IIDEs)- that causes sudden changes, and it persists shortly in their states at certain moments

(ii) Non-instantaneous impulse differential equations (NIIDEs)-an impulsive action which starts arbitrarily at a fixed point and remains active on a given time interval.

On the other hand, Control system is an interconnection of components forming a system configuration that will provide a desired system response. The ability to steer a dynamic system using the set of permitted controls from an arbitrary initial state to an arbitrary final state is known as controllability, and it is one of the structural features of dynamical systems. The concept of controllability [introduced by Kalman 1960] leads to some very important conclusions regarding the behavior of linear and nonlinear dynamical systems. The controllability of fractional dynamical systems represented by fractional differential equations is an essential subject for many practical applications because control theory produces better results using fractional order derivatives and integrals calculus. There are different notions of controllability like complete [23], approximate [24, 25], exact [26, 27], null controllability [28], etc. T-controllability, a modern concept of controllability, has been offered as a new direction in the domain of control theory. This new notion puts us in a position to answer several natural questions which arise in connection with control theory. In T-controllability problems, we look for a control which steers the system along a prescribed trajectory rather than a control steering a given initial state to a desired final state.

The advantages of studying T-controllability are as follows:

1. It may minimize certain cost involved in steering the system from initial state to final desired state.

2. It may also safeguard the system.

For example, while launching a rocket in space sometimes it may be desirable to have a precise path along with desired destination for cost effectiveness and collision avoidance.
Chalishajar et al.[30] faced challenges of T-controllability to identify control that steers the system along a prescribed trajectory to the final state instead of navigating a given initial state to the required final destination. It should be noted that Chalishajar et al. [30, 31] has demonstrated the formulation of first and second order T-controllability problems for the nonlinear integro-differential equation in both finite and infinite-dimensional spaces. In [32] authors have recently looked at the T-controllability for nonlinear fractional differential equations using Gronwall's inequality. In the present literature, many papers concerned to investigate T- controllability of FSNIIDEs, see [33, 34] and the references therein. Nevertheless, the T-controllability results on impulsive stochastic integro-differential equation with state dependent delay still exists, a fact that inspired the current work. The primary contribution and benefit of this manuscript are listed as follows:

- First, the control model is presented with a NIISIDEs with state dependent delay. We extracted this kind of SIDES with noise and fBm.
- To the best of author's knowledge, there is no work for solvability and T-controllability results of NIISIDEs with state dependent delay, which were inspired by the studies previously stated.
- The obtained results which will generalizes the existing work of [7, 34].

This work's innovations are as follows:

- T-controllability is new to the NIISIDEs with state dependent delay driven by fBm.
- To obtain the existence results, we have used the ideas of resolvent operator technique via stochastic techinque and fixed point theorems(FPTs). The T-controllability of the system under consideration is also established using the generalized Gronwall inequality under some appropriate assumptions.
- The obtained results are applied to the stochastic heat equation.

Now, we study the existence of mild solution and T-controllability for NIISIDEs with state dependent delay via fBm:

$$\begin{aligned}
d[\mathfrak{z}(\iota)+\mathfrak{h}(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)})] &= \mathfrak{A}\left[\mathfrak{z}(\iota)+\mathfrak{h}(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)})+\mathfrak{C}(\iota)\right]d\iota+\int_0^\iota \Theta(\iota-\varsigma)\left[\mathfrak{z}(\varsigma)+\mathfrak{h}(\varsigma,\mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})\right]d\varsigma d\iota\\
&+ \mathfrak{f}(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)})d\iota+\mathfrak{g}(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)})d\omega(\iota)\\
&+ \sigma(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)})d\mathrm{B}^{\mathrm{H}}(\iota), \qquad \iota\in\cup_{i=0}^{n}(\varsigma_i,\iota_{i+1}],\ n\in\mathbb{N};\\
\mathfrak{z}(\iota) &= \mathfrak{G}_i(\iota,\mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}), \qquad \iota\in\cup_{i=1}^{n}(\iota_i,\varsigma_i];\\
\mathfrak{z}(\iota) &= \phi(\iota)\in\mathscr{D}; \qquad \iota\in(-\infty,0]; \qquad (1.1)
\end{aligned}$$

were the state $\mathfrak{u}(.)$ takes values in Hilbert space $(\mathscr{H},(.,.),\|.\|)$. $\mathfrak{A}:\mathcal{D}(\mathfrak{A})\subset\mathscr{H}\to\mathscr{H}$ is the infinitesimal generator of a strongly continuous bounded linear operator $\{\mathcal{T}(\iota):\iota\geq 0\}$. Θ is a closed linear operator on $\mathscr{H}$ with domain $\mathcal{D}(\Theta)\supset\mathcal{D}(\mathfrak{A})$ which is independent of ι. Let $(\mathscr{K},(.,.),\|.\|)$ and $(\mathscr{Y},(.,.),\|.\|)$ be another separable Hilbert spaces. Let $\{\omega(\iota)\}_{\iota\geq 0}$ is a Wiener process defined on $(\Omega,\Im,\{\Im_\iota\}_{\iota\geq 0},\mathbb{P})$ with the values in $\mathscr{K}$. Let B^{H} is a fBm with Hurst indext $\mathrm{H}>\frac{1}{2}$ defined on $(\Omega,\Im,\{\Im_\iota\}_{\iota\geq 0},\mathbb{P})$ with the values in $\mathscr{Y}$. The history function $\mathfrak{z}_\iota:(-\infty,0]\to\mathscr{H}$ denotes the time history of the function $\mathfrak{z}$ from $-\infty$ to the present time ι and defined by $\mathfrak{z}_\iota(\theta)=\mathfrak{z}(\iota+\theta)$ for $\theta\in(-\infty,0]$ in the phase space $\mathscr{D}$ which is defined later. The map $\mathfrak{h}:J\times\mathscr{D}\to\mathscr{H}$, $\vartheta:J\times\mathscr{D}\to(-\infty,\mathfrak{b}]$, $\mathfrak{f}:J\times\mathscr{D}\to\mathscr{H}$, $\mathfrak{g}:J\times\mathscr{D}\to\mathcal{L}_2^0(\mathscr{K},\mathscr{H})$, $\sigma:J\times\mathscr{D}\to\mathcal{L}_2^0(\mathscr{Y},\mathscr{H})$ and $\mathfrak{G}_i:(\iota_i,\varsigma_i]\times\mathscr{D}\to\mathscr{H}$ are the considered functions . $0=\iota_0=s_0<\iota_1<\varsigma_1<\iota_2<\cdots<\iota_n<\varsigma_n<\iota_{n+1}<\cdots<\varsigma:=[0,\mathfrak{b}]=J$ are the impulsive points ι_i and the effect remains on the interval $(\iota_i,\varsigma_i]$. The control function $\mathfrak{C}\in\mathcal{L}^2_{\Im}(J,\mathscr{U})$. Here $\mathcal{L}^2_{\Im}(J,\mathscr{U})$ is the space of all admissible control functions, which is square integrable and $\Im_\iota$-adopted. $\mathfrak{C}$ is a bounded linear operator from Hilbert space $\mathscr{U}$ into $\mathscr{H}$.

In this article, for the aforementioned system (1.1), three existence results are investigated using semigroup theory and FPTs. Initially, the sufficient conditions are framed to prove the existence and uniqueness of the mild solution of (1.1) using Banach Contraction Principle under Lipschitz conditions on nonlinear terms. In the second and third existence results, we prove the existence of mild solution via Darbo and Darbo-Sadaovskii fixed point theorems under non-Lipschitz conditions on nonlinear terms. In the later part, the T-controllability of the proposed system (1.1) is examined by employing the Gronwall's inequality.

The following is how the paper is structured: In Sect 2, the fundamental concepts of the stochastic analysis and semigroup theory is given. In Sect 3, the solvability of (1.1) using FPTs is studied by three different ways. In the Sect 4, the T- controllability of the system (1.1) is proved. At the end of Sect 5, the application is demonstrating the obtained theoretical results.

2 Preliminary and Notations

Lemma 2.1. *[13] If $\sigma : [0,\iota] \to \mathcal{L}_2^0(\mathscr{Y},\mathscr{H})$ satisfies $\int_0^\iota \|\sigma(\varsigma)\|^2_{\mathcal{L}_2^0} < \infty$, then the sum in (2) in [13] is well defined as $\mathscr{H}$-valued random variable and we have*

$$\mathbb{E}\left\|\sigma(\varsigma)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2 \leq 2\mathrm{H}\iota^{\mathrm{H}-1}\int_0^\iota \|\sigma(\varsigma)\|^2_{\mathcal{L}_2^0}\, d\varsigma.$$

For details on basic preliminaries of fBm, one can refer to [10, 11, 12].

Consider the Banach space $\mathscr{PC}(\mathscr{H})$, which is the space of all $\Im_\iota-$adapted measurable, $\mathscr{H}-$valued stochastic processes $\{\mathfrak{z}(\iota) : \iota \in [0,\mathfrak{b}]\}$ such that $\mathfrak{z}$ is continuous at $\iota \neq \iota_i$, $\mathfrak{z}(\iota_i) = \mathfrak{z}(\iota_i^-)$ and $\mathfrak{z}(\iota_i^+)$ exists for all $i = 1, 2, \cdots, n$ endowed with the norm,

$$\|\mathfrak{z}\|_{\mathscr{PC}} = \left(\sup_{0\leq\iota\leq\mathfrak{b}} \mathbb{E}\,\|\mathfrak{u}(\iota)\|^2\right)^{1/2}.$$

In order to deal with the infinite delay, we will consider the phase space $\mathscr{D}$ which was described by Hale and Kato in [35]. The abstract phase space is a seminormed linear space $(\mathscr{D}, \|.\|_{\mathscr{D}})$ of $\Im_0-$measurable functions which maps $(-\infty, 0]$ into $\mathscr{H}$ which satisfies the following axioms:

(a) If $\mathfrak{z} : (-\infty,\mathfrak{b}] \to \mathscr{H}$ is a continuous function on J such that $\mathfrak{z}\mid_J \in \mathcal{C}(J,\mathscr{H})$ and $\mathfrak{z}_0 \in \mathscr{D}$, then for every $\iota \in J$ the following conditions hold:
(i) $\mathfrak{z}\iota$ is in $\mathscr{D}$,
(ii) $\|\mathfrak{z}(\iota)\| \leq \mathbb{K}\|\mathfrak{z}_\iota\|_{\mathscr{D}}$, where $\mathbb{K} \geq 0$,
(iii) $\|\mathfrak{z}_\iota\|_{\mathscr{D}} \leq \mathbb{H}(\iota)\|\mathfrak{z}_0\|_{\mathscr{D}} + \mathscr{N}(\iota)\sup\{\|\mathfrak{z}(\varsigma)\| : 0 \leq s \leq \iota\}$. $\mathbb{H} : [0,\infty) \to [0,\infty)$ is locally bounded, $\mathscr{N} : [0,\infty) \to [0,\infty)$ is continuous, $\mathbb{H}$ and $\mathscr{N}$ are independent of $\mathfrak{z}(.)$.

(b) The phase space $\mathscr{D}$ is complete.

(c) The function $\iota \to \varphi_\iota$ is well defined from the set $\mathbb{R}(\vartheta^-) = \{\vartheta(\varsigma,\varpi) : (\varsigma,\varpi) \in J \times \mathscr{D}\}$ into $\mathscr{D}$ and $\exists$ a continuous, bounded function $\mathbb{J}^\varphi : \mathbb{R}(\vartheta^-) \to (0,\infty) \ni \|\varphi_\iota\|_{\mathscr{D}} \leq \mathbb{J}^\varphi(\iota)\|\varphi\|_{\mathscr{D}}\ \forall\ \iota \in \mathbb{R}(\vartheta^-)$.

Lemma 2.2. *[36] Let $\mathfrak{z} \in (-\infty,\mathfrak{b}] \to \mathscr{H}$ be a function such that $\mathfrak{z}_0 = \varphi$ and $\mathfrak{z}\mid_J \in PC(J,\mathscr{H})$. Then*

$$\|\mathfrak{z}_\iota\|_{\mathscr{D}} \leq (\mathbb{H}_\mathfrak{b} + \mathbb{J}^\varphi)\|\varphi\|_{\mathscr{D}} + \mathscr{N}_\mathfrak{b}\sup\left\{\|\mathfrak{z}(\tau)\| : \tau \in [0,\max\{0,\iota\}]\right\}, \quad \iota \in \mathbb{R}(\vartheta^-) \cup J;$$

where $\mathbb{J}^\varphi = \sup_{\iota\in\mathbb{R}(\vartheta^-)} \mathbb{J}^\varphi(\iota)$, $\mathbb{H}_\mathfrak{b} = \sup_{\iota\in J}\mathbb{H}(\iota)$, $\mathscr{N}_\mathfrak{b} = \sup_{\iota\in J}\mathscr{N}(\iota)$.

Definition 2.1. *[7] The Hausdorff measure of noncompactness (HMNC) β of the set $\mathbb{B} \in \mathscr{H}$ is,*

$$\beta(\mathbb{B}) = \inf\{\epsilon > 0 : \mathbb{B} \textit{ has a finite } \epsilon-\textit{net in } \mathscr{H}\}$$

for every bounded subset $\mathbb{B}$ in $\mathscr{H}$.

Definition 2.2. *[7] A bounded and continuous map* $\mathcal{Q}:\mathbb{D}\subseteq\mathbb{X}\to\mathbb{X}$ *is known as* $\beta-$*contraction if* $\exists$ $0<\kappa<1\ni$

$$\beta(\mathcal{Q}(\mathbb{B}))\leq\kappa\beta(\mathbb{B})$$

for any noncompact bounded subset $\mathbb{B}\subset\mathbb{D}$. *To express the HMNC* $\mathcal{C}([0,\mathfrak{b}],\mathscr{H})$ *and* $PC(\mathscr{H})$, $\beta_{\mathcal{C}}$ *and* β_{PC} *are used.*

Lemma 2.3. *[7] For any bounded set* $\mathfrak{A}$, $\beta(\mathfrak{A})=0$ *if and only if* $\mathfrak{A}$ *is precompact.*

Lemma 2.4. *(Darbo)[13] If* $\mathbb{D}\subseteq\mathbb{X}$ *is convex and closed,* $0\in\mathbb{D}$, *the map* $\mathcal{Q}:\mathbb{D}\to\mathbb{D}$ *is continuous ad* $\beta-$*contraction set* $\{\mathfrak{u}\in\mathbb{D}:\mathfrak{u}=\lambda\mathcal{Q}\mathfrak{u}\}$ *is bounded for* $0<\lambda<1$, *then the map* $\mathcal{Q}$ *has a fixed point in* $\mathbb{D}$.

Lemma 2.5. *(Darbo-Sadovskii)[13] If* $\mathbb{D}\subseteq\mathbb{X}$ *be bounded closed, and convex. If the continuous map* $\mathcal{Q}:\mathbb{D}\to\mathbb{D}$ *is a* $\beta-$*contraction, then* $\mathcal{Q}$ *has a fixed point in* $\mathbb{D}$.

Lemma 2.6. *[11] For* $\mathfrak{r}\geq 1$ *and* $\mathscr{H}$*- valued predictable process* $\mathfrak{h}(.)$,

$$\sup_{\varsigma\in[0,\iota]}\mathbb{E}\left\|\int_0^{\varsigma}\mathfrak{h}(l)d\omega(l)\right\|_{\mathscr{H}}^2\leq\mathcal{C}_{\mathfrak{r}}\left(\int_0^{\iota}\mathbb{E}\left\|\mathfrak{h}(\varsigma)\right\|_{\mathcal{L}_2^0}^2d\varsigma\right)^{\mathfrak{r}},\quad\forall\quad\iota\in[0,\infty);$$

where $\mathcal{C}_{\mathfrak{r}}=(\mathfrak{r}(2\mathfrak{r}-1))^{\mathfrak{r}}$.

Definition 2.3. *[38] A one parameter family* $\{\mathscr{R}(\iota):\iota\geq 0\}$ *of bounded linear operators, is called resolvent operator for*

$$\frac{d\mathfrak{Z}}{d\iota}=\mathfrak{A}\left[\mathfrak{Z}(\iota)+\int_0^{\iota}\Theta(\iota-\varsigma)\mathfrak{Z}(\varsigma)d\varsigma\right],$$

if

(i) $\mathscr{R}(0)=\mathrm{I}$, $\|\mathscr{R}(\iota)\|\leq\mathrm{M}e^{\lambda\iota}$ *for some constant* λ *and* $\mathrm{M}\geq 1$.

(ii) $\forall\ \mathfrak{Z}\in\mathscr{H}$, $\mathscr{R}(\iota)\mathfrak{Z}$ *is strongly continuous for* $\iota\in l$.

(iii) $\forall\ \iota\in l$, $\mathscr{R}(\iota)\in\mathcal{L}(\mathscr{K})$. $\forall\ \mathfrak{Z}\in\mathscr{K}$, $\mathscr{R}(\cdot)\mathfrak{Z}\in\mathcal{C}^1(l,\mathscr{H})\cap\mathcal{C}(l,\mathscr{K})$ *and*

$$\begin{aligned}\frac{d}{d\iota}\mathscr{R}(\iota)\mathfrak{Z} &= \mathfrak{A}\left[\mathscr{R}(\iota)\mathfrak{Z}+\int_0^{\iota}\Theta(\iota-\varsigma)\mathscr{R}(\varsigma)\mathfrak{Z}d\varsigma\right]\\ &= \mathscr{R}(\iota)\mathfrak{A}\mathfrak{Z}+\int_0^{\iota}\mathscr{R}(\iota-\varsigma)\mathfrak{A}\Theta(\varsigma)d\varsigma,\ \iota\in l.\end{aligned}$$

For more details on the resolvent operator, we refer to [37, 38].

Definition 2.4. *Let* $\mathfrak{Z}:(-\infty,\mathfrak{b}]\to\mathscr{H}$ *is known as mild solution of (1.1) if*

(i) $\mathfrak{Z}(\iota)$ *is measurable and* $\Im_{\iota}-$*adapted for each* $\iota\geq 0$.

(ii) $\mathfrak{z}(\iota)$ *has cadlag paths on* $\iota \in [0, \mathfrak{b}]$ *a.s. satisfies the integral equation*

$$\mathfrak{z}(\iota) = \begin{cases} \phi(\iota), & \iota \in (-\infty, 0]; \\ \mathscr{R}(\iota)\left[\phi(0) + \mathfrak{h}(0, \phi(0))\right] - \mathfrak{h}(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}) & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{f}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\varsigma & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{C}(\varsigma)d\varsigma + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{g}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\omega(\varsigma) & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\sigma(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in [0, \iota_1]; \\ \mathfrak{G}_i(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}), & \iota \in \cup_{i=1}^n (\iota_i, \varsigma_i]; \\ \mathscr{R}(\iota - \varsigma_i)\left[\mathfrak{G}_i(\varsigma_i, \mathfrak{z}_{\vartheta(\varsigma_i,\mathfrak{z}_{\varsigma_i})}) + \mathfrak{h}(\varsigma_i, \mathfrak{z}_{\vartheta(\varsigma_i,\mathfrak{z}_{\varsigma_i})})\right] - \mathfrak{h}(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}) & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{f}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\varsigma + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{C}(\varsigma)d\varsigma & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{g}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\omega(\varsigma) & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\sigma(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in (\varsigma_i, \iota_{i+1}]. \end{cases} \tag{2.1}$$

3 Existence Results

Consider the space $\mathscr{D}_\mathfrak{b} : \mathfrak{z} : (-\infty, \mathfrak{b}] \to \mathscr{H}$ such that $\mathfrak{z}_0 \in \mathscr{D}, \mathfrak{z}\mid_{[0,\mathfrak{b}]} \in \mathscr{PC}([0, \mathfrak{b}], \mathscr{H})$ with seminorm $\|.\|_\mathfrak{b}$ is

$$\|\mathfrak{z}\|_\mathfrak{b} = \|\mathfrak{z}_0\|_\mathscr{D} + \sup_{\varsigma \in [0,\mathfrak{b}]} \left(\mathbb{E}\left(|\mathfrak{z}(\varsigma)|^2\right)\right)^{1/2}, \quad \mathfrak{z} \in \mathscr{D}_\mathfrak{b}.$$

Consider the operator $\Phi : \mathscr{D}_\mathfrak{b} \to \mathscr{D}_\mathfrak{b}$ is

$$(\Phi\mathfrak{z})(\iota) = \begin{cases} \phi(\iota), & \iota \in (-\infty, 0]; \\ \mathscr{R}(\iota)\left[\phi(0) + \mathfrak{h}(0, \phi(0))\right] - \mathfrak{h}(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}) & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{f}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\varsigma & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{C}(\varsigma)d\varsigma + \int_0^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{g}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\omega(\varsigma) & \\ + \int_0^\iota \mathscr{R}(\iota - \varsigma)\sigma(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in [0, \iota_1]; \\ \mathfrak{G}_i(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}), & \iota \in \cup_{i=1}^n (\iota_i, \varsigma_i]; \\ \mathscr{R}(\iota - \varsigma_i)\left[\mathfrak{G}_i(\varsigma_i, \mathfrak{z}_{\vartheta(\varsigma_i,\mathfrak{z}_{\varsigma_i})}) + \mathfrak{h}(\varsigma_i, \mathfrak{z}_{\vartheta(\varsigma_i,\mathfrak{z}_{\varsigma_i})})\right] - \mathfrak{h}(\iota, \mathfrak{z}_{\vartheta(\iota,\mathfrak{z}_\iota)}) & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{f}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\varsigma + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{C}(\varsigma)d\varsigma & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\mathfrak{g}(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\omega(\varsigma) & \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota - \varsigma)\sigma(\varsigma, \mathfrak{z}_{\vartheta(\varsigma,\mathfrak{z}_\varsigma)})d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in (\varsigma_i, \iota_{i+1}]. \end{cases} \tag{3.1}$$

Set $\overline{\phi}(.) : (-\infty, \mathfrak{b}] \to \mathscr{H}$ is defined by

$$\overline{\phi}(\iota) = \begin{cases} \phi(\iota), & \iota \in (-\infty, 0]; \\ \mathscr{R}(\iota)\phi(0), & \iota \in J. \end{cases}$$

$\therefore\ \overline{\phi}_0 = \phi.$

If $\mathfrak{Z}$ hold (2.1), we can decompose $\mathfrak{Z}(\iota) = \overline{\phi}(\iota) + \mathfrak{x}(\iota),\ \iota \in (-\infty, \mathfrak{b}]$ iff $\mathfrak{x}_0 = 0$ and

$$(\mathfrak{x})(\iota) = \begin{cases} \mathscr{R}(\iota)\left[\mathfrak{h}(0,\phi(0))\right] - \mathfrak{h}\big[\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big] \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\varsigma \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\omega(\varsigma) \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\sigma\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in [0,\iota_1]; \\ \mathfrak{G}_i\big[\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big], & \iota \in \cup_{i=1}^n(\iota_i,\varsigma_i]; \\ \mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{G}_i\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) \\ +\mathfrak{h}\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right)\Big] \\ -\mathfrak{h}\big[\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big] \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\varsigma \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\omega(\varsigma) \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\sigma\big[s, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\mathrm{B}^{\mathrm{H}}(\iota), & \iota \in (\varsigma_i,\iota_{i+1}]. \end{cases} \tag{3.2}$$

We may define $\mathscr{D}_{\mathfrak{b}}^0 = \{\mathfrak{x} \in \mathscr{D}_{\mathfrak{b}} : \mathfrak{x}_0 = 0 \in \mathscr{D}\}$. For any $\mathfrak{x} \in \mathscr{D}_{\mathfrak{b}}^0$,

$$\|\mathfrak{x}\|_{\mathfrak{b}} = \|\mathfrak{x}_0\|_{\mathscr{D}} + \sup_{\varsigma\in[0,\mathfrak{b}]}\left(\mathbb{E}\left(\|\mathfrak{x}(\varsigma)\|^2\right)\right)^{1/2} = \sup_{\varsigma\in[0,\mathfrak{b}]}\left(\mathbb{E}\left(\|\mathfrak{x}(\varsigma)\|^2\right)\right)^{1/2},\ \ \mathfrak{x} \in \mathscr{D}_{\mathfrak{b}}.$$

It is easy to verify that $\left(\mathscr{D}_{\mathfrak{b}}^0, \|.\|_{\mathscr{D}_{\mathfrak{b}}^0}\right)$ is a Banach space. For each $\mathfrak{r} > 0$,

$$\mathbb{B}_{\mathfrak{r}} = \{\mathfrak{x} \in \mathscr{D}_{\mathfrak{b}}^0 : \mathbb{E}\,\|\mathfrak{x}\|^2 \le \mathfrak{r}\},$$

then for each $\mathfrak{r}$, $\mathbb{B}_{\mathfrak{r}}$ is a bounded closed convex set in $\mathscr{D}_{\mathfrak{b}}^0$. By Lemma 2.3,

$$\begin{aligned} &\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right\|_{\mathscr{D}}^2 \\ &\le 2\left(\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\|_{\mathscr{D}}^2 + \|\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\|_{\mathscr{D}}^2\right) \\ &\le 4\Big[(\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^\phi)^2\mathbb{E}\|\overline{\phi_0}\|_{\mathscr{D}}^2 + \mathscr{N}_{\mathfrak{b}}^2 \sup_{0\le s\le\iota}\mathbb{E}\left\|\overline{\phi}(\varsigma)\right\|^2 \\ &\quad + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^\phi)^2\mathbb{E}\|\mathfrak{x}_0\|_{\mathscr{D}}^2 + \mathscr{N}_{\mathfrak{b}}^2 \sup_{0\le s\le\iota}\mathbb{E}\left\|\mathfrak{x}(\varsigma)\right\|^2\Big] \\ &\le 4\big[\mathscr{N}_{\mathfrak{b}}^2\mathfrak{r} + \big(\mathscr{N}_{\mathfrak{b}}^2\mathcal{M}^2\mathscr{K}^2 + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^\phi)^2\big)\|\phi\|_{\mathscr{D}}^2\big] = \hat{\mathfrak{r}},\ \ \iota \in [0,\mathfrak{b}]. \end{aligned} \tag{3.3}$$

Define the operator $\overline{\Phi}:\mathscr{D}^0_{\mathfrak{b}}\to\mathscr{D}^0_{\mathfrak{b}}$

$$(\overline{\Phi}\mathfrak{x})(\iota)=\begin{cases}\mathscr{R}(\iota)\left[\mathfrak{h}(0,\phi(0))\right]-\mathfrak{h}\big[\iota,\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}+\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big]\\ +\int_0^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{f}\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\varsigma\\ +\int_0^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma\\ +\int_0^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{g}\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\omega(\varsigma)\\ +\int_0^\iota\mathscr{R}(\iota-\varsigma)\sigma\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]dB^{H}(\iota), & \iota\in[0,\iota_1];\\ \mathfrak{G}_i\big[\iota,\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}+\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big], & \iota\in\cup_{i=1}^n(\iota_i,\varsigma_i];\\ \mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{G}_i\big(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}+\mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big)\\ +\mathfrak{h}\left(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}+\mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right)\Big]\\ -\mathfrak{h}\big[\iota,\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}+\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big]\\ +\int_{\varsigma_i}^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{f}\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\varsigma\\ +\int_{\varsigma_i}^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma\\ +\int_{\varsigma_i}^\iota\mathscr{R}(\iota-\varsigma)\mathfrak{g}\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]d\omega(\varsigma)\\ +\int_{\varsigma_i}^\iota\mathscr{R}(\iota-\varsigma)\sigma\big[s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big]dB^{H}(\iota), & \iota\in(\varsigma_i,\iota_{i+1}].\end{cases}\tag{3.4}$$

In order to show the existence results of system (1.1), we need to show that the operator $\overline{\Phi}$ has unique fixed point.

Existence result based on Banach FPT:

We may take into account the following assumptions:

(A1) $\mathfrak{h}:J\times\mathscr{D}\to\mathscr{H}$ is continuous and $\exists$ a constant $\mathscr{L}_{\mathfrak{h}}$ being +ve such that

$$\mathbb{E}\left\|\mathfrak{h}(\iota,\mathfrak{z}_1)-\mathfrak{h}(\iota,\mathfrak{z}_2)\right\|^2\leq\mathscr{L}_{\mathfrak{h}}\left\|\mathfrak{z}_1-\mathfrak{z}_2\right\|^2_{\mathscr{D}},$$

$\forall\ \iota\in J$ and $\mathfrak{z}_1,\mathfrak{z}_2\in\mathscr{D}$.

(A2) The continuous function $\mathfrak{f}:J\times\mathscr{D}\to\mathscr{H}$, $\exists$ a constant $\mathscr{L}_{\mathfrak{f}}$ being +ve such that

$$\mathbb{E}\left\|\mathfrak{f}(\iota,\mathfrak{z}_1)-\mathfrak{f}(\iota,\mathfrak{z}_2)\right\|^2\leq\mathscr{L}_{\mathfrak{f}}\left\|\mathfrak{z}_1-\mathfrak{z}_2\right\|^2_{\mathscr{D}},$$

$\forall\ \iota\in J$ and $\mathfrak{z}_1,\mathfrak{z}_2\in\mathscr{D}$.

(A3) The function $\mathfrak{g}:J\times\mathscr{D}\to\mathcal{L}^0_2(\mathscr{K},\mathscr{H})$ is continuous $\exists$ a constant $\mathscr{L}_{\mathfrak{g}}$ +ve such that

$$\mathbb{E}\left\|\mathfrak{g}(\iota,\mathfrak{z}_1)-\mathfrak{g}(\iota,\mathfrak{z}_2)\right\|^2_{\mathcal{L}^0_2}\leq\mathscr{L}_{\mathfrak{g}}\left\|\mathfrak{z}_1-\mathfrak{z}_2\right\|^2_{\mathscr{D}},$$

$\forall\ \iota\in J$ and $\mathfrak{z}_1,\mathfrak{z}_2\in\mathscr{D}$.

(A4) The function $\sigma : J \times \mathscr{D} \to \mathcal{L}_2^0(\mathscr{Y}, \mathscr{H})$ is continuous and $\exists$ a constant $\mathscr{L}_\sigma$ being +ve such that

$$\mathbb{E}\left\|\sigma(\iota, \mathfrak{z}_1) - \sigma(\iota, \mathfrak{z}_2)\right\|_{\mathcal{L}_2^0}^2 \leq \mathscr{L}_\sigma \left\|\mathfrak{z}_1 - \mathfrak{z}_2\right\|_{\mathscr{D}}^2,$$

$\forall\, \iota \in J$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathscr{D}$.

(A5) The map $\mathfrak{G}_i : (\iota_i, \varsigma_i] \times \mathscr{D} \to \mathscr{H}, \quad i = 1, 2, \cdots n$ are continuous and $\exists\, \mathscr{L}_i > 0$ such that

$$\mathbb{E}\left\|\mathfrak{G}_i(\iota, \mathfrak{z}_1) - \mathfrak{G}_i(\iota, \mathfrak{z}_2)\right\|^2 \leq \mathscr{L}_i \left\|\mathfrak{z}_1 - \mathfrak{z}_2\right\|_{\mathscr{D}}^2,$$

$\forall\, \iota \in J$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathscr{D}$.

Theorem 3.1. *Assume that (A1)-(A5) hold. Then $\exists$ unique mild solution of (1.1) on J, provided*

$$5\mathscr{N}_{\mathfrak{b}}^2\left[(1+2\mathbb{C}^2)\mathscr{L}_i + \mathfrak{b}(\mathbb{C}^2+1)[\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}] + \mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}}\mathscr{L}_\sigma + (\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right] < 1. \tag{3.5}$$

Proof. In this part, we need to prove $\overline{\Phi}$ has a unique fixed point. For $\mathfrak{x}, \mathfrak{y} \in \mathscr{D}_{\mathfrak{b}}^0$ and $\iota \in [0, \iota_1]$

$$\begin{aligned}
&\mathbb{E}\left\|(\overline{\Phi}\mathfrak{x})(\iota) - (\overline{\Phi}\mathfrak{y})(\iota)\right\|^2 \\
&\leq 4\mathbb{E}\left\|\mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big) - \mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)} + \mathfrak{y}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)}\big)\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Big[\mathfrak{f}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big)\right. \\
&- \left.\mathfrak{f}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)} + \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\big)\Big]d\varsigma\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Big[\mathfrak{g}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big)\right. \\
&- \left.\mathfrak{g}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)} + \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\big)\Big]d\omega(\varsigma)\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Big[\sigma\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\big)\right. \\
&- \left.\sigma\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)} + \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\big)\Big]d\mathrm{B}^{\mathrm{H}}(\iota)\right\|^2 \\
&\leq 4\mathbb{C}^2\mathscr{L}_{\mathfrak{h}}\left\|\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} - \mathfrak{y}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)}\right\|_{\mathscr{D}}^2 + 4\mathbb{C}^2\int_0^\iota \mathscr{L}_{\mathfrak{f}}\left\|\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} - \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\right\|_{\mathscr{D}}^2 d\varsigma \\
&+ 4\mathbb{C}^2\int_0^\iota \mathscr{L}_{\mathfrak{g}}\left\|\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} - \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\right\|_{\mathscr{D}}^2 d\varsigma \\
&+ 4\mathbb{C}^2\left(\mathcal{H}(2\mathcal{H}-1)\mathrm{T}^{2\mathcal{H}-1}\right)\int_0^\iota \mathscr{L}_\sigma\left\|\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} - \mathfrak{y}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{y}_\varsigma)}\right\|_{\mathscr{D}}^2 d\varsigma \\
&\leq 4\mathscr{N}_{\mathfrak{b}}^2\left[\mathbb{C}^2\left(\mathscr{L}_{\mathfrak{h}} + \iota_1(\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}) + \mathcal{H}(2\mathcal{H}-1)\iota_1^{2\mathcal{H}}\mathscr{L}_\sigma\right) + \mathscr{L}_{\mathfrak{h}}\right]\mathbb{E}\left\|\mathfrak{x}(\iota) - \mathfrak{y}(\iota)\right\|^2.
\end{aligned} \tag{3.6}$$

For $\iota \in \cup_{i=1}^n (\iota_i, \varsigma_i]$,

$$\begin{aligned}
\mathbb{E}\left\|(\overline{\Phi}\mathfrak{x})(\iota) - (\overline{\Phi}\mathfrak{y})(\iota)\right\|^2 &\leq \mathbb{E}\Big\|\mathfrak{G}_i\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\big) \\
&- \mathfrak{G}_i\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)} + \mathfrak{y}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)}\big)\Big\|^2 \\
&\leq \mathscr{L}_i\left\|\mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} - \mathfrak{y}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)}\right\|_{\mathscr{D}}^2 \\
&\leq \mathscr{L}_i\mathscr{N}_{\mathfrak{b}}^2\mathbb{E}\left\|\mathfrak{x}(\iota) - \mathfrak{y}(\iota)\right\|^2.
\end{aligned} \tag{3.7}$$

II^{ly}, for $\iota \in \cup_{i=1}^{n}(\varsigma_i, \iota_{i+1}]$,

$$\begin{aligned}
&\mathbb{E}\left\|(\overline{\Phi}\mathfrak{x})(\iota) - (\overline{\Phi}\mathfrak{y})(\iota)\right\|^2 \\
&\leq 5\mathbb{E}\Bigg\|\mathscr{R}(\iota-\varsigma_i)\left[\mathfrak{G}_i\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) - \mathfrak{G}_i\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})} + \mathfrak{y}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})}\big)\right] \\
&+ \mathscr{R}(\iota-\varsigma_i)\left[\mathfrak{h}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) - \mathfrak{h}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})} + \mathfrak{y}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})}\big)\right]\Bigg\|^2 \\
&+ 5\mathbb{E}\left\|\mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_{\iota}+\mathfrak{x}_{\iota})} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_{\iota}+\mathfrak{x}_{\iota})}\big) - \mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_{\iota}+\mathfrak{y}_{\iota})} + \mathfrak{y}_{\vartheta(\iota,\overline{\phi}_{\iota}+\mathfrak{y}_{\iota})}\big)\right\|^2 \\
&+ 5\mathbb{E}\Bigg\|\int_{\varsigma_i}^{\iota}\mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{f}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) \\
&- \mathfrak{f}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})} + \mathfrak{y}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})}\big)\Big]d\varsigma\Bigg\|^2 \\
&+ 5\mathbb{E}\Bigg\|\int_{\varsigma_i}^{\iota}\mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{g}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) \\
&- \mathfrak{g}\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})} + \mathfrak{y}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})}\big)\Big]d\omega(\varsigma)\Bigg\|^2 \\
&+ 5\mathbb{E}\Bigg\|\int_{\varsigma_i}^{\iota}\mathscr{R}(\iota-\varsigma_i)\Big[\sigma\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\big) \\
&- \sigma\big(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})} + \mathfrak{y}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{y}_{\varsigma_i})}\big)\Big]d\mathrm{B}^{\mathrm{H}}(\iota)\Bigg\|^2 \\
&\leq 5\mathbb{C}^2\mathscr{N}_{\mathfrak{b}}^2\Big[\mathscr{L}_i + \mathscr{L}_{\mathfrak{h}} + (\iota_{i+1}-\varsigma_i)[\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}] + \mathcal{H}(2\mathcal{H}-1)(\iota_{i+1}-\varsigma_i)^{2\mathcal{H}}\mathscr{L}_{\sigma}\Big] \\
&\times \mathbb{E}\left\|\mathfrak{x}(\iota) - \mathfrak{y}(\iota)\right\|^2. \qquad (3.8)
\end{aligned}$$

Therefore, $\forall\ \iota \in J$,

$$\begin{aligned}
\mathbb{E}\left\|(\overline{\phi}\mathfrak{x})(\iota) - (\overline{\phi}\mathfrak{y})(\iota)\right\|^2 \leq\ & 5\mathscr{N}_{\mathfrak{b}}^2\Big[(1+2\mathbb{C}^2)\mathscr{L}_i + \mathfrak{b}(\mathbb{C}^2+1)[\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}] + \mathcal{H}(2\mathcal{H}-1)\mathfrak{b}^{2\mathcal{H}}\mathscr{L}_{\sigma} \\
&+ (\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\Big]\mathbb{E}\left\|\mathfrak{x}(\iota) - \mathfrak{y}(\iota)\right\|^2. \qquad (3.9)
\end{aligned}$$

Thus $\overline{\phi}$ is a contraction mapping and has unique fixed point $\mathfrak{x} \in \mathscr{D}_{\mathfrak{b}}^0$.

□

In order to prove the existence results of system (1.1), under non-Lipschitz continuity of nonlinear term, we will utilize Darbo and Darbo-Sadovskii fixed point theorems.

Existence result based on Darbo's FPT:

Let us consider the following hypotheses:

(A'1) The function $\mathfrak{h} : J \times \mathscr{D} \to \mathscr{H}$ is continuous and $\exists\ \mathscr{L}_{\mathfrak{h}} > 0$ such that

$$\mathbb{E}\left\|\mathfrak{h}(\iota, \mathfrak{z})\right\|^2 \leq \mathscr{L}_{\mathfrak{h}}(1 + \|\mathfrak{z}\|_{\mathscr{D}}^2), \quad \forall \quad \iota \in [0, \mathfrak{b}] \text{ and } \mathfrak{z} \in \mathscr{D}.$$

(A'2) The map $\mathfrak{f} : J \times \mathscr{D} \to \mathscr{H}$ satisfies the following :

(a) $\mathfrak{f}$ hold the Carateŏdory type condition (i.e.) $\mathfrak{f}(\iota, .) : \mathscr{D} \to \mathscr{H}$ is continuous for a.e., $\iota \in J$

and $\forall\ \mathfrak{z} \in \mathscr{D}$, the function $\mathfrak{f}(.,\mathfrak{z}) : J \to \mathscr{H}$ is strongly measurable.

(b) $\exists$ a continuous nondecreasing function $\mathfrak{m}_{\mathfrak{f}} : [0,\infty) \to (0,+\infty)$ and a +ve integrable function $\varphi : \mathcal{L}^1([0,\mathfrak{b}],\mathbb{R}^+)$ such that

$$\mathbb{E}\,\|\mathfrak{f}(\iota,\mathfrak{z})\|^2 \le \varphi(\iota)\mathfrak{m}_{\mathfrak{f}}(\|\mathfrak{z}\|^2_{\mathscr{D}}).$$

(A'3) The map $\mathfrak{g} : J \times \mathscr{D} \to \mathcal{L}^0_2(\mathscr{K},\mathscr{H})$ satisfies the following

(a) $\mathfrak{g}$ satisfies the condition (i.e.) $\mathfrak{g}(\iota,.) : \mathscr{D} \to \mathcal{L}^0_2(\mathscr{K},\mathscr{H})$ is continuous for a.e., $\iota \in J, \forall\ \mathfrak{u} \in \mathscr{D}$, the function $\mathfrak{g}(.,\mathfrak{z}) : J \to \mathcal{L}^0_2(\mathscr{K},\mathscr{H})$ is strongly measurable.

(b) $\exists$ a continuous nondecreasing function $\mathfrak{m}_{\mathfrak{g}} : [0,\infty) \to (0,+\infty)$ and a +ve integrable function $\upsilon : \mathcal{L}^1([0,\mathfrak{b}],\mathbb{R}^+)$ such that

$$\mathbb{E}\,\|\mathfrak{g}(\iota,\mathfrak{z})\|^2 \le \upsilon(\iota)\mathfrak{m}_{\mathfrak{g}}(\|\mathfrak{z}\|^2_{\mathscr{D}}).$$

(A'4) The map $\sigma J \times \mathscr{D} \to \mathcal{L}^0_2(\mathscr{Y},\mathscr{H})$ hold the following

(a) σ satisfies the condition (i.e.) $\sigma(\iota,.) : \mathscr{D} \to \mathcal{L}^0_2(\mathscr{Y},\mathscr{H})$ is continuous for a.e., $\iota \in J, \forall\ \mathfrak{z} \in \mathscr{D}$, the function $\sigma(.,\mathfrak{u}) : J \to \mathcal{L}^0_2(\mathscr{Y},\mathscr{H})$ is strongly measurable.

(b) $\exists$ a continuous nondecreasing function $\mathfrak{m}_\sigma : [0,\infty) \to (0,+\infty)$ and a +ve integrable function $\beta : \mathcal{L}^1(J,\mathbb{R}^+)$ such that

$$\mathbb{E}\,\|\sigma(\iota,\mathfrak{z})\|^2 \le \beta(\iota)\mathfrak{m}_\sigma(\|\mathfrak{z}\|^2_{\mathscr{D}}).$$

(A'5) The function $\mathfrak{G}_i(\iota_i,\varsigma_i] \times \mathscr{D} \to \mathscr{H}, i = 1,2,\cdots,n$ are continuous and $\exists\ \mathscr{L}_i > 0,\ i = 1,2,\cdots,n$ such that

$$\mathbb{E}\,\|\mathfrak{G}_i(\iota,\mathfrak{z})\|^2 \le \mathscr{L}_i(1 + \|\mathfrak{z}\|^2_{\mathscr{D}}),\quad \forall\ \ \iota \in (\iota_i,\varsigma_i]\ \textit{and}\ \mathfrak{z} \in \mathscr{D}.$$

Theorem 3.2. *Let $\mathfrak{z}_0 \in \mathcal{L}^2_0(\Omega,\mathscr{H})$, (A1), (A5) and (A'1)-(A'5) gets satisfied. If $\int_1^\infty \frac{dx}{\mathfrak{m}_{\mathfrak{f}}(\varsigma)+\mathfrak{m}_{\mathfrak{g}}(\varsigma)+\mathfrak{m}_\sigma(\varsigma)} = \infty$ and*

$$2\mathscr{N}_{\mathfrak{b}}^2\left[(1+2\mathbb{C}^2)\mathscr{L}_i + (1+2\mathbb{C}^2)\mathscr{L}_{\mathfrak{h}}\right] < 1, \tag{3.10}$$

then system (1.1) has at least one mild solution on J.

Proof. To prove the main results. Our proof will split into several steps.

Step 1: To claim the set $\{\mathfrak{x} \in \mathscr{PC} : \mathfrak{x} = \lambda\overline{\phi}\mathfrak{x},\ \ for\ \ 0 < \lambda < 1\}$ is bounded.

Let $\bar{\mathfrak{x}}$ be a solution of $\lambda\overline{\phi}\mathfrak{x}$ for $0 < \lambda < 1$,

$$\sup_{0\le s\le \iota}\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}(\varsigma))} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}(\varsigma))}\right\|^2_{\mathscr{D}} \le 4\left[\mathscr{N}_{\mathfrak{b}}^2\mathbb{E}\|\bar{\mathfrak{x}}\|^2 + \left(\mathscr{N}_{\mathfrak{b}}^2\mathcal{M}^2\mathscr{K}^2 + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^\phi)^2\right)\|\phi\|^2_{\mathscr{D}}\right],$$

where $\|\bar{\mathfrak{x}}\|^2 = \sup\limits_{0\le s\le\iota}\|\bar{\mathfrak{x}}(\varsigma)\|^2$.

Consider,

$$\mathfrak{p}(\iota) = 4\left[\mathscr{N}_{\mathfrak{b}}^2\mathbb{E}\|\bar{\mathfrak{x}}\|^2 + \left(\mathscr{N}_{\mathfrak{b}}^2\mathcal{M}^2\mathscr{K}^2 + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^\phi)^2\right)\|\phi\|^2_{\mathscr{D}}\right],\quad \iota \in [0,\mathfrak{b}]. \tag{3.11}$$

For $\iota \in [0, \iota_1]$:

$$\begin{aligned}
&\mathbb{E}\left\|\bar{\mathfrak{x}}(\iota)\right\|^2 \\
&\leq 6\Big\{\mathbb{E}\left\|\mathscr{R}(\iota)\mathfrak{h}(0,\phi(0))\right\|^2 + \mathbb{E}\left\|\mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\big)\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)d\varsigma\right\|^2 + \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)d\omega(\varsigma)\right\|^2 \\
&\quad + \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\sigma\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)dB^{H}(\iota)\right\|^2\Big\} \\
&\leq 6\Big\{\mathbb{C}^2\mathscr{L}_{\mathfrak{h}}(1+\|\phi\|^2_{\mathscr{D}}) + \mathscr{L}_{\mathfrak{h}}\left(1+\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\right\|^2\right) \\
&\quad + \mathbb{C}^2\int_0^\iota \varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\varsigma))d\varsigma + \mathbb{C}^2\int_0^\iota \mathbb{E}\|\mathscr{C}(\varsigma)\|^2 d\varsigma \\
&\quad + \mathbb{C}^2\int_0^\iota \upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\varsigma))d\varsigma + \mathbb{C}^2 H(2H-1)\mathrm{T}^{2H-1}\int_0^\iota \beta(\varsigma)\mathfrak{m}_\sigma(\mathfrak{p}(\varsigma))d\varsigma\Big\} \\
&\leq 6\Big\{\mathbb{C}^2\mathscr{L}_{\mathfrak{h}}(1+\|\phi\|^2_{\mathscr{D}}) + \mathscr{L}_{\mathfrak{h}}\left(1+\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\right\|^2\right) \\
&\quad + \mathbb{C}^2\mathfrak{b}\mathbb{E}\|\mathscr{C}(\iota)\|^2_{\mathcal{L}^2_{\mathfrak{F}}} + \mathbb{C}^2\mathfrak{b}\big[\varphi(\iota)\mathfrak{m}_{\mathfrak{f}} + \upsilon(\iota)\mathfrak{m}_{\mathfrak{g}} \\
&\quad + H(2H-1)\mathfrak{b}^{2H}\beta_\iota\mathfrak{m}_\sigma\big]\left[\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\right\|^2_{\mathscr{D}}\right]\Big\}.
\end{aligned}$$

For $\iota \in \cup_{i=1}^n(\iota_i, \varsigma_i]$, we have

$$\begin{aligned}
\mathbb{E}\|\bar{\mathfrak{x}}(\mathfrak{x})\|^2 &\leq \mathscr{L}_i\left(1+\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\right\|^2_{\mathscr{D}}\right) \\
&\leq \mathscr{L}_i(1+\mathfrak{p}(\iota)).
\end{aligned}$$

II^{ly}, for $\iota \in \cup_{i=1}^n(\varsigma_i, \iota_{i+1}]$,

$$\begin{aligned}
\mathbb{E}\|\bar{\mathfrak{x}}(\iota)\|^2 \leq\ & 6\Big\{\mathbb{E}\Big\|\mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{G}_i\big(\varsigma_i + \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\bar{\mathfrak{x}}_{\varsigma_i})} + \bar{\mathfrak{x}}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\bar{\mathfrak{x}}_{\varsigma_i})}\big) \\
+\ & \mathfrak{h}\big(s_i + \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\bar{\mathfrak{x}}_{\varsigma_i})} + \bar{\mathfrak{x}}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\bar{\mathfrak{x}}_{\varsigma_i})}\big)\Big]\Big\|^2 + \mathbb{E}\left\|\mathfrak{h}\big(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)} + \bar{\mathfrak{x}}_{\vartheta(\iota,\overline{\phi}_\iota+\bar{\mathfrak{x}}_\iota)}\big)\right\|^2 \\
+\ & \mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)d\varsigma\right\|^2 + \mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{C}(\varsigma)d\varsigma\right\|^2 \\
+\ & \mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)d\omega(\varsigma)\right\|^2 \\
+\ & \mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\sigma\big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)} + \bar{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\bar{\mathfrak{x}}_\varsigma)}\big)dB^{H}(\iota)\right\|^2\Big\} \\
\leq\ & 6\Big\{2\mathbb{C}^2\mathscr{L}_i(1+\mathfrak{p}(\varsigma_i)) + 2\mathbb{C}^2\mathscr{L}_{\mathfrak{h}}(1+\mathfrak{p}(\varsigma_i)) + \mathscr{L}_{\mathfrak{h}}(1+\mathfrak{p}(\iota)) + \mathbb{C}^2(\iota_{i+1}-\varsigma_i) \\
\times\ & \int_{\varsigma_i}^\iota \varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\varsigma))d\varsigma + \mathbb{C}^2(\iota_{i+1}-\varsigma_i)\int_{\varsigma_i}^\iota \mathbb{E}\|\mathscr{C}\|^2_{\mathcal{L}^2_{\mathfrak{F}}}d\varsigma \\
+\ & \mathbb{C}^2(\iota_{i+1}-\varsigma_i)\int_{\varsigma_i}^\iota \upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\varsigma))d\varsigma
\end{aligned}$$

$$+ \quad \mathbb{C}^2 \mathrm{H}(2\mathrm{H}-1)(\iota_{i+1}-\varsigma_i)^{2\mathrm{H}} \int_{\varsigma_i}^{\iota} \beta(\varsigma)\mathfrak{m}_\sigma(\mathfrak{p}(\varsigma))d\varsigma \Big\}.$$

Thus $\forall \iota \in J$,

$$\begin{aligned} \mathbb{E}\|\bar{\mathfrak{x}}(\iota)\|^2 \quad &\leq \quad \mathscr{B}^* + \mathscr{L}_i \mathfrak{p}(\iota) + 12\mathbb{C}^2 \left[\mathscr{L}_i \mathfrak{p}(\iota) + \mathscr{L}_{\mathfrak{h}} \mathfrak{p}(\iota)\right] + 6\mathscr{L}_{\mathfrak{h}} \mathfrak{p}(\iota) + 6\mathbb{C}^2 \mathfrak{b} \int_0^{\iota} \varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\varsigma))d\varsigma \\ &+ \quad 6\mathbb{C}^2 \mathfrak{b} \int_0^{\iota} \upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\varsigma))d\varsigma + 6\mathbb{C}^2 \mathcal{H}(2\mathcal{H}-1)\mathfrak{b}^{2\mathcal{H}} \int_0^{\iota} \beta(\varsigma)\mathfrak{m}_\sigma(\mathfrak{p}(\varsigma))d\varsigma; \end{aligned}$$

where,

$$\mathscr{B}^* = (12\mathbb{C}^2+1)\mathscr{L}_i + 6\mathbb{C}^2 \mathscr{L}_{\mathfrak{h}}(1+\|\phi\|^2_{\mathscr{D}}) + 6[(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}] + 6\mathbb{C}^2 \mathfrak{b}\mathbb{E}\|\mathscr{C}\|^2_{\mathcal{L}^2_{\mathfrak{I}}}.$$

Substituting the values in (3.11), we obtain

$$\begin{aligned} &\mathfrak{p}(\iota) \\ &\leq 4\left[\mathscr{N}_{\mathfrak{b}}^2 \mathscr{B}^* + \left(\|\mathscr{N}_{\mathfrak{b}}^2 \mathcal{M}^2 \mathscr{K}^2 + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^{\phi})^2\right)\|^2_{\mathscr{D}}\right] + 4\mathscr{N}_{\mathfrak{b}}^2 \left[(1+2\mathbb{C}^2)\mathscr{L}_i + 6(1+2\mathbb{C}^2)\mathscr{L}_{\mathfrak{h}}\right]\mathfrak{p}(\iota) \\ &\quad + 6\mathbb{C}^2 \mathfrak{b} \int_0^{\iota} \varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\varsigma))d\varsigma + 6\mathbb{C}^2 \mathfrak{b} \int_0^{\iota} \upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\varsigma))d\varsigma \\ &\quad + 6\mathbb{C}^2 \mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}} \int_0^{\iota} \beta(\varsigma)\mathfrak{m}_\sigma(\mathfrak{p}(\varsigma))d\varsigma \\ &\leq \frac{\xi_2}{1-\xi_1} + \frac{\xi_3}{1-\xi_1}\int_0^{\iota} \varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\varsigma))d\varsigma + \frac{\xi_4}{1-\xi_1}\int_0^{\iota} \upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\varsigma))d\varsigma \\ &\quad + \frac{\xi_5}{1-\xi_1}\int_0^{\iota} \beta(\varsigma)\mathfrak{m}_\sigma(\mathfrak{p}(\varsigma))d\varsigma; \end{aligned}$$

where,

$$\begin{aligned} \xi_1 &= 4\mathscr{N}_{\mathfrak{b}}^2 \max_{1\leq i\leq \mathscr{N}} \left[(1+2\mathbb{C}^2)\mathscr{L}_i + 6(1+2\mathbb{C}^2)\mathscr{L}_{\mathfrak{h}}\right] < 1, \\ \xi_2 &= 4\left[\mathscr{N}_{\mathfrak{b}}^2 \mathscr{B}^* + \left(\|\mathscr{N}_{\mathfrak{b}}^2 \mathcal{M}^2 \mathscr{K}^2 + (\mathbb{H}_{\mathfrak{b}} + \mathbb{J}^{\phi})^2\right)\|^2_{\mathscr{D}}\right], \\ \xi_3 &= 6\mathbb{C}^2\mathfrak{b};\ \xi_4 = 6\mathbb{C}^2\mathfrak{b};\ \xi_5 = 6\mathbb{C}^2\mathcal{H}(2\mathcal{H}-1)\mathfrak{b}^{2\mathcal{H}}, \end{aligned}$$

with $\zeta(0) = \frac{\xi_2}{1-\xi_1}$.

$$\begin{aligned} \zeta'(\iota) \quad &\leq \quad \frac{\xi_3}{1-\xi_1}\varphi(\iota)\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\iota)) + \frac{\xi_4}{1-\xi_1}\upsilon(\iota)\mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\iota)) + \frac{\xi_5}{1-\xi_1}\beta(\iota)\mathfrak{m}_\sigma(\mathfrak{p}(\iota)) \\ &\leq \quad \max\left\{\frac{\xi_3}{1-\xi_1}\varphi(\iota), \frac{\xi_4}{1-\xi_1}\upsilon(\iota), \frac{\xi_5}{1-\xi_1}\beta(\iota)\right\}\left[\mathfrak{m}_{\mathfrak{f}}(\mathfrak{p}(\iota)) + \mathfrak{m}_{\mathfrak{g}}(\mathfrak{p}(\iota)) + \mathfrak{m}_\sigma(\mathfrak{p}(\iota))\right]. \end{aligned}$$

Thus,

$$\int_{\zeta(0)}^{\zeta(\iota)} \frac{d\varsigma}{\mathfrak{m}_{\mathfrak{f}}(\varsigma) + \mathfrak{m}_{\mathfrak{g}}(\varsigma) + \mathfrak{m}_\sigma(\varsigma)} \leq \int_0^{\mathfrak{b}} \max\left\{\frac{\xi_3}{1-\xi_1}\varphi(\iota), \frac{\xi_4}{1-\xi_1}\upsilon(\iota), \frac{\xi_5}{1-\xi_1}\beta(\iota)\right\} d\iota < \infty.$$

Thus $\zeta(\iota)$ is bounded on $\mathfrak{b}$ which implies $\mathfrak{p}(\iota)$ is bounded thereby $\mathfrak{x}(.)$ is bounded in $\mathfrak{b}$.

Step 2: To claim $\overline{\phi} : \mathscr{D}^0_{\mathfrak{b}} \to \mathscr{D}^0_{\mathfrak{b}}$ is continuous.

Let $\{\mathfrak{x}^{(\mathfrak{r})}\}_{\mathfrak{r}\in\mathbb{N}} \subset \mathscr{D}_{\mathfrak{b}}^0$ such that $\{\mathfrak{x}^{(\mathfrak{r})}\} \to \mathfrak{x} \in \mathscr{D}_{\mathfrak{b}}^0$ as $\mathfrak{r} \to \infty$. $\exists\ \mathfrak{q} > 0$ such that $\|\mathfrak{x}^{(\mathfrak{r})}(\iota)\| \leq \mathfrak{q}\ \forall\ \mathfrak{r}$ and a.s. $\iota \in [0, \mathfrak{b}]$. By (3.3)

$$\left\|\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}(\iota))} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}(\iota))}\right\|_{\mathscr{D}}^2 \leq \hat{\mathfrak{r}}.$$

Using Lemma 2.2,

$$\begin{aligned}
\left\|\mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} - \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right\|_{\mathscr{D}}^2 &\leq 2\left[(\mathscr{N}(\iota))^2 \sup_{\varsigma\in[0,\iota]}\{\left\|\mathfrak{x}^{(\mathfrak{r})}(\varsigma) - \mathfrak{x}(\varsigma)\right\|^2\} + [\mathbb{H}(\iota)]^2\left\|\mathfrak{x}_0^{(\mathfrak{r})} - \mathfrak{x}_0\right\|_{\mathscr{D}}^2\right] \\
&= 2(\mathscr{N}(\iota))^2 \sup_{\varsigma\in[0,\iota]}\{\left\|\mathfrak{x}^{(\mathfrak{r})}(\varsigma) - \mathfrak{x}(\varsigma)\right\|^2\} \\
&\leq 2\mathscr{N}_{\mathfrak{b}}^2\left\|\mathfrak{x}^{(\mathfrak{r})} - \mathfrak{x}\right\|_{\mathfrak{b}}^2 \to 0\ as\ n \to \infty.
\end{aligned}$$

By the Caratheödory functions $\mathfrak{f}$, $\mathfrak{g}$ and σ,

$$\begin{aligned}
\lim_{\mathfrak{r}\to\infty}\mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) &= \mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right), \\
\lim_{\mathfrak{r}\to\infty}\mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) &= \mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right), \\
\lim_{\mathfrak{r}\to\infty}\sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) &= \sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right).
\end{aligned}$$

By apply Young's inequality and Lebesgue dominated convergence theorem,

for $\iota \in [0, \iota_1]$,

$\mathbb{E}\left\|(\overline{\phi}\mathfrak{x}^{(\mathfrak{r})})(\iota) - (\overline{\phi}\mathfrak{x})(\iota)\right\|^2$

$$\begin{aligned}
&\leq 4\Bigg[\mathbb{E}\left\|\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) - \mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 \\
&+ \mathbb{E}\Bigg\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Bigg[\mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) \\
&- \mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\Bigg]d\varsigma\Bigg\|^2 \\
&+ \mathbb{E}\Bigg\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Bigg[\mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) \\
&- \mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\Bigg]d\omega(\varsigma)\Bigg\|^2 \\
&+ \mathbb{E}\Bigg\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Bigg[\sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) \\
&- \sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\Bigg]d\mathcal{B}^{\mathcal{H}}(\varsigma)\Bigg\|^2\Bigg] \\
&\leq 4\Bigg[\mathbb{E}\left\|\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) - \mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 \\
&+ \mathbb{C}^2\int_0^\iota \mathbb{E}\left\|\mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}^{(\mathfrak{r})}_\iota)}\right) - \mathfrak{f}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 d\varsigma
\end{aligned}$$

$$+ \quad \mathbb{C}^2 \int_0^\iota \mathbb{E} \left\| \mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})}\right) - \mathfrak{g}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 d\varsigma$$

$$+ \quad \mathbb{C}^2\mathcal{H}(2\mathcal{H}-1)\mathrm{T}^{2\mathcal{H}-1} \int_0^\iota \mathbb{E} \left\| \sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})}\right)\right.$$

$$- \quad \left.\sigma\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 d\varsigma\Bigg] \to 0 \quad as \quad \mathfrak{r} \to \infty.$$

For $\iota \in \cup_{i=1}^n (\iota_i, \varsigma_i]$,

$$\begin{aligned}
\mathbb{E}\left\|(\overline{\phi}\mathfrak{x}^{(\mathfrak{r})})(\iota) - (\overline{\phi}\mathfrak{x})(\iota)\right\|^2 &= \mathbb{E}\left\|\mathfrak{G}_i\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{G}_i\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{y}_\iota)}\right)\right\|^2 \\
&\to 0 \quad as \quad \mathfrak{r} \to \infty.
\end{aligned}$$

II^{ly}, for $\iota \in \cup_{i=1}^n (\varsigma_i, \iota_{i+1}]$,

$\mathbb{E}\left\|(\overline{\phi}\mathfrak{x}^{(\mathfrak{r})})(\iota) - (\overline{\phi}\mathfrak{x})(\iota)\right\|^2$

$$\begin{aligned}
&\le 5\Bigg[2\mathbb{C}^2\Bigg[\mathbb{E}\left\|\mathfrak{G}_i\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i}^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i}^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{G}_i\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right)\right\|^2 \\
&+ \mathbb{E}\left\|\mathfrak{h}\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i}^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i}^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{h}\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right)\right\|^2\Bigg] \\
&+ \mathbb{E}\left\|\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 \\
&+ \mathbb{C}^2 \int_{\varsigma_i}^\iota \mathbb{E}\left\|\mathfrak{f}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{f}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)\right\|^2 d\varsigma \\
&+ \mathbb{C}^2 \int_{\varsigma_i}^\iota \mathbb{E}\left\|\mathfrak{g}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})}\right)\right. \\
&- \left.\mathfrak{g}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)\right\|^2 d\varsigma \\
&+ \mathbb{C}^2\mathcal{H}(2\mathcal{H}-1)\mathrm{T}^{2\mathcal{H}-1}\mathbb{C}^2 \int_{\varsigma_i}^\iota \mathbb{E}\left\|\sigma\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})} + \mathfrak{x}^{(\mathfrak{r})}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma^{(\mathfrak{r})})}\right)\right. \\
&- \left.\sigma\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)\right\|^2 d\varsigma\Bigg] \to 0 \quad as \quad \mathfrak{r} \to \infty.
\end{aligned}$$

This implies that $\overline{\phi} : \mathscr{D}^0_{\mathfrak{b}} \to \mathscr{D}^0_{\mathfrak{b}}$ is continuous.

Step 3: To claim $\overline{\phi}$ is $\beta-$contraction.

We my decompose $\overline{\phi} = \overline{\phi}_1 + \overline{\phi}_2$ for $\iota \in J$ where

$$(\overline{\phi}_1\mathfrak{x})(\iota) = \begin{cases} \mathscr{R}(\iota)\left[\phi(0) + \mathfrak{h}(0, \phi(0))\right] - \mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right) \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathscr{C}(\varsigma)d\varsigma, & \iota \in [0, \iota_1]; \\ \mathfrak{G}_i\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right), & \iota \in \cup_{i=1}^n(\iota_i, \varsigma_i]; \\ \mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{G}_i\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right) \\ +\mathfrak{h}\left(\varsigma_i, \overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})} + \mathfrak{x}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{x}_{\varsigma_i})}\right)\Big] \\ -\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right) + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathscr{C}(\varsigma)d\varsigma, & \iota \in (\varsigma_i, \iota_{i+1}]. \end{cases}$$

$$(\overline{\phi}_1\mathfrak{x})(\iota) = \begin{cases} \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma) \\ + \int_0^\iota \mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma), & \iota \in [0, \iota_1]; \\ 0, & \iota \in \cup_{i=1}^n(\iota_i, \varsigma_i]; \\ \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\mathfrak{g}\Big(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} \\ +\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\Big)d\omega(\varsigma) \\ + \int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma, \overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)} + \mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma), & \iota \in (\varsigma_i, \iota_{i+1}]. \end{cases}$$

Claim 1: To prove $\overline{\phi}_1$ is Lipschitz continuous.

For $\iota \in [0, \iota_1]$, (A1), Lemma 2.3 and 2.5 and $\mathfrak{u}, \mathfrak{v} \in \mathscr{D}_{\mathfrak{b}}^0$,

$$\begin{aligned} \mathbb{E}\left\|(\overline{\phi}_1\mathfrak{u})(\iota) - (\overline{\phi}_1\mathfrak{v})(\iota)\right\|^2 &\leq \Big[\mathbb{E}\Big\|\mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)} + \mathfrak{u}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)}\right) \\ &- \mathfrak{h}\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)} + \mathfrak{v}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}\right)\Big\|^2\Big] \\ &\leq \mathscr{L}_{\mathfrak{h}}\left\|\mathfrak{u}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)} - \mathfrak{v}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}\right\|_{\mathscr{D}}^2 \\ &\leq \mathscr{N}_{\mathfrak{b}}^2\mathscr{L}_{\mathfrak{h}}\sup_{\iota\in[0,\iota_1]}\mathbb{E}\left\|\mathfrak{u}(\iota) - \mathfrak{v}(\iota)\right\|^2. \end{aligned}$$

For $\iota \in \cup_{i=1}^n(\iota_i, \varsigma_i]$,

$$\begin{aligned} \mathbb{E}\left\|(\overline{\phi}_1\mathfrak{u})(\iota) - (\overline{\phi}_1\mathfrak{v})(\iota)\right\|^2 &\leq \mathbb{E}\left\|\mathfrak{G}_i\left(\iota, \overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)} + \mathfrak{x}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{x}_\iota)}\right)\right\|^2 \\ &\leq \mathscr{L}_i\left\|\mathfrak{u}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)} - \mathfrak{v}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}\right\|^2 \\ &\leq \mathscr{N}_{\mathfrak{b}}^2\mathscr{L}_i\sup_{\iota\in(\iota_i,\varsigma_i]}\mathbb{E}\|\mathfrak{u}(\iota) - \mathfrak{v}(\iota)\|^2. \end{aligned}$$

II^{ly}, for $\iota \in \cup_{i=1}^n(\varsigma_i, \iota_{i+1}]$,

$\mathbb{E}\left\|(\overline{\phi}_1\mathfrak{u})(\iota)-(\overline{\phi}_1\mathfrak{v})(\iota)\right\|^2$

$$\begin{aligned}
&\leq 2\Bigg[\mathbb{E}\Bigg\|\mathscr{R}(\iota-\varsigma_i)\Bigg[\mathfrak{G}_i\left(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}+\mathfrak{u}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}\right)\\
&- \mathfrak{G}_i\left(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{v}_{\varsigma_i})}+\mathfrak{v}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{v}_{\varsigma_i})}\right)\\
&+ \mathfrak{h}\left(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}+\mathfrak{u}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}\right)\\
&- \mathfrak{h}\left(\varsigma_i,\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{v}_{\varsigma_i})}+\mathfrak{v}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{v}_{\varsigma_i})}\right)\Bigg]\Bigg\|^2\\
&+ \mathbb{E}\left\|\mathfrak{h}\left(\iota,\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)}+\mathfrak{u}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\overline{\phi}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}+\mathfrak{v}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}\right)\right\|^2\Bigg]\\
&\leq 2\mathbb{C}^2\Bigg[2\mathscr{L}_i\|\mathfrak{u}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}-\mathfrak{v}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}\|^2_{\mathscr{D}}\\
&+ 2\mathscr{L}_{\mathfrak{h}}\left\|\mathfrak{u}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}-\mathfrak{v}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\mathfrak{u}_{\varsigma_i})}\right\|^2_{\mathscr{D}}\Bigg]\\
&+ \mathscr{L}_{\mathfrak{h}}\left\|\mathfrak{u}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{u}_\iota)}-\mathfrak{v}_{\vartheta(\iota,\overline{\phi}_\iota+\mathfrak{v}_\iota)}\right\|^2_{\mathscr{D}}\\
&\leq 2\mathscr{N}_{\mathfrak{b}}^2\left[2\mathbb{C}^2\mathscr{L}_i+(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right]\sup_{\iota\in(\varsigma_i,\iota_{i+1}]}\mathbb{E}\|\mathfrak{u}(\iota)-\mathfrak{v}(\iota)\|^2.
\end{aligned}$$

Thus, $\forall\iota\in J$,

$$\mathbb{E}\left\|(\overline{\phi}_1\mathfrak{u})(\iota)-(\overline{\phi}_1\mathfrak{v})(\iota)\right\|^2\leq 2\mathscr{N}_{\mathfrak{b}}^2\left[(1+2\mathbb{C}^2)\mathscr{L}_i+(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right]\sup_{\iota\in[0,\mathfrak{b}]}\mathbb{E}\|\mathfrak{u}(\iota)-\mathfrak{v}(\iota)\|^2.$$

Taking sup over ι,

$$\|\overline{\phi}_1\mathfrak{u}-\overline{\phi}_1\mathfrak{v}\|^2_{\mathfrak{b}}\leq\xi^*\|\mathfrak{u}-\mathfrak{v}\|^2_{\mathfrak{b}},$$

where $\xi^*=2\mathscr{N}_{\mathfrak{b}}^2\left[(1+2\mathbb{C}^2)\mathscr{L}_i+(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right].$

Thus from (3.10), $\xi^*<1$, we get $\overline{\phi}_1$ is Lipschitz continuous.

Claim 2: To claim $\{\overline{\phi}_2\mathfrak{x},\mathfrak{x}\in\mathscr{D}^0_{\mathfrak{b}}\}$ is an equicontinuous family of function on $\mathfrak{b}$.

Let $s_i<\epsilon<\iota<\iota_{i+1},\quad i=0,1,2,\cdots,n$ and $\delta>0\ni\|\gamma(\mathfrak{m}_1)-\gamma(\mathfrak{m}_2)\|<\epsilon$, for every $\mathfrak{m}_1,\mathfrak{m}_2\in\cup_{i=1}^n(\varsigma_i,\iota_{i+1}]$ with $|\mathfrak{m}_1-\mathfrak{m}_2|<\delta$. For $\mathfrak{x}\in\mathscr{D}^0_{\mathfrak{b}}$, $0<|\mathfrak{h}|<\delta$, $\iota+\varrho\in\cup_{i=1}^n(\varsigma_i,\iota_{i+1}]$,

$\mathbb{E}\left\|(\overline{\phi}_2\mathfrak{x})(\iota+\varrho)-(\overline{\phi}_2\mathfrak{x})(\iota)\right\|^2$

$$\begin{aligned}
&\leq 6\mathbb{E}\left\|\int_\iota^{\iota+\varrho}\mathscr{R}(\iota+\varrho-\varsigma)\mathfrak{f}\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\right\|^2\\
&+ 6\mathbb{E}\left\|\int_{\varsigma_i}^{\iota}\left[\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)\right]\mathfrak{f}\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\right\|^2\\
&+ 6\mathbb{E}\left\|\int_\iota^{\iota+\varrho}\mathscr{R}(\iota+\varrho-\varsigma)\mathfrak{g}\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\right\|^2\\
&+ 6\mathbb{E}\left\|\int_{\varsigma_i}^{\iota}\left[\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)\right]\mathfrak{g}\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\right\|^2\\
&+ 6\mathbb{E}\left\|\int_\iota^{\iota+\varrho}\mathscr{R}(\iota+\varrho-\varsigma)\sigma\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\quad 6\mathbb{E}\left\| \int_{\varsigma_i}^{\iota} [\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)]\,\sigma\left(s,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2\\
&\leq\quad 6\mathbb{C}^2\mathfrak{m}_{\mathfrak{f}}(\hat{\mathfrak{r}})\int_{\iota}^{\iota+\varrho}\varphi(\varsigma)d\varsigma+6\mathfrak{m}_{\mathfrak{f}}(\hat{\mathfrak{r}})\int_{\varsigma_i}^{\iota}[\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)]\,\varphi(\varsigma)d\varsigma\\
&+\quad 6\mathbb{C}^2\mathfrak{m}_{\mathfrak{g}}(\hat{\mathfrak{r}})\int_{\iota}^{\iota+\varrho}\upsilon(\varsigma)d\varsigma+6\mathfrak{m}_{\mathfrak{g}}(\hat{\mathfrak{r}})\int_{\varsigma_i}^{\iota}[\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)]\,\upsilon(\varsigma)d\varsigma\\
&+\quad 6\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathfrak{m}_{\sigma}(\hat{\mathfrak{r}})\int_{\iota}^{\iota+\varrho}\beta(\varsigma)d\varsigma\\
&+\quad 6\mathcal{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathfrak{m}_{\sigma}(\hat{\mathfrak{r}})\int_{\varsigma_i}^{\iota}[\mathscr{R}(\iota+\varrho-\varsigma)-\mathscr{R}(\iota-\varsigma)]\,\beta(\varsigma)d\varsigma.
\end{aligned}
$$

For sufficiently small ϵ, (3.12) $\to 0$ as $\mathfrak{h}\to 0$. Hence, $\{\overline{\phi}_2\mathfrak{x},\ \mathfrak{x}\in\mathscr{D}_{\mathfrak{b}}^0\}$ is equicontinuous.

Claim 3: To prove $\overline{\phi}_2$ maps $\mathscr{D}_{\mathfrak{b}}^0$ onto a precompact set in $\mathscr{D}_{\mathfrak{b}}^0$.

Consider the set $\mathbb{Q}(\iota)=\{(\overline{\phi}_2\mathfrak{x})(\iota):\mathfrak{x}\in\mathscr{D}_{\mathfrak{b}}^0\}$ is relatively compact.

Clearly, $\mathbb{Q}(0)=\{0\}$ is compact. Let ξ is real number and $\iota\in(\varsigma_j,\iota_{j+1}]$ be fixed with $0<\xi<\iota$.

$$
\begin{aligned}
(\overline{\phi}_2^{\xi}\mathfrak{x})(\iota) \quad &= \quad \int_{\varsigma_j}^{\iota-\epsilon}\mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\\
&+\quad \int_{\varsigma_j}^{\iota-\epsilon}\mathscr{R}(\iota-\varsigma)\mathfrak{g}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\\
&+\quad \int_{\varsigma_j}^{\iota-\epsilon}\mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathcal{B}^{\mathcal{H}}(\varsigma).
\end{aligned}
$$

Since $\mathscr{R}(\iota)$ is compact, the set $\mathbb{Q}^{\xi}(\iota)=\{(\overline{\phi}_2\mathfrak{x})(\iota):\mathfrak{x}\in\mathscr{D}_{\mathfrak{b}}^0\}$ is relatively compact set $\forall\ \xi$.

$$
\begin{aligned}
\mathbb{E}\left\|(\overline{\phi}_2\mathfrak{x})(\iota)-(\overline{\phi}_2^{\xi}\mathfrak{x})(\iota)\right\|^2 \quad &\leq \quad 3\mathbb{E}\left\|\int_{\varsigma_j}^{\iota}\mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\right.\\
&-\quad \left.\int_{\varsigma_j}^{\iota-\xi}\mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\right\|^2\\
&+\quad 3\mathbb{E}\left\|\int_{\varsigma_j}^{\iota}\mathscr{R}(\iota-\varsigma)\mathfrak{g}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\right.\\
&-\quad \left.\int_{\varsigma_j}^{\iota-\xi}\mathscr{R}(\iota-\varsigma)\mathfrak{g}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\right\|^2\\
&+\quad 3\mathbb{E}\left\|\int_{\varsigma_j}^{\iota}\mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right.\\
&-\quad \left.\int_{\varsigma_j}^{\iota-\xi}\mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2\\
&\leq\quad 3\mathbb{E}\left\|\int_{\iota-\xi}^{\iota}\mathscr{R}(\iota-\varsigma)\mathfrak{f}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\varsigma\right\|^2\\
&+\quad 3\mathbb{E}\left\|\int_{\iota-\xi}^{\iota}\mathscr{R}(\iota-\varsigma)\mathfrak{g}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\omega(\varsigma)\right\|^2\\
&+\quad 3\mathbb{E}\left\|\int_{\iota-\xi}^{\iota}\mathscr{R}(\iota-\varsigma)\sigma\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}+\mathfrak{x}_{\vartheta(\varsigma,\overline{\phi}_\varsigma+\mathfrak{x}_\varsigma)}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad 3\mathbb{C}^2\xi\mathfrak{m}_{\mathfrak{f}}(\mathfrak{r})\int_{\iota-\xi}^{\iota}\varphi(\varsigma)d\varsigma+3\mathbb{C}^2\xi\mathfrak{m}_{\mathfrak{g}}(\mathfrak{r})\int_{\iota-\xi}^{\iota}\upsilon(\varsigma)d\varsigma+3\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\iota_{j+1}^{2\mathrm{H}-1}\int_{\iota-\xi}^{\iota}\beta(\varsigma)d\varsigma\\
&\to \quad 0 \ \ as \ \ \xi\to 0.
\end{aligned}
$$

Hence relatively compact $\{(\overline{\phi}_2^{\xi}\mathfrak{x})(\iota):\mathfrak{x}\in\mathscr{D}_{\mathfrak{b}}^0\}$ are arbitrarily close to the set $\{(\overline{\phi}_2\mathfrak{x})(\iota):\mathfrak{x}\in\mathscr{D}_{\mathfrak{b}}^0\}$. Thus the set is precompact in $\mathscr{D}_{\mathfrak{b}}^0$.

Let B be an arbitrary bounded subset of $\mathscr{D}_{\mathfrak{b}}^0$. Using precompactness of $\overline{\phi}_2$, we get

$$
\begin{aligned}
\beta_{PC}(\overline{\phi}B) &= \beta_{PC}(\overline{\phi}_1B+\overline{\phi}_2B)\\
&\leq \beta_{PC}(\overline{\phi}_1B)+\beta_{PC}(\overline{\phi}_2B)\\
&= \beta_{PC}(\overline{\phi}_1B)\\
&\leq \xi^*\beta_{PC}(B).
\end{aligned}
$$

Thus $\overline{\phi}$ is $\beta-$contraction. Hence by Lemma 2.4, $\overline{\phi}$ has at least one fixed point $\mathfrak{x}^*\in\mathcal{U}\subset\mathscr{D}_{\mathfrak{b}}^0$. Let $\mathfrak{u}(\iota)=\overline{\phi}(\iota)+\mathfrak{x}^*(\iota)$, $\iota\in(-\infty,\mathfrak{b}]$. Then $\mathfrak{u}$ is a fixed point of $\overline{\phi}$, i.e. $\mathfrak{u}$ is the mild solution of (1.1).

□

Now, we may prove the existence of solution with the help of Lemma 2.5.

Existence result based on Darbo-Sadoskii's FPT:

Theorem 3.3. *Let $\mathfrak{u}_0\in\mathcal{L}_0^2(\Omega,\mathcal{H})$ and the hypotheses (A1), (A5) and (A'1)-(A'5) hold on J, provided that*

$$
\begin{aligned}
4\mathscr{N}_{\mathfrak{b}}^2\max_{1\leq i\leq\mathscr{N}}\Bigg(&(10\mathbb{C}^2\mathcal{H})\mathscr{L}_i+5\left((2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right)+5\mathbb{C}^2\int_0^{\iota}\varphi(\varsigma)d\varsigma\lim_{\tau\to\infty}\frac{\mathfrak{m}_{\mathfrak{f}}(\tau)}{\tau}+5\mathbb{C}^2\int_0^{\iota}\upsilon(\varsigma)d\varsigma\\
&\times\lim_{\tau\to\infty}\frac{\mathfrak{m}_{\mathfrak{g}}(\tau)}{\tau}+5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\int_0^{\iota}\beta(\varsigma)d\varsigma\lim_{\tau\to\infty}\frac{\mathfrak{m}_{\sigma}(\tau)}{\tau}\Bigg)<1, \qquad (3.12)
\end{aligned}
$$

then system (1.1) has at least one mild solution on J.

Proof. By similar proof of Theorem 3.2, we may conclude that $\overline{\phi}:\mathscr{D}_{\mathfrak{b}}^0\to\mathscr{D}_{\mathfrak{b}}^0$ is continuous. Now for $\mathfrak{r}>0$, we need to show that $\overline{\phi}(\mathbb{B}_{\mathfrak{r}})\subset\mathbb{B}_{\mathfrak{r}}$. Let us suppose the contrary, then for any $\mathfrak{r}>0\ \exists\ \hat{\mathfrak{x}}\in\mathbb{B}_{\mathfrak{r}}$ and $\hat{\iota}\in[0,\mathfrak{b}]\ni\mathfrak{r}<\mathbb{E}\left\|(\overline{\phi}\hat{\mathfrak{x}})(\hat{\iota})\right\|^2$. Thus for $\hat{\iota}\in[0,\iota_1]$ and $\hat{\mathfrak{x}}\in\mathbb{B}_{\mathfrak{r}}$,

$$
\begin{aligned}
\mathfrak{r} &< \mathbb{E}\left\|(\overline{\phi}\hat{\mathfrak{x}})(\hat{\iota})\right\|^2\\
&\leq 5\Bigg[\mathbb{E}\left\|\mathscr{R}(\hat{\iota}-\varsigma)[\phi(0)+\mathfrak{h}(0,\phi(0))]\right\|^2+\mathbb{E}\left\|\mathfrak{h}\left(\hat{\iota},\overline{\phi}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}+\hat{\mathfrak{x}}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}\right)\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\iota}\mathscr{R}(\hat{\iota}-\varsigma)\mathscr{C}(\varsigma)d\varsigma\right\|^2+\mathbb{E}\left\|\int_0^{\iota}\mathscr{R}(\hat{\iota}-\varsigma)\mathfrak{f}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right)d\varsigma\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\iota}\mathscr{R}(\hat{\iota}-\varsigma)\mathfrak{g}\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right)d\omega(\varsigma)\right\|^2\\
&+ \mathbb{E}\left\|\int_0^{\iota}\mathscr{R}(\hat{\iota}-\varsigma)\sigma\left(\varsigma,\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right)d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & 5\bigg[\mathbb{C}^2\mathscr{L}_{\mathfrak{h}}(1+\|\phi\|^2_{\mathscr{D}})+\mathscr{L}_{\mathfrak{h}}\left(1+\left\|\mathfrak{h}\left(\hat{\iota},\overline{\phi}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}+\hat{\mathfrak{x}}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}\right)\right\|^2_{\mathscr{D}}\right)\\
+ \quad & \mathbb{C}^2\int_0^{\iota}\mathbb{E}\|\mathscr{C}\|^2 d\varsigma+\mathbb{C}^2\int_0^{\iota}\varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma\\
+ \quad & \mathbb{C}^2\int_0^{\iota}\upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma\\
+ \quad & \mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathrm{T}^{2\mathrm{H}-1}\int_0^{\iota}\beta(\varsigma)\mathfrak{m}_{\sigma}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma\bigg].
\end{aligned}
$$

For $\hat{\iota}\in(\iota_i,\varsigma_i]$ and $\hat{\mathfrak{x}}\in\mathbb{B}_{\mathfrak{r}}$,

$$
\mathfrak{r}<\mathbb{E}\left\|(\overline{\phi}\hat{\mathfrak{x}})(\hat{\iota})\right\|^2=\mathscr{L}_i\left(1+\left\|\mathfrak{h}\left(\hat{\iota},\overline{\phi}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}+\hat{\mathfrak{x}}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}\right)\right\|^2_{\mathscr{D}}\right).
$$

II^{ly}, for $\hat{\iota}\in\cup_{i=1}^{n}(\varsigma_i,\iota_{i+1}]$,

$$
\begin{aligned}
\mathfrak{r} \quad < \quad & \mathbb{E}\left\|(\overline{\phi}\hat{\mathfrak{x}})(\hat{\iota})\right\|^2\\
\leq \quad & 10\mathbb{C}^2\bigg[\mathscr{L}_i\left(1+\left\|\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\hat{\mathfrak{x}}_{\varsigma_i})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\hat{\mathfrak{x}}_{\varsigma_i})}\right\|^2_{\mathscr{D}}\right)\\
+ \quad & \mathscr{L}_{\mathfrak{h}}\left(1+\left\|\overline{\phi}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\hat{\mathfrak{x}}_{\varsigma_i})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma_i,\overline{\phi}_{\varsigma_i}+\hat{\mathfrak{x}}_{\varsigma_i})}\right\|^2_{\mathscr{D}}\right)\bigg]\\
+ \quad & 5\mathscr{L}_{\mathfrak{h}}\left(1+\left\|\overline{\phi}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}+\hat{\mathfrak{x}}_{\vartheta(\hat{\iota},\overline{\phi}_{\hat{\iota}}+\hat{\mathfrak{x}}_{\hat{\iota}})}\right\|^2_{\mathscr{D}}\right)\\
+ \quad & \mathbb{C}^2\int_{\varsigma_i}^{\iota}\varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma\\
+ \quad & \mathbb{C}^2\int_{\varsigma_i}^{\iota}\upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma\\
+ \quad & \mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathrm{T}^{2\mathrm{H}-1}\int_{\varsigma_i}^{\iota}\beta(\varsigma)\mathfrak{m}_{\sigma}\left(\left\|\overline{\phi}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}+\hat{\mathfrak{x}}_{\vartheta(\varsigma,\overline{\phi}_{\varsigma}+\hat{\mathfrak{x}}_{\varsigma})}\right\|^2_{\mathscr{D}}\right)d\varsigma.
\end{aligned}
$$

$\therefore\ \forall\ \hat{\iota}\in[0,\mathfrak{b}]$,

$$
\begin{aligned}
&\mathfrak{r}\\
&<\mathbb{E}\left\|(\overline{\phi}\hat{\mathfrak{x}})(\hat{\iota})\right\|^2(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\\
&\leq\mathscr{B}^*+10\mathscr{L}_i\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]\\
&\quad+10\mathbb{C}^2\mathscr{L}_i\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]\\
&\quad+4\mathscr{L}_{\mathfrak{h}}\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]\\
&\quad+10\mathscr{L}_{\mathfrak{h}}\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]\\
&\quad+5\mathbb{C}^2\int_0^{\iota}\varphi(\varsigma)\mathfrak{m}_{\mathfrak{f}}\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]d\varsigma\\
&\quad+5\mathbb{C}^2\int_0^{\iota}\upsilon(\varsigma)\mathfrak{m}_{\mathfrak{g}}\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]d\varsigma\\
&\quad+5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathcal{H}-1}\int_0^{\iota}\beta(\varsigma)\mathfrak{m}_{\sigma}\left[4\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathfrak{r}+\left(\mathscr{N}_{\mathfrak{b}}{}^2\mathcal{M}^2\mathscr{K}^2+(\mathbb{H}_{\mathfrak{b}}+\mathbb{J}^{\phi})^2\right)\right)\|\phi\|^2_{\mathscr{D}}\right]d\varsigma. \qquad (3.13)
\end{aligned}
$$

Dividing (3.13), by $\mathfrak{r}$ and letting $\mathfrak{r} \to \infty$, we get

$$\begin{aligned} 1 &< 4\mathscr{N}_{\mathfrak{b}}^2 \max_{1\leq i\leq \mathscr{N}} \Big\{ (1+10\mathbb{C}^2)\mathscr{L}_i + 5(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}} + 5\mathbb{C}^2 \int_0^\iota \varphi(\varsigma)d\varsigma \lim_{\tau\to\infty} \frac{\mathfrak{m}_{\mathfrak{f}}\tau}{\tau} \\ &+ 5\mathbb{C}^2 \int_0^\iota \upsilon(\varsigma)d\varsigma \lim_{\tau\to\infty} \frac{\mathfrak{m}_{\mathfrak{g}}\tau}{\tau} \\ &+ 5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1} \int_0^\iota \beta(\varsigma)d\varsigma \lim_{\tau\to\infty} \frac{\mathfrak{m}_{\sigma}\tau}{\tau} \Big\}, \end{aligned} \tag{3.14}$$

which contradicts (3.12). Therefore $\exists$ a +ve constant $\mathfrak{r} > 0 \ni$

$$\overline{\phi}(\mathbb{B}_{\mathfrak{r}}) \subset \mathbb{B}_{\mathfrak{r}}.$$

Similar to Theorem 3.2, by proceeding the same way, we may conclude that system (1.1) has a mild solution.

□

4 T- Controllability

This section is devoted to the study the T-Controllability for the aforementioned system (1.1) by using generalized Gronwall's inequality.

Definition 4.1. *The system (1.1) is known as T-controllable on* $[0, \mathfrak{b}]$*, if for every* $\rho \in \mathcal{V}$*, such that the mild solution* $\mathfrak{x}(.)$ *of (1.1) satisfies* $\rho(t) = \mathfrak{x}(t)$ *a.e.*

Lemma 4.1. *(Generalized Gronwall's inequality [34]): If* $\beta > 0$*,* $\tilde{a}(\iota)$ *is a non-negative function locally integrable on* $0 \leq \iota < \mathfrak{b}$*, some* $\mathfrak{b} < +\infty$ *and* $\mathfrak{q}(\iota)$ *is a non-decreasing continuous function on* $0 \leq \iota < \mathfrak{b}$*,* $\mathfrak{q}(\iota) \leq \mathfrak{c}$ *and suppose* $\tilde{\mathfrak{u}}(\iota) \leq \tilde{a}(\iota) + \mathfrak{q}(\iota)\int_0^\iota (\iota-\varsigma)^{\beta-1}\tilde{\mathfrak{u}}(\varsigma)d\varsigma$*, on this interval. Then*

$$\tilde{\mathfrak{u}(\iota)} \leq \tilde{a}(\iota) + \int_0^\iota \sum_{n=1}^{\infty} \frac{(\mathfrak{q}(\iota)\Gamma(\beta))^n}{\Gamma(n\beta)} (\iota-\varsigma)^{\beta-1}\tilde{a}(\varsigma)d\varsigma, \quad 0 \leq \iota \leq \mathfrak{b}.$$

In particular, when $\tilde{a}(\iota) = 0$*, then* $\tilde{\mathfrak{u}}(\iota) = 0 \ \forall \ 0 \leq \iota < \mathfrak{b}$*.*

Theorem 4.1. *Assume the hypotheses (A1)-(A5) holds, then the system (1.1) is trajectory controllable on* J *and*

$$10(\mathbb{C}^2+1)\mathscr{L}_i\mathscr{N}_{\mathfrak{b}}^2 + 5\mathscr{N}_{\mathfrak{b}}^2(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}} < 1. \tag{4.1}$$

Proof. Let $\mu(\iota)$ be the given trajectory on $\mathcal{V}$. We may choose the suitable feedback control $\mathscr{C}(\iota)$ as

$$\begin{aligned} \mathscr{C}(\iota) &= d\left[\mu(\iota) + \mathfrak{h}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right)\right] - \mathfrak{A}\left[\mu(\iota) + \mathfrak{h}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right)\right] d\iota \\ &- \int_0^\iota \Theta(\iota-\varsigma)\left[\mu(\varsigma) + \mathfrak{h}(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)})\right] d\varsigma d\iota \\ &- \mathfrak{f}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right) d\iota - \mathfrak{g}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right) d\omega(\iota) - \sigma\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right) d\mathrm{B}^{\mathrm{H}}(\iota). \end{aligned}$$

Hence (1.1) becomes

$$
\begin{aligned}
& d[\mathfrak{u}(\iota)+\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)] \\
& = \quad \mathfrak{A}[\mathfrak{u}(\iota)+\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)]d\iota+\int_0^\iota \Theta(\iota-\varsigma)[\mathfrak{u}(\varsigma)+\mathfrak{h}\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)]d\varsigma d\iota+\mathfrak{f}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)d\iota \\
& + \quad \Big\{d\left[\mu(\iota)+\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right]-\mathfrak{A}\left[\mu(\iota)+\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right]d\iota \\
& - \quad \int_0^\iota \Theta(\iota-\varsigma)\left[\mu(\varsigma)+\mathfrak{h}(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)})\right]d\varsigma d\iota \\
& - \quad \mathfrak{f}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)d\iota-\mathfrak{g}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)d\omega(\iota)-\sigma\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)d\mathrm{B}^{\mathrm{H}}(\iota)\Big\} \\
& + \quad \mathfrak{g}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)d\omega(\iota)+\sigma\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)d\mathrm{B}^{\mathrm{H}}(\iota).
\end{aligned}
$$

Let $\Xi(\iota)=\mathfrak{u}(\iota)-\mu(\iota)$, we obtain

$$
\begin{aligned}
& d\left[\Xi(\iota)+\left(\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right)\right] \\
& =\mathfrak{A}\left[\Xi(\iota)+\left(\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right)\right] \\
& +\int_0^\iota \Theta(\iota-\varsigma)\Big[\Xi(\varsigma)+\left(\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right)\Big]d\varsigma d\iota \\
& +\left[\mathfrak{f}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{f}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right]d\iota+\left[\mathfrak{g}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{g}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right]d\omega(\iota) \\
& +\left[\sigma\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\sigma\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right]d\mathrm{B}^{\mathrm{H}}(\iota) \\
& \mathcal{I}_{0^+}^{(1-\delta)}\Xi(\iota)=\mathcal{I}_{0^+}^{(1-\delta)}\phi-\mathcal{I}_{0^+}^{(1-\delta)}\phi=0, \quad \iota\in(-\infty,0].
\end{aligned} \tag{4.2}
$$

The mild solution of (4.2) is

$$
\Xi=\begin{cases}
\left[\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right] \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{f}\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\mathfrak{f}\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\varsigma \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{g}\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\mathfrak{g}\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\omega(\varsigma) \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\sigma\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\sigma\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\mathcal{B}^{\mathcal{H}}(\varsigma); & \iota\in(0,\iota_1], \\
\mathfrak{G}_i\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{G}_i\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right); & \iota\in[\iota_i,\varsigma_i), \\
\mathscr{R}(\iota-\varsigma_i)\Big\{\left[\mathfrak{G}_i\left(\varsigma_i,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{G}_i\left(\varsigma_i,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)\right] \\
+\left[\mathfrak{h}\left(\varsigma_i,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\varsigma_i,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)\right]\Big\} \\
-\left[\mathfrak{h}\left(\iota,\mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right)-\mathfrak{h}\left(\iota,\mu_{\vartheta(\iota,\mu_\iota)}\right)\right] \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{f}\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\mathfrak{f}\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\varsigma \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{g}\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\mathfrak{g}\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\omega(\varsigma) \\
+\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\sigma\left(\varsigma,\mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right)-\sigma\left(\varsigma,\mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right]d\mathrm{B}^{\mathrm{H}}(\varsigma); & \iota\in(0,\iota_1].
\end{cases}
$$

Hence for $\iota\in(-\infty,0]$, the initial data to be zero, we obtain

$$
\Phi(\iota)=0, \qquad \iota\in(-\infty,0].
$$

For $\iota \in [0, \iota_1]$,

$$\begin{aligned}
\mathbb{E}\left\|\Xi(\iota)\right\|^2 &\leq 4\Big[\mathbb{E}\left\|\mathfrak{h}\left(\iota, \mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right) - \mathfrak{h}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right)\right\|^2 \\
&+ \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{f}\left(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right) - \mathfrak{f}\left(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right] d\varsigma\right\|^2 \\
&+ \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{g}\left(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right) - \mathfrak{g}\left(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right] d\omega(\varsigma)\right\|^2 \\
&+ \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\varsigma)\Big[\sigma\left(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right) - \sigma\left(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\Big] d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2\Big] \\
&\leq 4\mathscr{L}_{\mathfrak{h}}\mathscr{N}_{\mathfrak{b}}{}^2\mathbb{E}\|\Xi(\iota)\|^2 + \left[\mathbb{C}^2\mathscr{L}_{\mathfrak{f}} + \mathbb{C}^2\mathscr{L}_{\mathfrak{g}} + \mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathscr{L}_\sigma\right]\int_0^\iota \mathbb{E}\left\|\Xi(\varsigma)\right\|_{\mathscr{D}}^2.
\end{aligned}$$

For $\iota \in \cup_{i=1}^n (\iota_i, \varsigma_i]$,

$$\begin{aligned}
\mathbb{E}\left\|\Xi(\iota)\right\|^2 &\leq \mathbb{E}\left\|\mathfrak{G}_i\left(\iota, \mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right) - \mathfrak{G}_i\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right)\right\|^2 \\
&\leq \mathscr{L}_i\mathscr{N}_{\mathfrak{b}}^2 \sup_{\iota\in(\iota_i,\varsigma_i]} \mathbb{E}\left\|\mathfrak{u}(\iota) - \mu(\iota)\right\|^2.
\end{aligned}$$

For $\iota \in \cup_{i=1}^n (\varsigma_i, \iota_{i+1}]$,

$$\begin{aligned}
\mathbb{E}\left\|\Xi(\iota)\right\|^2 &\leq 5\mathbb{E}\Big[\left\|\mathscr{R}(\iota-\varsigma_i)\left[\mathfrak{G}_i\left(s_i, \mathfrak{u}_{\vartheta(\varsigma_i,\mathfrak{u}_{\varsigma_i})}\right) - \mathfrak{G}_i\left(\varsigma_i, \mu_{\vartheta(\varsigma_i,\mu_{\varsigma_i})}\right)\right]\right\|^2 \\
&+ \left\|\mathscr{R}(\iota-\varsigma_i)\Big[\mathfrak{h}\left(s_i, \mathfrak{u}_{\vartheta(\varsigma_i,\mathfrak{u}_{\varsigma_i})}\right) - \mathfrak{h}\left(\varsigma_i, \mu_{\vartheta(\varsigma_i,\mu_{\varsigma_i})}\right)\Big]\right\|^2\Big] \\
&+ 5\mathbb{E}\left\|\mathfrak{h}\left(\iota, \mathfrak{u}_{\vartheta(\iota,\mathfrak{u}_\iota)}\right) - \mathfrak{h}\left(\iota, \mu_{\vartheta(\iota,\mu_\iota)}\right)\right\|^2 \\
&+ 5\mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\Big[\mathfrak{f}\Big(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\Big) - \mathfrak{f}\Big(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\Big)\Big] d\varsigma\right\|^2 \\
&+ 5\mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\left[\mathfrak{g}\left(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right) - \mathfrak{g}\left(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right] d\omega(\varsigma)\right\|^2 \\
&+ 5\mathbb{E}\left\|\int_{\varsigma_i}^\iota \mathscr{R}(\iota-\varsigma)\left[\sigma\left(\varsigma, \mathfrak{u}_{\vartheta(\varsigma,\mathfrak{u}_\varsigma)}\right) - \sigma\left(\varsigma, \mu_{\vartheta(\varsigma,\mu_\varsigma)}\right)\right] d\mathrm{B}^{\mathrm{H}}(\varsigma)\right\|^2 \\
&\leq \mathscr{N}_{\mathfrak{b}}^2\left[10\mathbb{C}^2(\mathscr{L}_i + \mathscr{L}_{\mathfrak{h}}) + 5\mathscr{L}_{\mathfrak{h}}\right]\mathbb{E}\|\Xi(\iota)\|^2 + \mathscr{N}_{\mathfrak{b}}^2\Big[5\mathbb{C}^2[\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}] \\
&+ 5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathscr{L}_\sigma\Big]\int_{\varsigma_i}^\iota \mathbb{E}\left\|\mathbb{E}\|\Xi(\varsigma)\|^2\right\|^2 d\varsigma.
\end{aligned}$$

Thus for $\iota \in J$,

$$\begin{aligned}
\mathbb{E}\|\Xi(\iota)\|^2 &\leq \left[(10\mathbb{C}^2+1)\mathscr{L}_i\mathscr{N}_{\mathfrak{b}}{}^2 + 5\mathscr{N}_{\mathfrak{b}}{}^2(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}}\right]\mathbb{E}\|\Xi(\iota)\|^2 + \Big[5\mathbb{C}^2(\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}) \\
&+ 5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathscr{L}_\sigma\Big]\mathscr{N}_{\mathfrak{b}}^2\int_0^{\mathfrak{b}} \mathbb{E}\|\Xi(\varsigma)\|^2 d\varsigma. \\
&\leq \mathscr{V}_1 + \mathscr{V}_2\int_0^{\mathfrak{b}} \mathbb{E}\|\Xi(\varsigma)\|^2 d\varsigma \\
&\leq \frac{\mathscr{V}_2}{1-\mathscr{V}_1}\int_0^{\mathfrak{b}} \mathbb{E}\|\Xi(\varsigma)\|^2 d\varsigma,
\end{aligned}$$

where,

$$\begin{aligned}
\mathscr{V}_1 &= (10\mathbb{C}^2+1)\mathscr{L}_i\mathscr{N}_{\mathfrak{b}}{}^2 + 5\mathscr{N}_{\mathfrak{b}}{}^2(2\mathbb{C}^2+1)\mathscr{L}_{\mathfrak{h}} \\
\mathscr{V}_2 &= 5\mathbb{C}^2(\mathscr{L}_{\mathfrak{f}} + \mathscr{L}_{\mathfrak{g}}) + 5\mathbb{C}^2\mathrm{H}(2\mathrm{H}-1)\mathfrak{b}^{2\mathrm{H}-1}\mathscr{L}_\sigma.
\end{aligned}$$

Using generalized Gronwall's inequality, $\mathbb{E}\|\Xi\|^2 \to 0$, i.e, $\mathfrak{u}(\iota) = \mu(\iota)$. Thus the control problem is T-controllable.

□

5 Illustration

Consider the following impulsive neutral stochastic integro-differential equation with fBm

$$\begin{aligned}
& d\left[\mathfrak{Z}(\iota,\mathfrak{x}) + \int_{-\infty}^{\iota} e^{2(\varsigma-\iota)}\frac{\mathfrak{Z}\left(\varsigma-\vartheta(\|\mathfrak{Z}(\iota)\|),\mathfrak{x}\right)}{25}d\varsigma\right] \\
& = \frac{\partial^2}{\partial\mathfrak{x}^2}\mathfrak{Z}(\iota,\mathfrak{x})\left[\mathfrak{Z}(\iota,\mathfrak{x}) + \int_{-\infty}^{\iota} e^{2(\varsigma-\iota)}\frac{\mathfrak{Z}\left(\varsigma-\vartheta(\|\mathfrak{Z}(\iota)\|),\mathfrak{x}\right)}{25}d\varsigma\right]d\iota + \mathfrak{C}(\iota,\mathfrak{x}) \\
& + \int_{-\infty}^{\iota}\widetilde{\Theta}(\varsigma)\frac{\partial^2}{\partial\mathfrak{x}^2}\mathfrak{Z}(\iota,\mathfrak{x})\left[\mathfrak{Z}(\iota,\mathfrak{x}) + \int_{-\infty}^{\iota} e^{2(\varsigma-\iota)}\frac{\mathfrak{Z}\left(\varsigma-\vartheta(\|\mathfrak{Z}(\iota)\|),\mathfrak{x}\right)}{25}d\varsigma\right]d\iota \\
& + \int_{-\infty}^{\iota} e^{2(\varsigma-\iota)}\frac{\mathfrak{Z}\left(\varsigma-\vartheta(\|\mathfrak{Z}(\iota)\|),\mathfrak{x}\right)}{9}d\varsigma \\
& + \int_{0}^{\iota}\sin(\iota-\varsigma)\int_{-\infty}^{\varsigma} e^{2(\mathfrak{z}-\varsigma)}\frac{\mathfrak{Z}\left(\mathfrak{z}-\vartheta(\|\mathfrak{Z}(\varsigma)\|),\mathfrak{x}\right)}{9}d\omega(\mathfrak{z})d\varsigma \\
& + \int_{-\infty}^{\iota}\Pi(\varsigma)d\mathrm{B}^{\mathrm{H}}(\varsigma), \qquad (\iota,\mathfrak{x}) \in \cup_{i=1}^{n}(\varsigma_i,\iota_{i+1}] \times [0,\pi]; \\
& \mathfrak{Z}(\iota,\mathfrak{x}) = \int_{-\infty}^{\iota} e^{2(\varsigma-\iota)}\frac{\mathfrak{Z}\left(\varsigma-\vartheta(\|\mathfrak{Z}(\iota)\|),\mathfrak{x}\right)}{49}d\varsigma, \quad (\iota,\mathfrak{x}) \in (\iota_i,\varsigma_i] \times [0,\pi]; \\
& \mathfrak{Z}(\iota,0) = \mathfrak{Z}(\iota,\pi) = 0, \quad \iota \geq 0, \\
& \mathfrak{Z}(\iota,\mathfrak{x}) = \phi(\iota,\mathfrak{x}), \qquad \iota \in (-\infty,0].
\end{aligned} \tag{5.1}$$

$0 = \iota_0 = s_0 < \iota_1 < \varsigma_1 < \iota_2 < \cdots < \iota_n < \varsigma_n < \iota_{n+1} = 1$ are fixed real numbers. Define the operator $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathscr{H} \to \mathscr{H}$ by $\mathfrak{A}\mathfrak{x} = x''$ provided,

$$\mathscr{D}(\mathfrak{A}) = \{\mathfrak{x} \in \mathscr{H} : \mathfrak{x}, \mathfrak{x}' \text{ are absolutely continuous}, \mathfrak{x}'' \in \mathscr{H},\ \mathfrak{x}(0) = \mathfrak{x}(\pi) = 0\}.$$

then $\mathfrak{A}$ generates an analytic semigroup $\mathcal{T}(\iota) \in \mathscr{H}$. Moreover $\mathfrak{A}$ has a discrete spectrum with eigenvalues $-n^2, n \in \mathbb{N}$ with the corresponding normalized eigen functions $\mathfrak{z}_n(\mathfrak{x}) = \sqrt{\frac{2}{\pi}}\sin(n\mathfrak{x})$. Then, the operator $(\mathfrak{A})^{1/2}$ is given by

$$(\mathfrak{A})^{1/2}\mathfrak{x} = \sum_{n=1}^{\infty} n(\mathfrak{x},\mathfrak{z}_n)\mathfrak{z}_n$$

on the space $\mathscr{D}((\mathfrak{A})^{1/2}) = \{\mathfrak{x}(.) \in \mathscr{H},\ \sum_{n=1}^{\infty} n(\mathfrak{x},\mathfrak{z}_n)\mathfrak{z}_n \in \mathscr{H}\}$. Moreover $\mathcal{T}(\iota)$ is given by

$$\mathcal{T}(\iota)\mathfrak{x} = \sum_{n=1}^{\infty} e^{n^2\iota}(\mathfrak{x},\mathfrak{z}_n)\mathfrak{z}_n.$$

Let $\Theta : \mathscr{D}(\mathfrak{A}) \subset \mathscr{H} \to \mathscr{H}$ be $\Theta(\iota)(\mathfrak{x}) = \widetilde{\Theta}(\mathfrak{x})\mathfrak{A}\mathfrak{x}$ and $\mathfrak{x} \in \mathscr{D}(\mathfrak{A})$. Set $\mathfrak{p} \in [1,\infty)$, $\mathfrak{r} \in [0,\infty)$, $\mathfrak{f} : (-\infty,-\mathfrak{r}) \to \mathbb{R}$ and $\mathfrak{g} : (-\infty,-\mathfrak{r}) \to \mathbb{R}$ be a nonnegative Borel measurable function which satisfies the conditions (H5) and (H6) in the terminology, see Hino et al. [39] i.e. $\mathfrak{g}$ is locally integrable function and $\exists$ a nonnegative locally bounded function ϑ on $(-\infty, 0]$ such that $\mathfrak{g}(\theta+\psi) \leq \vartheta(\theta)\mathfrak{g}(\psi)\ \forall\ \theta \leq 0$ and $\psi \in (-\infty, -\iota)/\mathscr{N}_\theta$, where $\mathscr{N}_\theta \subseteq (-\infty, -\mathfrak{r})$ is a set with Lebesgue measure 0.

Consider the space $\mathscr{D} = \mathcal{C}_\mathfrak{r} \times \mathcal{L}^2(\mathfrak{g}, \mathscr{H})$ of all classes of function $\psi : (-\infty, 0] \to \mathbb{X}$ such that $\psi\mid_{[-\mathfrak{r},0]} \in \mathcal{C}([-\mathfrak{r}, 0], \mathscr{H})$, $\psi(.)$ is Lebesgue measurable on $(-\infty, -\mathfrak{r})$ with the seminorm

$$\|\psi\|_{\mathscr{D}} = \sup_{\tau\in[-\mathfrak{r},0]} \|\psi(\tau)\| + \left(\int_{-\infty}^{-\mathfrak{r}} \mathfrak{g}(\tau)\|\psi(\tau)\|^{\mathfrak{p}} d\tau\right)^{1/\mathfrak{p}}.$$

As in the proof of [[39], Theorem 1.3.8], we may conclude that $\mathscr{D}$ is a phase space. Moreover, for $\mathfrak{r} = 0$ and $\mathfrak{p} = 2$, we obtain $\mathscr{D} = \mathcal{C}_0 \times \mathcal{L}^2(\mathfrak{g}, \mathscr{H})$ with $\mathscr{H} = 1$, $\mathcal{M}(\iota) = \vartheta(\iota)$ and $\mathscr{K}(\iota) = 1 + \int_{-\iota}^{0} \mathfrak{g}(\tau)d\tau$ for $\iota \geq 0$.

Set $\phi(\tau)(\mathfrak{x}) = \phi(\tau, \mathfrak{x}) \in \mathscr{D}$ and define

$$\begin{aligned}
\mathfrak{Z}(\iota)(\mathfrak{x}) &= \mathfrak{Z}(\iota, \mathfrak{x}), \\
\vartheta(\iota, \phi)(\mathfrak{x}) &= \vartheta(\|\phi(0)\|), \\
\mathfrak{h}(\iota, \phi)(\mathfrak{x}) &= \int_{-\infty}^{0} e^{2s}\frac{\phi}{24}d\varsigma, \\
\mathfrak{f}(\iota, \phi)(\mathfrak{x}) &= \int_{-\infty}^{0} e^{2s}\frac{\phi}{9}d\varsigma, \\
\int_0^\iota \mathfrak{k}(\iota, \varsigma, \phi)(\mathfrak{x})d\varsigma &= \int_0^\iota \sin(\iota-\varsigma)\int_{-\infty}^{0} e^{2\rho}\frac{\phi}{36}d\rho d\varsigma, \\
\mathfrak{G}_i(\iota, \phi)(\mathfrak{x}) &= \int_{-\infty}^{0} e^{2s}\frac{\phi}{49}d\varsigma \\
\sigma(\iota, \phi)(\mathfrak{x}) &= \int_{-\infty}^{\iota} \Pi(\varsigma)dB^{H}(\varsigma).
\end{aligned}$$

Then, (5.1) can be written in the abstract formulation of (1.1). We can check the values $\mathfrak{h}, \mathfrak{f}, \mathfrak{k}, \sigma$ and $\mathfrak{G}_i$ hold the hypothese (A1)-(A5). Hence (5.1) has a unique mild solution on [0,1] by implying Theorem 3.1.

6 Conclusion

A new control model is presented with a SIDEs with state dependent delay driven by mixed Brownian motion suffered by non-instantaneous impulses in Hilbert spaces. The solvability of the proposed stochastic system is obtained using stochastic analysis, fixed point theorems, and the resolvent operator approach. Furthermore, given some reasonable assumptions, the T-controllability of the investigated system is established using extended Gronwall's inequality. Finally, an example is provided to demonstrate the theoretical findings gained. In future, the authors plan to develop theoretical results in SIDEs with Lev'y noise. The above result can also be extended to second order system using sine and cosine operators with Lev'y noise. Fractional order state dependent SIDEs with Lev'y noice and the numerical estimations of SIDEs will be an interesting work.

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