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An Exploration of the Qualitative Analysis of the Generalized Pantograph Equation with the *q*-Hilfer Fractional Derivative

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Abstract: This manuscript tries to show that there is only one solution to the problem of the q-Hilfer fractional generalized pantograph differential equations with a nonlocal condition, and it does so by employing a particular technique known as Schaefer's fixed point theorem and the Banach contraction principle. Then, we verify that the Ulam-type stability is valid. To illustrate the results, an example is provided.

Keywords: *q*-Hilfer fractional derivative; pantograph equation; existence; uniqueness; Ulam stable; nonlocal condition

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1. Introduction

Fractional calculus and its potential applications have gained increasing importance because fractional calculus has become a powerful tool with highly accurate and successful results in modeling many complex phenomena in several seemingly diverse and widespread fields of science and engineering. Many areas, such as aerodynamics, control systems, signal processing, bioengineering, and biomedical sciences, benefit from the application of fractional differential equations and dynamical systems. In recent years, these mathematical tools have proven to be highly effective in modeling various phenomena across engineering, physics, and economics. They have found significant applications in fields such as viscoelasticity, heat conduction in materials with memory, and fluid dynamic traffic models. For more details, refer to [1–4]. Two well-known definitions that have attracted significant attention in the study of differential equations are the Caputo fractional derivative and the Riemann–Liouville (R-L) fractional derivative. Buliding on these, Hilfer [5] later formulated a more general definition of the fractional derivative by interpolating between these two operators.

The study of calculus without the concept of limits is commonly called quantum calculus or q-calculus. The so-called q-integral and q-derivative were initially established by Jackson [6]. There are many applications of quantum calculus, such as in physics, number theory, integer partitions, vertical polynomials, and hypergeometric functions, see [7,8]. Al-Salam [9] and Agarwal [10] have also generalized q-derivative and q-integrals into orders other than integers, which is utilized in the construction of the q-difference calculus. For details on q-fractional calculus and equations, see the monograph of [11]. q-calculus on finite intervals was also introduced by Tariboon and Ntouyas [12]. Furthermore, the definition of the q-derivative and q-integral was examined and gradually developed by numerous researchers in [13–15]. The definition of q-derivative and q-integral have been developed, which are based on the fractional integral in the sense of R-L. Hilfer [5] suggested a general operator for fractional q-derivative, called the q-Hilfer fractional derivative (q-HFD), which is a composite of the Caputo and R-L fractional q-derivatives [16].

Pantograph differential equations are a special type of delay differential equations. This type allows the delay term to be introduced after the initial value but before the computation of the desired approximations; see [17,18]. Among deterministic problems, there is a very special type of delayed differential equation called the pantograph equation. Historically, it was used by Ockendon and Taylor to study how electricity is collected by the pantograph of an electric locomotive, from which it gets its name. In fact, a device that rejects scaling and drawing from another device is called a pantograph. Moreover, this device has been modified by scientists, and recently they have been using it in laser modeling, especially quantum dot lasers, the modeling of objects, electric trains, etc. [19]. The pantograph was originally developed to aid in drawing and scaling tasks. As the technology evolved, the device was refined and found broader applications. Today, pantograph mechanisms are employed in various fields, including electric trains, material modeling, and the simulation of lasers, especially quantum dot lasers. Pantograph-type equations are gaining increasing attention due to their relevance in various areas of mathematics and physics, such as quantum mechanics, number theory, probability, electrodynamics, and control systems. Recognizing their importance, many researchers have extended these equations into generalized forms and explored their solvability using both theoretical and numerical approaches; see [20–26]. In contrast, very few works have been proposed on pantograph fractional differential equations. Balachandran et al. [20] initiated an overview of the various types of pantograph equations and their existence. In [27], the authors studied the attractivity of solutions for Hilfer–Hadamard delay differential equations with a nonlocal condition. Lacchouri et al. [28] investigated fractional pantograph q-difference equations with nonlocal conditions. The authors in [29] established the existence and stability of the solution to the Hilfer type fractional implicit q-differential equations with nonlocal conditions. In [30], the authors studied the existence and uniqueness theorem of the nonlocal problem for Hilfer fractional q-difference equations. In the existing literature, there are no results on the generalized pantograph q-derivative of Hilfer type with a nonlocal condition. Motivated by the above, here we discuss efficient results on the q-Hilfer generalized pantograph system with a nonlocal condition.

Here, the generalized pantograph q-differential equation is considered as

$$\begin{cases} {}^{H}_{q}\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}u(\tau) = g(\tau, u(\sigma_1(\tau)), u(\sigma_2(\tau)), \cdots, u(\sigma_n(\tau))), \quad \tau \in \mathfrak{J} := [0,T], \\ {}_{q}\mathfrak{I}^{1-\vartheta}_{0^+}u(0) = \sum_{i=1}^{m} \lambda_i u(\tau_i), \quad \vartheta = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \tau_i \in [0,T], \end{cases} \tag{1}$$

where ${}^{H}_{q}\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}$ is the q-HFD of order $\zeta_1 \in (0,1)$ and type $\zeta_2 \in [0,1]$, ${}_{q}\mathfrak{I}^{1-\vartheta}_{0^+}$ is the q-R-L fractional integral of order $1-\vartheta$, $\sigma_i : \mathfrak{J} \to \mathfrak{J}$, $(i = 1,2,\cdots,n)$ are continuous functions, $g : \mathfrak{J} \times \mathbb{R}^n \to \mathbb{R}$ is the level-wise continuous function and $\sigma_i(\tau) \le \tau$ for every $\tau \in \mathfrak{J}$, and $\tau_i(i = 1,2,\cdots,m)$ are prefixed points satisfying $0 < \tau \le \tau_2 \le \cdots \le \tau_m \le T$.

The manuscript is constructed as follows. In Section 2, we give some basic definitions are summarized for q-fractional calculus. In Section 3, certain essential conditions are derived for the existence and uniqueness of (1). In Section 4, the Ulam-type stability of solution will be studied. An example is given in Section 5 to illustrate our results. A conclusion is drawn in Section 6.

2. Preliminaries

Let us start this section by introducing certain preliminary notions of q-fractional calculus.

Let $\mathscr{C}(\mathfrak{J}, \mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm $\|u\|_{\mathscr{C}} = \max\{|u(\tau)| : \tau \in \mathfrak{J}\}$. For $0 \leq \vartheta < 1$, we denote the space $\mathscr{C}_\vartheta(\mathfrak{J})$ as

$$\mathscr{C}_\vartheta(\mathfrak{J}) = \left\{u : (0, T] \to \mathbb{R} | (\tau - 0)_q^\vartheta | u(\tau) \in \mathscr{C}(\mathfrak{J}, \mathbb{R})\right\},$$

where $\mathscr{C}_\vartheta(\mathfrak{J})$ is the weighted space of all continuous functions u on the finite interval $\mathfrak{J}$.

Obviously, $\mathscr{C}_\vartheta(\mathfrak{J})$ is the Banach space with the norm

$$\|u\|_{\mathscr{C}_\vartheta} = \|(\tau - 0)_q^\vartheta u(t)\|_{\mathscr{C}}.$$

Meanwhile, $\mathscr{C}_\vartheta^n(\mathfrak{J}) = \left\{u \in \mathscr{C}^{n-1}(\mathfrak{J}) : u^{(n)} \in \mathscr{C}_\vartheta(\mathfrak{J})\right\}$ is the Banach space with the norm

$$\|u\|_{\mathscr{C}_\vartheta^n} = \sum_{k=0}^{n-1} \|u^{(k)}\|_{\mathscr{C}} + \|u^{(k)}\|_{\mathscr{C}}, \quad n \in \mathbb{N}.$$

In addition, $\mathscr{C}_\vartheta^0(\mathfrak{J}) := \mathscr{C}_\vartheta(\mathfrak{J})$.

In order to solve our problem, the subsequent spaces are presented.

$$\mathscr{C}_{1-\vartheta}^{\zeta_1,\zeta_2}(\mathfrak{J}) = \left\{u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}), {}_q^H\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2} u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J})\right\}$$

and

$$\mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J}) = \left\{u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}), {}_q\mathfrak{D}_{0^+}^{\vartheta} u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J})\right\}.$$

It is obvious that

$$\mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J}) \subset C_{1-\vartheta}^{\zeta_1,\zeta_2}(\mathfrak{J}).$$

Definition 1 ([15])**.** *Let $q \in (0,1)$ and $\zeta_1 > 0$. Then the q-RL fractional integral is defined as*

$${}_q\mathfrak{I}_{0^+}^{\zeta_1} u(\tau) = \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} u(\eta) d_q\eta,$$

where

$$(n - m)^{(k)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{i+k}}, \quad n \neq 0, \quad k \in \mathbb{R},$$

and

$$\Gamma_q(\tau) = \frac{(1-q)^{(\tau-1)}}{(1-q)^{\tau-1}}, \quad \tau \in \mathbb{R} - \{0, -1, -2, \cdots\},$$

where $\Gamma_q(\tau + 1) = [\tau]_q \Gamma_q(\tau)$ with

$$[k]_q = \frac{1 - q^k}{1 - q}, \quad k \in \mathbb{R}.$$

Definition 2 ([11]). *Let $n-1<\zeta_1<n$, the R-L fractional q-derivative of the function u is defined by*

$$ {}_q\mathfrak{D}^{\zeta_1}_{0^+}u(\tau)=\mathfrak{D}^n_q {}_q\mathfrak{I}^{n-\zeta_1}_{0^+}u(\tau), $$

where $\mathfrak{D}^n_q=\left(\dfrac{d_q}{d_q\tau}\right)^n$.

Definition 3 ([16]). *The Caputo fractional q-derivative of order $\zeta_1>0$ of function u is described as*

$$ {}^C_q\mathfrak{D}^{\zeta_1}_{0^+}u(\tau)={}_q\mathfrak{I}^{n-\zeta_1}_{0^+}\mathfrak{D}^n_q u(\tau),\quad t\in\mathfrak{J}. $$

Definition 4 ([30]). *The q-HFD of a function u of order $0<\zeta_1<1$ and type $0\le\zeta_2\le1$ is defined by*

$$ {}^H_q\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}u(\tau)=\Big({}_q\mathfrak{I}^{\zeta_2(1-\zeta_1)}_{0^+}\mathfrak{D}_q\big({}_q\mathfrak{I}^{(1-\zeta_2)(1-\zeta_1)}_{0^+}u\big)\Big)(\tau). $$

Remark 1 ([30]). *The q-HFD can be viewed as a generalization of the q-RL and q-Caputo derivative:*

(i) *The operator ${}_q\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}$ also can be rewritten as*

$$ \begin{aligned} {}^H_q\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}u(\tau)=&\Big({}_q\mathfrak{I}^{\zeta_2(1-\zeta_1)}_{0^+}\mathfrak{D}_q\big({}_q\mathfrak{I}^{(1-\zeta_2)(1-\zeta_1)}_{0^+}u\big)\Big)(\tau)\\ =&{}_q\mathfrak{I}^{\zeta_2(1-\zeta_1)}_{0^+}{}_q\mathfrak{D}^{\vartheta}_{0^+}u(\tau),\quad \vartheta=\zeta_1+\zeta_2-\zeta_1\zeta_2. \end{aligned} $$

(ii) *Let $\zeta_1=0$, the R-L fractional q-derivative, be presented as ${}_q\mathfrak{D}^{\zeta_1}_{0^+}={}_q\mathfrak{D}^{\zeta_1,0}_{0^+}$.*

(iii) *Let $\zeta_2=0$, the Caputo fractional q-derivative, be presented as ${}^C_q\mathfrak{D}^{\zeta_1}_{0^+}={}_q\mathfrak{I}^{1-\zeta_1}_{0^+}\mathfrak{D}_q$.*

Lemma 1 ([15]). *Let $\zeta_1,\zeta_2>0$. Then*

(i) $\big({}_q\mathfrak{I}^{\zeta_1}_{0^+}\eta^{\zeta_2-1}\big)(\tau)=\dfrac{\Gamma_q(\zeta_2)}{\Gamma_q(\zeta_2+\zeta_1)}\tau^{\zeta_2+\zeta_1-1}.$

(ii) $\big({}_q\mathfrak{D}^{\zeta_1}_{0^+}\eta^{\zeta_1-1}\big)(\tau)=0,\quad 0<\zeta_1<1.$

Lemma 2 ([15]). *Let $\zeta_1,\zeta_2>0$ and $u\in\mathscr{L}^1(\mathfrak{J})$, for $\tau\in\mathfrak{J}$. The following properties exist:*

(i) $\big({}_q\mathfrak{I}^{\zeta_1}_{0^+}{}_q\mathfrak{I}^{\zeta_2}_{0^+}u\big)(\tau)=\big({}_q\mathfrak{I}^{\zeta_1+\zeta_2}_{0^+}u\big)(\tau).$

(ii) $\big({}_q\mathfrak{D}^{\zeta_1}_{0^+}{}_q\mathfrak{I}^{\zeta_1}_{0^+}u\big)(\tau)=u(\tau).$

Lemma 3 ([11]). *Let $0<\zeta_1<1$, and $0\le\vartheta<1$. If $u\in\mathscr{C}_\vartheta(\mathfrak{J},\mathbb{R})$ and ${}_q\mathfrak{I}^{1-\zeta_1}_{0^+}u\in\mathscr{C}^1_\vartheta(\mathfrak{J},\mathbb{R})$, then*

$$ \big({}_q\mathfrak{I}^{\zeta_1}_{0^+}{}_q\mathfrak{D}^{\zeta_1}_{0^+}u\big)(\tau)=u(\tau)-\frac{{}_q\mathfrak{I}^{1-\zeta_1}_{0^+}u(0)}{\Gamma_q(\zeta_1)}\tau^{\zeta_1-1},\quad \tau\in\mathfrak{J}. $$

Lemma 4 ([11]). *For $0\le\vartheta<1$ and $u\in\mathscr{C}_\vartheta(\mathfrak{J},\mathbb{R})$,*

$$ \big({}_q\mathfrak{I}^{\zeta_1}_{0^+}u\big)(0)=\lim_{\tau\to0}{}_q\mathfrak{I}^{\zeta_1}_{0^+}u(\tau)=0,\quad 0\le\vartheta<\zeta_1. $$

Lemma 5 ([11]). *Let $\zeta_1,\zeta_2>0$ and $\vartheta=\zeta_1+\zeta_2-\zeta_1\zeta_2$. If $u\in\mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$, then*

$$ {}_q\mathfrak{I}^{\vartheta}_{0^+}{}_q\mathfrak{D}^{\vartheta}_{0^+}u={}_q\mathfrak{I}^{\vartheta}_{0^+}{}_q\mathfrak{D}^{\zeta_1,\zeta_2}_{0^+}u,\quad {}_q\mathfrak{D}^{\vartheta}_{0^+}{}_q\mathfrak{I}^{\zeta_1}_{0^+}u={}_q\mathfrak{D}^{\zeta_2(1-\zeta_1)}_{0^+}u(\tau). $$

Lemma 6 ([30])**.** *Let $u \in \mathscr{L}^1(\mathfrak{J})$ and ${}_q\mathfrak{D}_{0^+}^{\zeta_2(1-\zeta_1)}u \in \mathscr{L}^1(\mathfrak{J})$. Then*

$$ {}_q^H\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2}\, {}_q\mathfrak{J}_{0^+}^{\zeta_1} u = {}_q\mathfrak{J}_{0^+}^{\zeta_2(1-\zeta_1)}\, {}_q\mathfrak{D}_{0^+}^{\zeta_2(1-\zeta_1)} u. $$

3. Main Results

To obtain our findings, we want the following hypotheses.

$(\mathcal{H}_1)$Let $g : \mathfrak{J} \times \mathbb{R}^n \to \mathbb{R}$ be a level-wise continuous function such that $g \in \mathscr{C}_{1-\vartheta}^{\zeta_2(1-\zeta_1)}(\mathfrak{J}, \mathbb{R})$ for any $u \in \mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J}, \mathbb{R})$, and there is a constant $L_1 > 0$ such that

$$ \begin{aligned} &|g(\tau_1, u_1, u_2, \cdots, u_n) - g(\tau_2, v_1, v_2, \cdots, v_n)| \\ &\quad \le L_1\{|\tau_1 - \tau_2| + |u_1 - v_1| + |u_2 - v_2| + \cdots + |u_n - v_n|\}, \end{aligned} $$

for $\tau_1, \tau_2 \in (0, T]$ and all $u_i, v_i \in \mathbb{R}, (i = 1, 2, \cdots, n)$.

$(\mathcal{H}_2)$There is an increasing function $\varphi \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$, and there is $\mu_\varphi > 0$ such that for any $\tau \in \mathfrak{J}$,

$$ {}_q\mathfrak{J}_{0^+}^{\zeta_1}\varphi(\tau) \le \mu_\varphi \varphi(\tau). $$

Lemma 7 ([29], Lemma 3.6)**.** *Let $g : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$ be a function such that $g \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ for any $u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$. A function $u \in \mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J}, \mathbb{R})$ is a solution of a nonlocal IVP:*

$$ \begin{aligned} &{}_q^H\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2} u(\tau) = g(\tau, u(\tau)), \quad \tau \in \mathfrak{J}, \\ &{}_q\mathfrak{J}_{0^+}^{1-\vartheta} u(0) = u_0, \quad \vartheta = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \end{aligned} $$

if and only if u satisfies the following integral equation:

$$ u(\tau) = \frac{u_0 \tau^{\vartheta-1}}{\Gamma_q(\vartheta)} + \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1-1} g(\eta, u(\eta)) d_q\eta. $$

Lemma 8. *Let $g : \mathfrak{J} \times \mathbb{R}^n \to \mathbb{R}$ be a level-wise continuous function such that $g \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ for any $u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$. A function $u \in \mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J}, \mathbb{R})$ is a solution to the problem (1) if and only if u satisfies the following integral equation:*

$$ \begin{cases} u(\tau) = & \dfrac{\Omega \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \displaystyle\sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + \dfrac{1}{\Gamma_q(\zeta_1)} \displaystyle\int_0^\tau (\tau - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta, \end{cases} \tag{2} $$

where

$$ \Omega := \frac{1}{\Gamma_q(\vartheta) - \sum_{i=1}^m \lambda_i (\tau_i)^{\vartheta-1}}, \quad \text{if} \quad \Gamma_q(\vartheta) \ne \sum_{i=1}^m \lambda_i (\tau_i)^{\vartheta-1}. \tag{3} $$

Proof. Indeed, from Lemma 7, a solution to the problem (1) can be expressed by

$$ \begin{cases} u(\tau) = & \dfrac{{}_q\mathfrak{J}_{0^+}^{1-\vartheta} u(0)}{\Gamma_q(\vartheta)} \tau^{\vartheta-1} + \dfrac{1}{\Gamma_q(\zeta_1)} \displaystyle\int_0^\tau (\tau - q\eta)^{\zeta_1-1} \\ & \times \quad g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \end{cases} \tag{4} $$

Substituting $\tau = \tau_i$ and multiplying both sides by λ_i in Equation (4) gives

$$\begin{cases} \lambda_i u(\tau_i) = & \dfrac{{}_q\Im_{0^+}^{1-\vartheta} u(0)}{\Gamma_q(\vartheta)} \lambda_i (\tau_i)^{\vartheta-1} + \dfrac{\lambda_i}{\Gamma_q(\zeta_1)} \displaystyle\int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1 - 1} \\ & \times g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \end{cases} \tag{5}$$

From the nonlocal initial condition, ${}_q\Im_{0^+}^{1-\vartheta} u(0) = \sum_{i=1}^{m} \lambda_i u(\tau_i)$, and in view of Equation (5), we obtain

$$\begin{aligned} {}_q\Im_{0^+}^{1-\vartheta} u(0) =& \frac{{}_q\Im_{0^+}^{1-\vartheta} u(0)}{\Gamma_q(\vartheta)} \sum_{i=1}^{m} \lambda_i (\tau_i)^{\vartheta-1} + \frac{\sum_{i=1}^{m} \lambda_i}{\Gamma_q(\zeta_1)} \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \\ & \times g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta, \end{aligned}$$

which gives

$${}_q\Im_{0^+}^{1-\vartheta} u(0) = \frac{\Gamma_q(\vartheta)}{\Gamma_q(\zeta_1)} \Omega \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \tag{6}$$

Inserting Equation (6) in Equation (4), we obtain

$$\begin{aligned} u(\tau) =& \frac{\Omega \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \\ & \times g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + \frac{1}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \end{aligned}$$

Conversely, applying ${}_q\Im_{0^+}^{1-\vartheta}$ to both sides of Equation (2), and by using Lemma's 1 and 2, we obtain

$$\begin{aligned} {}_q\Im_{0^+}^{1-\vartheta} u(\tau) =& {}_q\Im_{0^+}^{1-\vartheta} (\tau)^{\vartheta-1} \frac{\Omega}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \\ & \times g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + {}_q\Im_{0^+}^{1-\vartheta} {}_q\Im_{0^+}^{\zeta_1} g(\tau, u(\sigma_1(\tau)), u(\sigma_2(\tau)), \cdots, u(\sigma_n(\tau))) \\ =& \frac{\Gamma_q(\vartheta)}{\Gamma_q(\zeta_1)} \Omega \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \\ & \times g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + {}_q\Im_{0^+}^{1-\zeta_2(1-\zeta_1)} g(\tau, u(\sigma_1(\tau)), u(\sigma_2(\tau)), \cdots, u(\sigma_n(\tau))). \end{aligned}$$

Since $1 - \vartheta < 1 - \zeta_2(1 - \zeta_1)$, Lemma 4 can be used when taking the limit as $\tau \to 0$:

$${}_q\Im_{0^+}^{1-\vartheta} u(0) = \frac{\Gamma_q(\vartheta)}{\Gamma_q(\zeta_1)} \Omega \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \tag{7}$$

Substituting $\tau - \tau_i$ in Equation (2), we have

$$\begin{aligned} u(\tau_i) =& \frac{\Omega \tau_i^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + \frac{1}{\Gamma_q(\zeta_1)} \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \end{aligned}$$

This implies

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i u(\tau_i) =&\frac{\Omega}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}g(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta\\&\times\sum_{i=1}^{m}\lambda_i\tau_i^{\vartheta-1}+\frac{\sum_{i=1}^{m}\lambda_i}{\Gamma_q(\zeta_1)}\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}\\&\times g(\eta,u(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta\\=&\frac{1}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}g(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta\\&\times\left(1+\Omega\sum_{i=1}^{m}\lambda_i(\tau_i)^{\vartheta-1}\right)\\=&\frac{\Gamma_q(\vartheta)}{\Gamma_q(\zeta_1)}\Omega\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}g(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{m}\lambda_i u(\tau_i)=\frac{\Gamma_q(\vartheta)}{\Gamma_q(\zeta_1)}\Omega\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}g(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta. \tag{8}$$

It follows that Equations (7) and (8) gives

$${}_q\mathfrak{I}_{0^+}^{1-\vartheta}u(0)=\sum_{i=1}^{m}\lambda_i u(\tau_i).$$

Operating ${}_q\mathfrak{D}_{0^+}^{\vartheta}$ both sides of Equation (2) and in view of Lemmas 1 and 5 gives

$${}_q\mathfrak{D}_{0^+}^{\vartheta}u(\tau)={}_q\mathfrak{D}_{0^+}^{\zeta_2(1-\zeta_1)}g(\tau,u(\sigma_1(\tau)),u(\sigma_2(\tau)),\cdots,u(\sigma_n(\tau))). \tag{9}$$

Since $u\in\mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J},\mathbb{R})$ and in view of Definition of $\mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J},\mathbb{R})$, we have

$${}_q\mathfrak{D}_{0^+}^{\zeta_2(1-\zeta_1)}g=\mathfrak{D}_q{}_q\mathfrak{I}_{0^+}^{1-\zeta_2(1-\zeta_1)}g\in\mathscr{C}_{1-\vartheta}(\mathfrak{J},\mathbb{R}).$$

For $g\in\mathscr{C}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$, it is well-known that ${}_q\mathfrak{I}_{0^+}^{1-\zeta_2(1-\zeta_1)}g\in\mathscr{C}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$, so ${}_q\mathfrak{I}_{0^+}^{1-\zeta_2(1-\zeta_1)}g\in\mathscr{C}_{1-\vartheta}^{1}(\mathfrak{J},\mathbb{R})$ satisfies the condition of Lemma 3.

Next, operating ${}_q\mathfrak{I}_{0^+}^{1-\zeta_2(1-\zeta_1)}$ on both sides of Equation (9) and by using Lemma 3, we obtain

$$\begin{aligned}{}_q\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2}u(\tau)=\ &g(\tau,u(\sigma_1(\tau)),u(\sigma_2(\tau)),\cdots,u(\sigma_n(\tau)))\\&-\frac{{}_q\mathfrak{I}_{0^+}^{1-\zeta_2(1-\zeta_2)}g(0,u(\sigma_1(0)),u(\sigma_2(0)),\cdots,u(\sigma_n(0)))}{\Gamma_q(\zeta_2(1-\zeta_1))}\tau^{\zeta_2(1-\zeta_1)},\end{aligned}$$

where

$${}_q\mathfrak{I}_{0^+}^{\zeta_2(1-\zeta_2)}g(0,u(\sigma_1(0)),u(\sigma_2(0)),\cdots,u(\sigma_n(0)))=0.$$

Thus, it reduces to ${}_q\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2}u(\tau)=g(\tau,u(\sigma_1(\tau)),u(\sigma_2(\tau)),\cdots,u(\sigma_n(\tau)))$. The proof is complete. □

Theorem 1. *Suppose that $g:\mathfrak{J}\times\mathbb{R}^n\to\mathbb{R}$ is level-wise continuous and bounded, and $\sigma_i:\mathfrak{J}\to\mathbb{R}(i=1,2,\cdots,m)$ are continuous. Then the problem (1) has at least one solution in the space $\mathscr{C}_{1-\vartheta}^{\vartheta}(\mathfrak{J},\mathbb{R})\subset\mathscr{C}_{1-\vartheta}^{\zeta_1,\zeta_2}(\mathfrak{J},\mathbb{R})$.*

Proof. Define an operator $\mathscr{P} : \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R}) \to \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ by

$$\begin{cases} (\mathscr{P}u)(\tau) = & \dfrac{\Omega\tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \\ & + \dfrac{1}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta. \end{cases} \tag{10}$$

It follows that the operator $\mathscr{P}$ is well-defined.

Step 1: $\mathscr{P}$ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$. Then, for each $\tau \in \mathfrak{J}$,

$$\begin{aligned} &\left| ((\mathscr{P}u_n)(\tau) - (\mathscr{P}u)(\tau))\tau^{1-\vartheta} \right| \\ &\le \frac{|\Omega|}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \big| g(\eta, u_n(\sigma_1(\eta)), u_n(\sigma_2(\eta)), \cdots, u_n(\sigma_n(\eta))) \\ &\quad - g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta + \frac{\tau^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} \\ &\quad \times \big| g(\eta, u_n(\eta, u_n(\sigma_1(\eta)), u_n(\sigma_2(\eta)), \cdots, u_n(\sigma_n(\eta))) \\ &\quad - g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta. \end{aligned}$$

Taking into account the fact that g is level-wise continuous, it follows that

$$\big| g(\eta, u_n(\sigma_1(\eta)), u_n(\sigma_2(\eta)), \cdots, u_n(\sigma_n(\eta))) - g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| \to 0$$

as $u_n \to u$. We obtain from the foregoing inequality the following:

$$\|\mathscr{P}u_n - \mathscr{P}u\| \to 0 \quad \text{as} \quad u_n \to u.$$

Thus, $\mathscr{P}$ is continuous.

Step 2: $\mathscr{P}$ maps bounded sets into bounded sets in $\mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$.

For $\rho > 0$, there is a constant $K > 0$ such that $u \in \mathfrak{B}_\rho = \{u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R}) : \|u\| \le \rho\}$. Then, for each $\tau \in \mathfrak{J}$, we have

$$\|\mathscr{P}u\|_{\mathscr{C}_{1-\vartheta}} \le K.$$

$$\begin{aligned} \big| (\mathscr{P}u)(\tau)\tau^{1-\vartheta} \big| \le & \frac{|\Omega|}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \big| g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta \\ & + \frac{\tau^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} \big| g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta \end{aligned}$$

which gives

$$\big| (\mathscr{P}u)(\tau)\tau^{1-\vartheta} \big| := \mathscr{P}_1 + \mathscr{P}_2, \tag{11}$$

where

$$\begin{aligned} \mathscr{P}_1 =& \frac{|\Omega|}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} \big| g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta, \\ \mathscr{P}_2 =& \frac{\tau^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} \big| g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta. \end{aligned}$$

This implies

$$\begin{cases} \mathscr{P}_1 = & \sum_{i=1}^{m} \lambda_i \tau_i^{\zeta_1+\vartheta+1} \dfrac{|\Omega| M^* \Gamma_q(\vartheta)}{\Gamma_q(\zeta_1+\vartheta)} \\ \mathscr{P}_2 = & \dfrac{M^* T^{\zeta_1} \Gamma_q(\zeta_1)}{\Gamma_q(\zeta_1+\vartheta)}, \end{cases} \tag{12}$$

where $M^* = \max_{\tau \in \mathfrak{J}} \left| g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \right|$. Inserting the Equation (12) in Equation (11), we get

$$\begin{aligned} \left|(\mathscr{P}u)(\tau)\tau^{1-\vartheta}\right| &\leq \sum_{i=1}^{m} \lambda_i \tau_i^{\zeta_1+\vartheta+1} \frac{|\Omega| M^* \Gamma_q(\vartheta)}{\Gamma_q(\zeta_1+\vartheta)} + \frac{M^* T^{\zeta_1} \Gamma_q(\zeta_1)}{\Gamma_q(\zeta_1+\vartheta)} \\ &\leq \frac{M^*}{\Gamma_q(\zeta_1+\vartheta)} \left(|\Omega| \sum_{i=1}^{m} \lambda_i \tau_i^{\zeta_1+\vartheta-1} \Gamma_q(\vartheta) + T^{\zeta_1} \Gamma_q(\zeta_1) \right) = K. \end{aligned}$$

Thus, $\|\mathscr{P}u\|_{\mathscr{C}_{1-\vartheta}} \leq K$.

Step 3: $\mathscr{P}$ maps bounded sets into an equicontinuous set of $\mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$.

For $\tau_1, \tau_2 \in \mathfrak{J}$ with $\tau_2 \leq \tau_1$, and $u \in \mathfrak{B}_\rho$, we have

$$\begin{aligned} &\left|\tau_1^{1-\vartheta} \mathscr{P}(u)(\tau_1) - \tau_2^{1-\vartheta} \mathscr{P}(u)(\tau_2)\right| \\ &\leq \left| \frac{\tau_1^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_0^{\tau_1} (\tau_1 - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right. \\ &\quad \left. - \frac{\tau_2^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_0^{\tau_2} (\tau_2 - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right| \\ &\leq \left| \frac{1}{\Gamma_q(\zeta_1)} \int_0^{\tau_1} \left[\tau_1^{1-\vartheta} (\tau_1 - q\eta)^{\zeta_1-1} - \tau_2^{1-\vartheta} (\tau_2 - q\eta)^{\zeta_1-1}\right] \right. \\ &\quad \left. \times g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right| \\ &\quad + \left| \frac{\tau_2^{1-\vartheta}}{\Gamma_q(\zeta_1)} \int_{\tau_1}^{\tau_2} (\tau_2 - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right|, \end{aligned}$$

which tends to zero as $\tau_1 \to \tau_2$ independently of u. As a consequence of Step $1-3$, together with the Arzela–Ascoli theorem, the operator $\mathscr{P}$ is completely continuous.

Step 4: A priori bounds.

It will be verified that the set $\varpi = \{u \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R}) : u = \beta(\mathscr{P}u), \beta \in [0,1]\}$ is bounded. Let $u \in \varpi$. Then $u = \beta(\mathscr{P}u)$. For any $\tau \in [0,1]$, we have

$$\begin{aligned} u(\tau) =& \beta \left[\frac{\Omega \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1-1} g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right. \\ & \left. + \frac{1}{\Gamma_q(\zeta_1)} \int_0^{\tau} (\tau - q\eta)^{\zeta_1-1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right]. \end{aligned}$$

Then

$$\left|u(\tau)\tau^{1-\vartheta}\right| \leq \frac{M^*}{\Gamma_q(\zeta_1+\vartheta)} \left(|\Omega| \sum_{i=1}^{m} \lambda_i \tau_i^{\zeta_1+\vartheta-1} \Gamma_q(\vartheta) + T^{\zeta_1} \Gamma_q(\zeta_1) \right) = K^*,$$

which proves that ϖ is bounded. According to Scheafer's fixed point theorem, the operator $\mathscr{P}$ has at least one fixed point. Thus, the nonlocal problem (1) has at least one solution. The proof is complete. □

Theorem 2. *Suppose that $(\mathcal{H}_1) - (\mathcal{H}_2)$ are satisfied. If*

$$\frac{nL_1}{\Gamma_q(\zeta_1+1)}\left(|\Omega|(m\lambda)T^{\zeta_1+\vartheta-1}+T^{\zeta_1}\right)<1, \tag{13}$$

then the problem (1) has a unique solution.

Proof. We will use the Banach contraction principle to prove that $\mathscr{P}$, defined by (10), has a unique fixed point. Now, we show that $\mathscr{P}$ is a contraction mapping. For $u_1, u_2 \in \mathscr{C}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$ and $\tau \in \mathfrak{J}$, we have
$|((\mathscr{P}u_1)(\tau) - (\mathscr{P}u_2)(\tau))\tau^{1-\vartheta}|$

$$\begin{aligned}
&\leq \frac{|\Omega|}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}|g(\eta,u_1(\sigma_1(\eta)),u_1(\sigma_2(\eta)),\cdots,u_1(\sigma_n(\eta)))\\
&\quad - g(\eta,u_2(\sigma_1(\eta)),u_2(\sigma_2(\eta)),\cdots,u_2(\sigma_n(\eta)))|d_q\eta + \frac{\tau^{1-\vartheta}}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau-q\eta)^{\zeta_1-1}\\
&\quad \times |g(\eta,u_1(\sigma_1(\eta)),u_1(\sigma_2(\eta)),\cdots,u_1(\sigma_n(\eta)))\\
&\quad - g(\eta,u_2(\sigma_1(\eta)),u_2(\sigma_2(\eta)),\cdots,u_2(\sigma_n(\eta)))|d_q\eta\\
&\leq \frac{|\Omega|}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}nL_1|u_1(\eta)-u_2(\eta)|d_q\eta\\
&\quad + \frac{\tau^{1-\vartheta}}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau-q\eta)^{\zeta_1-1}nL_1|u_1(\eta)-u_2(\eta)|d_q\eta\\
&\leq \frac{|\Omega|}{\Gamma_q(\zeta_1+1)}\left(nL_1(m\lambda)T^{\zeta_1+\vartheta-1}\|u_1-u_2\|_{\mathscr{C}_{1-\vartheta}}\right)\\
&\quad + \frac{T^{\zeta_1}nL_1}{\Gamma_q(\zeta_1+1)}\|u_1-u_2\|_{\mathscr{C}_{1-\vartheta}}\\
&\leq \frac{nL_1}{\Gamma_q(\zeta_1+1)}\left(|\Omega|(m\lambda)T^{\zeta_1+\vartheta-1}+T^{\zeta_1}\right)\|u_1-u_2\|_{\mathscr{C}_{1-\vartheta}}.
\end{aligned}$$

This implies that

$$\|\mathscr{P}u_1-\mathscr{P}u_2\|_{\mathscr{C}_{1-\vartheta}} \leq \frac{nL_1}{\Gamma_q(\zeta_1+1)}\left(|\Omega|(m\lambda)T^{\zeta_1+\vartheta-1}+T^{\zeta_1}\right)\|u_1-u_2\|_{\mathscr{C}_{1-\vartheta}}.$$

Then, $\mathscr{P}$ is a contraction mapping. From the Banach contraction principle, $\mathscr{P}$ has a unique fixed point. Therefore, the nonlocal problem (1) has a unique solution. The proof is completed. □

4. Stability Theory

In this section, we study the Ulam–Hyers (U-H), the generalized Ulam–Hyers (G-U-H), the Ulam–Hyers–Rassias (U-H-R), and the generalized Ulam–Hyers–Rassias (G-U-H-R) stability of the solution to the problem (1).

Definition 5. *Equation (1) is said to be U-H stable if there is a real constant $C_g > 0$ such that for all $\epsilon > 0$ and for every solution $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$ of the inequality*

$$\left|{}_q\mathfrak{D}_{0^+}^{\zeta_1,\zeta_2}v(\tau) - g(\tau,v(\sigma_1(\tau)),v(\sigma_2(\tau)),\cdots,v(\sigma_n(\tau)))\right| \leq \epsilon, \quad \tau\in\mathfrak{J}, \tag{14}$$

there is a solution $u \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ to the problem (1) with

$$|v(\tau) - u(\tau)| \leq C_g \epsilon, \quad \tau \in \mathfrak{J}. \tag{15}$$

Definition 6. *Equation (1) is said to be G-U-H stable if there is $\chi_g \in \mathscr{C}([0, \infty], [0, \infty])$ and $\chi_g(0) = 0$ such that for every solution $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ of (14), there is a solution $u \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ to the problem (1) such that*

$$|v(\tau) - u(\tau)| \leq \chi_g(\epsilon), \quad \tau \in \mathfrak{J}. \tag{16}$$

Definition 7. *Equation (1) is said to be U-H-R stable with respect to $\varphi \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$, if there is a real constant $C_g > 0$ such that for each $\epsilon > 0$ and for each solution $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ to the inequality*

$$\left| {}_q\mathfrak{D}^{\zeta_1, \zeta_2}_{0^+} v(\tau) - g(\tau, v(\sigma_1(\tau)), v(\sigma_2(\tau)), \cdots, v(\sigma_n(\tau))) \right| \leq \epsilon \varphi(\tau), \quad \tau \in \mathfrak{J}, \tag{17}$$

there is a solution $u \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ to the problem (1) with

$$|v(\tau) - u(\tau)| \leq C_g \epsilon \varphi(\tau), \quad \tau \in \mathfrak{J}. \tag{18}$$

Definition 8. *Equation (1) is said to be G-U-H-R stable with respect to $\varphi \in \mathscr{C}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ if there is a real constant $C_{g,\varphi} > 0$ such that for each solution $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ of the inequality (17) there is a solution $u \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ to the problem (1) with*

$$|v(\tau) - u(\tau)| \leq C_{g,\varphi} \varphi(\tau), \quad \tau \in \mathfrak{J}. \tag{19}$$

Remark 2. *A function $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ is a solution to (1) if and only if there is a function $g_1 \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ such that*

(i) $|g_1(\tau)| < \epsilon, \quad \tau \in \mathfrak{J},$

(ii) ${}_q\mathfrak{D}^{\zeta_1, \zeta_2}_{0^+} v(\tau) = g(\tau, v(\sigma_1(\tau)), v(\sigma_2(\tau)), \cdots, v(\sigma_n(\tau))) + g_1(\tau), \quad \tau \in \mathfrak{J}.$

Lemma 9. *Let $0 < \zeta_1 < 1$, $0 \leq \zeta_2 \leq 1$. If a function $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ is a solution to the inequality (14), then v is a solution to the following integral inequality:*

$$\begin{aligned} &\left| v(\tau) - \mathcal{A}_v - \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) d_q\eta \right| \\ &\qquad \leq \left(\frac{|\Omega|(m\lambda) T^{\vartheta + \zeta_1 - 1}}{\Gamma_q(\zeta_1 + 1)} + \frac{T^{\zeta_1}}{\Gamma_q(\zeta_1 + 1)} \right) \epsilon, \end{aligned} \tag{20}$$

where

$$\mathcal{A}_v = \frac{\Omega \tau^{\vartheta - 1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1 - 1} g(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) d_q\eta.$$

Proof. From Remark 2, it is evident that

$${}_q\mathfrak{D}^{\zeta_1, \zeta_2}_{0^+} v(\tau) = g(\tau, v(\sigma_1(\tau)), v(\sigma_2(\tau)), \cdots, v(\sigma_n(\tau))) + g_1(\tau), \quad \tau \in \mathfrak{J}.$$

Then

$$
\begin{aligned}
v(\tau) =& \frac{\Omega\tau_i^{\vartheta-1}}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\bigg(\int_0^{\tau_i}(\tau_i - q\eta)^{\zeta_1-1}g(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta)))d_q\eta \\
&+ \int_0^{\tau_i}(\tau_i - q\eta)^{\zeta_1-1}g_1(\eta)d_q\eta\bigg) + \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1} \\
&\times g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta)))d_q\eta \\
&+ \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1}g_1(\eta)d_q\eta,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left|v(\tau) - \mathcal{A}_v - \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1}g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta)))d_q\eta\right| \\
&= \left|\frac{\Omega\tau^{\vartheta-1}}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - q\eta)^{\zeta_1-1}g_1(\eta)d_q\eta + \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1}g_1(\eta)d_q\eta\right| \\
&\le \frac{|\Omega|\tau^{\vartheta-1}}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - q\eta)^{\zeta_1-1}|g_1(\eta)|d_q\eta + \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1}|g_1(\eta)|d_q\eta \\
&\le \left(\frac{|\Omega|(m\lambda)T^{\vartheta+\zeta_1-1}}{\Gamma_q(\zeta_1+1)} + \frac{T^{\zeta_1}}{\Gamma_q(\zeta_1+1)}\right)\epsilon,
\end{aligned}
$$

□

Lemma 10. *Let $0 < \zeta_1 < 1$, $0 \le \zeta_2 \le 1$. If a function $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ is a solution to the inequality (14), then v is a solution to the following integral inequality:*

$$
\begin{aligned}
&\left|v(\tau) - \mathcal{A}_v - \frac{1}{\Gamma_q(\zeta_1)}\int_0^{\tau}(\tau - q\eta)^{\zeta_1-1}g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta)))d_q\eta\right| \\
&\qquad \le \left(\Omega\tau^{\vartheta-1}(m\lambda) + 1\right)\epsilon\mu_{\varphi}\varphi(\tau),
\end{aligned}
\tag{21}
$$

where

$$
\mathcal{A}_v = \frac{\Omega\tau^{\vartheta-1}}{\Gamma_q(\zeta_1)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - q\eta)^{\zeta_1-1}g(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta)))d_q\eta.
$$

Proof. The proof of the lemma directly follows from Remark 1 and Lemma 9. □

Lemma 11 ([4] (**Gronwall's inequality**)). *Let $w_1 : \mathfrak{J} \to [0, \infty)$ be a real function, let $w_2(\cdot)$ be a non-negative, locally integrable function on $\mathfrak{J}$, and assume there are constants $b > 0$ such that*

$$
w_1(\tau) \le w_2(\tau) + b\int_0^{\tau}\frac{w_1(\eta)}{(\eta - q\eta)^{\zeta_1}}d_q\eta, \quad 0 < \zeta_1 < 1.
$$

Then there are constants $\mathcal{N}^* = \mathcal{N}^*(\zeta_1)$ such that

$$
w_1(\tau) \le w_2(\tau) + \mathcal{N}^* b\int_0^{\tau}\frac{w_2(\eta)}{(\eta - q\eta)^{\zeta_1}}d_q\eta, \quad \textit{for all} \quad \tau \in \mathfrak{J}.
$$

We are ready to establish our stability results for problem (1).

Theorem 3. *Suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and (13) are satisfied. Then the problem (1) is U-H stable and accordingly G-U-H stable.*

Proof. Let $\epsilon > 0$, let $v \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ be the solution to the inequality (14), and let $u \in \mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J}, \mathbb{R})$ be the unique solution to the problem (1). Using Lemma 8, we have

$$v(\tau) = \mathcal{A}_v + \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) d_q\eta,$$

where

$$\mathcal{A}_v = \frac{|\Omega| \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1 - 1} g(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) d_q\eta.$$

On the other hand, if $u(\tau_i) = v(\tau_i)$ and ${}_q\mathfrak{J}^{1-\vartheta}_{0^+} u(0) = {}_q\mathfrak{J}^{1-\vartheta}_{0^+} v(0)$, then $\mathcal{A}_u = \mathcal{A}_v$.

$$\begin{aligned}
|\mathcal{A}_u - \mathcal{A}_v| \leq & \frac{\Omega \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1 - 1} \big| g(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \\
& - g(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) \big| d_q\eta \\
\leq & \frac{\Omega \tau^{\vartheta-1}}{\Gamma_q(\zeta_1)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - q\eta)^{\zeta_1 - 1} n L_1 |u(\eta) - v(\eta)| d_q\eta \\
\leq & \frac{n L_1 |\Omega|}{\Gamma_q(\zeta_1)} \tau^{\vartheta-1} \sum_{i=1}^{m} \lambda_{iq} \mathfrak{J}^{\zeta_1}_{0^+} |u(\tau_i) - v(\tau_i)| \\
= & 0.
\end{aligned}$$

This implies that

$$\mathcal{A}_u = \mathcal{A}_v.$$

Then

$$u(\tau) = \mathcal{A}_v + \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta.$$

Thus, $|v(\tau) - u(\tau)|$

$$\begin{aligned}
\leq & \left| v(\tau) - \mathcal{A}_v - \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right| \\
& + \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} \big| g(\eta, v(\eta, v(\sigma_1(\eta)), v(\sigma_2(\eta)), \cdots, v(\sigma_n(\eta))) \\
& - g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) \big| d_q\eta \\
\leq & \left| v(\tau) - \mathcal{A}_v - \frac{1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} g(\eta, u(\eta, u(\sigma_1(\eta)), u(\sigma_2(\eta)), \cdots, u(\sigma_n(\eta))) d_q\eta \right| \\
& + \frac{n L_1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} |v(\eta) - u(\eta)| d_q\eta.
\end{aligned}$$

With the help of (20),

$$\begin{aligned}
|v(\tau) - u(\tau)| \leq & \left(\frac{|\Omega| (m\lambda) T^{\vartheta + \zeta_1 - 1}}{\Gamma_q(\zeta_1 + 1)} + \frac{T^{\zeta_1}}{\Gamma_q(\zeta_1 + 1)} \right) \epsilon \\
& + \frac{n L_1}{\Gamma_q(\zeta_1)} \int_0^\tau (\tau - q\eta)^{\zeta_1 - 1} |v(\eta) - u(\eta)| d_q\eta.
\end{aligned}$$

With the help of Lemma 11,

$$|v(\tau)-u(\tau)| \leq \left(\frac{|\Omega|(m\lambda)T^{\vartheta+\zeta_1-1}}{\Gamma_q(\zeta_1+1)}+\frac{T^{\zeta_1}}{\Gamma_q(\zeta_1+1)}\right)\epsilon+\frac{nL_1T^{\zeta_1}}{\Gamma_q(\zeta_1+1)}|v(\tau)-u(\tau)|$$
$$|v(\tau)-u(\tau)| \leq \left(\frac{|\Omega|(m\lambda)T^{\vartheta+\zeta_1-1}}{\Gamma_q(\zeta_1+1)}+\frac{T^{\zeta_1}}{\Gamma_q(\zeta_1+1)}\right)\left(1+\frac{\mathscr{N}^*nL_1}{\Gamma_q(\zeta_1+1)}T^{\zeta_1}\right)\epsilon,$$

where $\mathscr{N}^*=\mathscr{N}^*(\zeta_1)$ is a constant. Hence, we conclude that the problem (1) is U-H stable. Moreover, setting $\chi_g(\epsilon)=C_g\epsilon$, and $\chi_g(0)=0$, then the problem (1) is G-U-H stable. □

Theorem 4. *Suppose that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, *and* (13) *are satisfied, then the problem* (1) *is U-H-R stable.*

Proof. Let $\epsilon>0$, let $v\in\mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$ be the solution to the inequality (17), and let $u\in\mathscr{C}^{\vartheta}_{1-\vartheta}(\mathfrak{J},\mathbb{R})$ be the unique solution to the problem (1). Using Lemma 8, we have

$$v(\tau)=\mathcal{A}_v+\frac{1}{\Gamma_q(\zeta_1)}\int_0^\tau(\tau-q\eta)^{\zeta_1-1}g(\eta,u(\eta,u(\sigma_1(\eta)),u(\sigma_2(\eta)),\cdots,u(\sigma_n(\eta)))d_q\eta,$$

where

$$\mathcal{A}_v=\frac{|\Omega|\tau^{\vartheta-1}}{\Gamma_q(\zeta_1)}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-q\eta)^{\zeta_1-1}g(\eta,v(\sigma_1(\eta)),v(\sigma_2(\eta)),\cdots,v(\sigma_n(\eta)))d_q\eta.$$

With the help of Lemma 10,

$$\left|v(\tau)-\mathcal{A}_v-\frac{1}{\Gamma_q(\zeta_1)}\int_0^\tau(\tau-q\eta)^{\zeta_1-1}g(\eta,v(\eta,v(\sigma_1(\eta)),v(\sigma_2(\eta)),\cdots,v(\sigma_n(\eta)))d_q\eta\right| \leq\left(\Omega\tau^{\vartheta-1}(m\lambda)+1\right)\epsilon\mu_\varphi\varphi(\tau).$$

On the other hand, we have
$|v(\tau)-u(\tau)|$

$$\leq\left|v(\tau)-\mathcal{A}_v-\frac{1}{\Gamma_q(\zeta_1)}\int_0^\tau(\tau-q\eta)^{\zeta_1-1}g(\eta,v(\eta,v(\sigma_1(\eta)),v(\sigma_2(\eta)),\cdots,v(\sigma_n(\eta)))d_q\eta\right| +\frac{nL_1}{\Gamma_q(\zeta_1)}\int_0^\tau(\tau-q\eta)^{\zeta_1-1}|v(\eta)-u(\eta)|d_q\eta.$$

With the help of the inequality (21),

$$|v(\tau)-u(\tau)|\leq\left(\Omega\tau^{\vartheta-1}(m\lambda)+1\right)\epsilon\mu_\varphi\varphi(\tau)+\frac{nL_1}{\Gamma_q(\zeta_1)}\int_0^\tau(\tau-q\eta)^{\zeta_1-1}|v(\eta)-u(\eta)|d_q\eta.$$

This implies that

$$|v(\tau)-u(\tau)|\leq\left[\left(\Omega\tau^{\vartheta-1}(m\lambda)+1\right)\left(1+L_1L_2\mathscr{N}_1^*\mu_\varphi\right)\mu_\varphi\right]\epsilon\varphi(\tau),$$

where $\mathscr{N}_1^*=\mathscr{N}_1^*(\zeta_1)$ is a constant. Then, for any $\tau\in\mathfrak{J}$,

$$|v(\tau)-u(\tau)|\leq C_g\epsilon\varphi(\tau),\quad \tau\in\mathfrak{J}.$$

Hence, we conclude that the problem (1) is U-H-R stable. □

5. An Example

Taking the following nonlocal problem for the q-Hilfer fractional generalized pantograph equation:

$$\begin{cases} {}^{H}_{\frac{1}{2}}\mathfrak{D}^{\frac{3}{5},\frac{1}{5}}_{0^+}u(\tau) = & \frac{1}{10}(u(t-4)-u(t-6)), \quad \tau \in \mathfrak{J} := [0,1], \\ {}_{\frac{1}{2}}\mathfrak{I}^{1-\frac{11}{15}}_{0^+}u(0) = & \dfrac{1}{3}u(\dfrac{3}{2}). \end{cases} \tag{22}$$

Set $g(\tau, u(\sigma_1(\tau)), u(\sigma_2(\tau))) = a(u(\sigma_1(\tau)) - u(\sigma_2(\tau)))$, for all $\tau, \sigma_1, \sigma_2 \in \mathfrak{J}$.

Let $a = \frac{1}{10}$, and choose $\zeta_1 = \frac{3}{5}$, $\zeta_2 = \frac{1}{3}$, $\vartheta = \frac{11}{15}$, $q = \frac{1}{2}$, $\lambda = \frac{1}{3}$, $\tau = \frac{3}{2}$, $T = m = 1$. Denote $\sigma_1(\cdot) = -4$, $\sigma_2(\cdot) = -6$, $g(\cdot, \sigma_1(u(\cdot)), \sigma_2(u(\cdot))) = \dfrac{1}{10}(u(\cdot - 4) - u(\cdot - 6))$.

Clearly, the function g is continuous. For each $u, v \in \mathbb{R}$ and $\sigma_1, \sigma_2, \tau \in \mathfrak{J}$,

$$|g(\tau, u(\sigma_1(\tau)), u(\sigma_2(\tau))) - g(\tau, v(\sigma_1(\tau)), v(\sigma_2(\tau)))| \leq \frac{1}{5}|u - v|.$$

Hence, the hypothesis $(\mathcal{H}_1)$ holds with $n = 2$, $L_1 = \frac{1}{5}$, and the condition

$$|\Omega| = \left| \frac{1}{\Gamma_{\frac{1}{2}}(\frac{11}{15}) - \frac{1}{3}(\frac{3}{2} - 0)^{\frac{-4}{15}}} \right| \approx \left| \frac{1}{0.831 - \frac{1}{3}(\frac{3}{2})^{\frac{-4}{15}}} \right| \approx 0.5318,$$

so

$$\frac{nL_1}{\Gamma_q(\zeta_1 + 1)} \left(|\Omega|(m\lambda)T^{\zeta_1 + \vartheta - 1} + T^{\zeta_1} \right) \approx 0.31082 < 1.$$

Thus, the conditions of Theorems 2 and 3 are satisfied. Therefore, the problem (22) has a U-H stable solution that is unique.

Furthermore, by taking $\varphi(\tau) = \tau^2$, for any $\tau \in \mathfrak{J}$, we have

$$\begin{aligned} {}_{\frac{1}{2}}\mathfrak{I}^{\frac{3}{5}}_{0^+}\varphi(\tau) \leq & \frac{\tau^2}{\Gamma_{\frac{1}{2}}(\frac{3}{5})} \int_0^\tau (\tau - \frac{1}{2}\eta)^{\frac{3}{5}-1} d_q\eta \\ \leq & \frac{\varphi(\tau)}{1.37}. \end{aligned}$$

Thus,

$${}_{\frac{1}{2}}\mathfrak{I}^{\frac{3}{5}}_{0^+}\varphi(\tau) \leq \frac{\varphi(\tau)}{1.37} := \mu_\varphi \varphi(\tau).$$

Thus condition $(\mathcal{H}_2)$ is fulfilled with $\varphi(\tau) = \tau^2$ and $\mu_\varphi = \frac{1}{1.37}$. It follows from Theorem 4 that the problem (22) is U-H-R stable.

6. Conclusions

In this work, we looked at a nonlocal problem for generalized pantograph fractional q-differential equations involving q-HFD. We have discussed two important outcomes about the existence and uniqueness of solutions for the problem (1) by using Schaefer's fixed point theorem and the Banach fixed point theorem. We have also considered and studied the Ulam-type stability of the problem (1). Finally, we have provided an example to illustrate our results.

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