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Conformable Fractional Stochastic Differential Inclusions Driven by Poisson Jumps with Optimal Control and Clarke Subdifferential

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Abstract
This manuscript is devoted to analysing the solvability and optimal control of a conformable fractional stochastic differential inclusion with Clarke subdifferential and deviated argument. The proposed conformable fractional impulsive inclusion system's solvability in Hilbert space is established by employing fractional calculus, multivalued analysis, stochastic analysis, semigroup theory and a multivalued fixed point theorem. Furthermore, under some suitable assumptions, the existence of optimal control is derived by employing Balder's theorem. Lastly, an application is provided to validate the developed theoretical results.

Keywords Conformable fractional stochastic differential inclusion · Multivalued analysis · Optimal control · Poisson jumps · Deviated argument

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1 Introduction

Fractional calculus has the advantage of modelling natural events because of the nonlocal quality of the fractional operator, which makes it deserving of proving long memory or nonlocal effects to more accurately validate the physical phenomena. Fractional differential equations (FDEs) arise in problems in electrical circuits, input enhancers, electro expository science, fragmentary multi-poles, and neuron demonstrating distinctive branches of material science and organic sciences [1–4]. Nowadays, there are many fractional derivatives in the literature, for instance, Riemann–Liouville, Caputo, Hilfer, Grünwald–Letnikov, Hadamard, Coimbra, Atangana–Baleanu, and Caputo–Hadamard derivative. The best fractional derivative relies on the experimental data that best suit the theoretical model. Khalil et al. [5] addressed these issues with outdated definitions of fractional-order derivatives by proposing a new fractional derivative known as the conformable fractional derivative (CFD). The conventional limit representation concept of the ordinary derivative is expanded upon by this fractional derivative. The derivative of the exponential functions, the quotient rule, the chain rule, and the product rule are all satisfied by the CFD. Moreover, the theory and results on the CFD are rapidly developed by the researchers and also give their notable contributions [6–11].

Stochastic is one of the essential properties of the real world. Environmental noise and uncertainty in the real world has been a hot issue for many researchers. Stochastic differential equations (SDEs) contains the stochastic term composed of noise, which has the advantage of describing uncertain factors of environmental noise in the real world, see [12]. It is known that the fractional-order dynamical model is affected by random noises or stochastic noises. Hence, the stochastic dynamical system also has long-term memory [13]. The fractional-order stochastic derivative is related to the entire interval of the dynamical system compared to the integer-order stochastic derivative. In recent days, the authors initially developed the existence, uniqueness, and continuous dependence of solutions to conformable stochastic differential equations (SDEs) using the usual Picard's iteration method [14]. Ahmed et al. [15] discussed the conformable fractional stochastic differential equation with nonlinear noise and nonlocal condition via Rosenblatt process and Poisson jump is studied. Moreover, many practical systems (such as sudden price variations (jumps) due to market crashes, earthquakes, hurricanes, and epidemics) may undergo some jump-type stochastic perturbations. The sample paths of such systems are not continuous. Therefore, it is more appropriate to consider stochastic processes with jumps to describe such models. In general, these jump models are derived from Poisson random measure. The sample paths of such systems are right continuous and possess left limits. Recently, many researchers have been focusing their attention towards the theory and applications of fractional SDEs with Poisson jumps. From an application viewpoint, there are limits on Gaussian noise and to deal with more general situations one can exchange Gaussian noise by a Poisson random measure. For example, the model of river pollution is studied by Hausenblas and Marchis [16], in that the number of deposits on a bounded region of the river at infinitesimal length is $d\delta$, where δ is the distance coordinate along the river which behaves according to the Poisson process. Clarke subdifferential arises from the applied fields, such as filtration in porous media, thermo-viscoelasticity and have fascinating applications in non-smooth analysis and optimisation [17].

On the other hand, it is known that optimal control plays an important role in the analysis and in the design of control systems and engineering [18, 20–23]. For a different class of fractional optimal control problem, the solvability and optimal control of semi-linear nonlocal fractional evolution inclusion with Clarke subdifferential are studied in [24]. The same purpose has been attracted by the authors [25], for an optimal control of Sobolev-type Hilfer fractional non-instantaneous impulsive differential inclusion driven by Poisson jumps and Clarke subdifferential. Chalishajar [26] studied the optimal controls for stochastic integrodifferential equation in Hilbert space. Strongly inspired by the aforementioned works in the literature [9, 27, 28], to the best of the authors' knowledge, there is no research regarding the theoretical approach for solvability and optimal control of conformable fractional stochastic differential inclusions governed by the Clarke subdifferential with deviated argument and Poisson jumps in Hilbert space remain open, which serves as a motivation for this present work.

Remark: Comparisons:

(i) The CFD behaves well in the product rule and chain rule while complicated formulas appear in case of usual fractional calculus.
(ii) The CFD of a constant function is zero while it is not the case for Riemann fractional derivatives.
(iii) Mittag–Leffler functions play an important rule in fractional calculus as a generalization to exponential functions, while the fractional exponential function $f(t) = e^{\frac{t^\alpha}{\alpha}}$ appears in case of conformable fractional calculus.
(iv) Conformable fractional derivatives, conformable chain rule, conformable integration by parts, conformable Gronwall's inequality, conformable exponential function, conformable Laplace transform and so forth, all tend to the corresponding ones in usual calculus.
(v) In case of usual calculus, there some functions that do not have Taylor power series representations about certain points but in the theory of conformable fractional they do have.

The main contributions are listed as follows:

- First, we model the stochastic optimal control model with Poisson jumps of Clarke subdifferential type with conformable fractional derivative, which is the generalized fractional derivative.
- The presence of a Clarke subdifferential, conformable fractional derivative, deviated argument and Poisson jumps play the role of a novel conformable fractional stochastic inclusion model in the Hilbert space.
- The solvability of the proposed system is studied by employing stochastic analysis, properties of Clarke subdifferential and a multivalued fixed point theorem. The optimal control result is addressed for the considered system in the Hilbert space by employing Balder's theorem. The results obtained in this manuscript will generalize existing results with multivalued fractional SDEs involving Caputo and R-L derivatives [9, 27–29].

Now, we study the existence of a mild solution and optimal control for the following Clarke subdifferential-type conformable fractional stochastic differential inclusions with deviated argument and Poisson jumps:

$$
\begin{aligned}
&\mathscr{D}^{\alpha}\vartheta(\mathfrak{t}) \in \Lambda\vartheta(\mathfrak{t}) + \mathscr{B}\varpi(\mathfrak{t}) + \partial\Upsilon(\mathfrak{t}, \vartheta(\mathfrak{t})) \\
&\quad +\sigma(\mathfrak{t}, \vartheta(\mathfrak{t}), \vartheta(\mathfrak{e}(\vartheta(\mathfrak{t}), \mathfrak{t}))) \\
&\quad + \int_{\mathcal{Z}} \lambda(\mathfrak{t}, \vartheta(\mathfrak{t}), \delta)\, \tilde{\mathbb{N}}(\mathrm{d}\mathfrak{t}, \mathrm{d}\delta), \\
&\mathfrak{t} \in \bigcup_{i=0}^{m} (\mathfrak{a}_i, \mathfrak{b}_{i+1}] \subset \mathcal{J}' := (0, \mathrm{T}] \\
&\mathfrak{a}_0 := 0,\ \mathfrak{b}_{m+1} := \mathrm{T},\ \mathrm{T} > 0, \\
&\vartheta(\mathfrak{t}) = \mathfrak{h}_i\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-)\right),\ \mathfrak{t} \in (\mathfrak{b}_i, \mathfrak{a}_i],\ i = 1, 2, 3, \cdots m, \\
&\vartheta(0) = \vartheta_0, \quad \vartheta_0 \in \mathscr{H},
\end{aligned} \tag{1.1}
$$

where $\mathscr{D}^{\alpha}$ is the CFD of order α with $\frac{1}{2} < \alpha < 1$ and $\mathcal{J} := [0, \mathrm{T}]$, $\mathrm{T} > 0$. $\vartheta(\cdot) \in \mathscr{H}$ is the state variable in the Hilbert space $\mathscr{H}$ with the inner product $\langle\cdot,\cdot\rangle$ provided with the norm $\|\cdot\|$. Let $\Lambda : \mathscr{D}(\Lambda) \subset \mathscr{H} \to \mathscr{H}$ is the infinitesimal generator of an analytic semigroup of bounded linear operator $\mathbb{T}(\mathfrak{t})$, $\mathfrak{t} \geq 0$ on $\mathscr{H}$. The deviating argument $\mathfrak{e}(\mathfrak{t}, \cdot)$ is the mapping from $\mathscr{H} \times \mathscr{R}^+ \to \mathscr{R}^+$. The notation $\partial\Upsilon$ represents the Clarke subdifferential of a locally Lipschitz function $\Upsilon(\mathfrak{t}, \cdot) : \mathscr{H} \to \mathscr{R}$. $\mathfrak{a}_i$ and $\mathfrak{b}_i$ represent the fixed points that satisfy $\mathfrak{a}_0 < \mathfrak{b}_1 < \mathfrak{a}_1 < \mathfrak{b}_2 < \cdots < \mathfrak{b}_i < \mathfrak{a}_i < \mathfrak{b}_{i+1}$, $i = 0, 1, \cdots, m$. Also, $\mathfrak{h}_i : (\mathfrak{b}_i, \mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$ and $\vartheta(\mathfrak{b}_j^-)$ represent the left limit of ϑ at $\mathfrak{b}_i$. Let $(\Omega, \Im, \mathscr{P})$ be the complete probability space with the probability measure $\mathscr{P}$ on Ω and the normal filtration $\{\Im_{\mathfrak{t}}, \mathfrak{t} \geq 0\}$. Let $\mathbb{N}(\mathrm{d}\mathfrak{t}, \mathrm{d}\delta)$ be the Poisson counting process in the measurable space $(\mathcal{Z}, \mathbb{B}(\mathcal{Z}))$ defined on $(\Omega, \Im, \mathscr{P})$. $\tilde{\mathbb{N}}(\mathrm{d}\mathfrak{t}, \mathrm{d}\delta) = \mathbb{N}(\mathrm{d}\mathfrak{t}, \mathrm{d}\delta) - \mathfrak{h}(\mathrm{d}\delta)\mathrm{d}\mathfrak{t}$ is the compensated martingale measure with a sigma-finite intensity measure $\mathfrak{h}(\mathrm{d}\delta)$. The nonlinear functions $\lambda : \mathcal{J} \times \mathscr{H} \times \mathcal{Z} \setminus \{0\} \to \mathscr{H}$ and $\sigma : \mathcal{J} \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ are continuous. $\mathscr{B}$ is the linear operator from a separable reflexive Banach space $\mathscr{W}$ onto $\mathscr{H}$. The control variable $\varpi(\cdot)$ takes values in a set of all admissible controls $\mathscr{W}_{ad}$.

This paper is organized as follows: Section 2 provides some basic definitions and preliminary results. In Sect. 3, the proposed systems solvability is obtained by employing a multivalued fixed point theorem. Section 4 presents optimal control results for the considered conformable fractional stochastic differential inclusions with deviated argument and Poisson jumps of the system (1.1) in the mean-square moment by employing Balder's theorem. The results are illustrated with an example in Sect. 5.

2 Preliminaries

In this section, we provide some basic essential definitions and Lemmas, which are needed to examine the main results.

Definition 2.1 [5] Let $0 < \alpha < 1$. The CFD of order α of a function $\mathfrak{f}(\cdot)$ for $\mathfrak{t} > 0$ is defined as follows:

$$\frac{d^{\alpha}\mathfrak{f}(\mathfrak{t})}{d\mathfrak{t}^{\alpha}} = \lim_{\upsilon \to 0} \frac{\mathfrak{f}(\mathfrak{t} + \upsilon \mathfrak{t}^{1-\alpha}) - \mathfrak{f}(\mathfrak{t})}{\upsilon}.$$

For $\mathfrak{t} = 0$, we adopt the following definition:

$$\frac{d^{\alpha}\mathfrak{f}(0)}{d\mathfrak{t}^{\alpha}} = \lim_{\mathfrak{t} \to 0^{+}} \frac{d^{\alpha}\mathfrak{f}(\mathfrak{t})}{d\mathfrak{t}^{\alpha}}.$$

The fractional integral $\mathbb{I}^{\alpha}(\cdot)$ associated with the CFD is defined by:

$$\mathbb{I}^{\alpha}(\mathfrak{f})(\mathfrak{t}) = \int_0^{\mathfrak{t}} \mathfrak{s}^{\nu-1}\mathfrak{f}(\mathfrak{s})d\mathfrak{s}.$$

Throughout the article, $\mathcal{L}^2(\Im, \mathscr{H}) = \mathcal{L}^2(\Omega, \Im_{\mathfrak{t}}, \mathscr{P}, \mathscr{H})$ $(\mathfrak{t} > 0)$ represents the Hilbert space of all square integrable, strongly $\Im_{\mathfrak{t}}-$measurable $\mathscr{H}-$valued random variable with $\mathbb{E}\|\vartheta\|^2 < \infty$. Let $\mathscr{C}\left(\mathcal{J}, \mathcal{L}^2(\Im, \mathscr{H})\right)$ be the Banach space of all continuous maps from $\mathcal{J}$ into $\mathcal{L}^2(\Im, \mathscr{H})$ furnished with the supremum norm. Denote $\mathcal{L}^2(\Im, \mathscr{H})$, the Hilbert space of all stochastic processes $\Im_{\mathfrak{t}}$-adapted measurable function defined on $\mathcal{J}$ with the values in $\mathscr{H}$ and its norm, $\|\vartheta\|_{\mathcal{L}^2_{\Im}(\mathcal{J},\mathscr{H})} = \left[\int_0^T \mathbb{E}\|\vartheta(\mathfrak{t})\|^2 d\mathfrak{t}\right]^{1/2} < \infty$. Let $\mathcal{PC}\left(\mathcal{J}, \mathcal{L}^2(\Im, \mathscr{H})\right) = \Big\{\vartheta : \mathcal{J} \to \mathscr{H}$ such that $\vartheta|\mathcal{J}_i \in \mathscr{C}\left(\mathcal{J}_i, \mathcal{L}^2(\Im, \mathscr{H})\right), \mathcal{J}_i := (\mathfrak{b}_i, \mathfrak{b}_{i+1}],\ i = 0, 1, 2, \cdots, m$ and $\vartheta(\mathfrak{b}_i^{+})$ and $\vartheta(\mathfrak{b}_i^{-})$ exists for each $i = 1, 2, \cdots, m\Big\}$.

Now, $\mathfrak{X} = \mathcal{PC}\left(\mathcal{J}, \mathcal{L}^2(\Im, \mathscr{H})\right)$ is a Banach space furnished with the $\mathcal{PC}$-norm

$$\|\vartheta\|_{\mathcal{PC}} = \left[\sup_{\mathfrak{t}\in\mathcal{J}} \mathbb{E}\|\vartheta(\mathfrak{t})\|^2\right]^{1/2}.$$

$\mathcal{L}_{\infty}(\mathcal{J}, \mathcal{L}(\mathscr{W}, \mathscr{H}))$ denote the space of operator-valued functions, which are measurable in the strong operator topology and uniformly bounded in $\mathcal{J}$. Let $\mathcal{L}^2_{\Im}(\mathcal{J}, \mathscr{W})$ be the closed subspace of $\mathscr{W}$ consisting of all measurable and $\Im_{\mathfrak{t}}-$adapted, $\mathscr{W}-$valued stochastic processes satisfying $\mathbb{E}\int_0^{\mathfrak{t}} \|\varpi(\mathfrak{t})\|^2_{\mathscr{W}} d\mathfrak{t} < \infty$ and furnished with

$$\|\varpi\|_{\mathcal{L}^2_{\Im}(\mathcal{J},\mathscr{W})} = \left[\int_0^T \mathbb{E}\|\varpi(\mathfrak{t})\|^2 d\mathfrak{t}\right]^{1/2} < \infty.$$

Let $\mathbb{W}$ be a nonempty closed bounded convex subset of $\mathscr{W}$. Define $\mathscr{W}_{ad} = \Big\{\varpi(\cdot) \in \mathcal{L}^2_{\Im}(\mathcal{J}, \mathscr{W}) : \varpi(\mathfrak{t}) \in \mathscr{W}$ a.e. $\mathfrak{t} \in \mathcal{J}\Big\}$ be the set of all admissible controls where the control variables take its values.

For the sake of convenience, the following notations are defined: Denote the Banach spaces as $\mathscr{X}$ and $\mathscr{Y}$. Consider $\mathbb{P}(\mathscr{X})$ is the set of all nonempty subsets of $\mathscr{X}$.

$\mathbb{P}_{cl,bd}(\mathscr{X}) = \{\mathbb{A} \in \mathbb{P}(\mathscr{X}) : \mathbb{A}$ is closed and bounded $\}$, $\mathbb{P}_{cp}(\mathscr{X}) = \{\mathbb{A} \in \mathbb{P}(\mathscr{X}) : \mathbb{A}$ is compact $\}$, $\mathbb{P}_{cd}(\mathscr{X}) = \{\mathbb{A} \in \mathbb{P}(\mathscr{X}) : \mathbb{A}$ is compact and acyclic $\}$ $\mathbb{P}_{cv}(\mathscr{X}) = \{\mathbb{A} \in \mathbb{P}(\mathscr{X}) : \mathbb{A}$ is convex $\}$.

Definition 2.2 [10] Let $\mathscr{X}^*$ be the dual of the Banach space $\mathscr{X}$. The Clarke's generalized directional derivative of a locally Lipschitz function $\Upsilon : \mathscr{X} \to \mathscr{R}$ at ϑ in the direction ν, denoted by $\Upsilon^0(\vartheta; \nu)$, is represented by:

$$\Upsilon^0(\vartheta; \nu) = \lim_{y \to \vartheta} \sup_{\lambda \to 0^+} \frac{\Upsilon(y + \lambda\nu) - \Upsilon(y)}{\lambda}.$$

The generalized Clarke subdifferential of Υ at ϑ, denoted by $\partial\Upsilon$, is a subset of $\mathscr{X}^*$ given by $\partial\Upsilon(\vartheta) = \{\vartheta^* \in \mathscr{X}^* : \Upsilon^0(\vartheta; \nu) \geq \langle \vartheta^*, \nu \rangle, \forall \nu \in \mathscr{X}\}$.

Lemma 2.1 *[10] Let the two Banach spaces be $\mathscr{X}$ and $\mathscr{Y}$. If $\mathscr{F} : \mathscr{X} \to \mathbb{P}_{cp}(\mathscr{Y})$ is a closed compact multivalued function, then $\mathscr{F}$ is upper semicontinuous (u.s.c).*

Lemma 2.2 ((Bochner's Theorem)*[32] A measurable function $\beta : \mathcal{J} \to \mathscr{Y}$ is Bochner integrable if $\|\beta\|$ is Lebesgue integrable.*

Theorem 2.1 *[30, 31] Let $\mathscr{U}$ be an open subset of a Banach space $\mathscr{X}$. Let $\Psi_1 : \overline{\mathscr{U}} \to \mathscr{X}$ ($\overline{\mathscr{U}}$ denotes the closure of $\mathscr{U}$ in $\mathscr{X}$) be single-valued and $\Psi_2 : \overline{\mathscr{U}} \to \mathbb{P}_{cp,cv}(\mathscr{X})$ be a multivalued operator with $\Psi_1(\overline{\mathscr{U}}) + \Psi_2(\overline{\mathscr{U}})$ is bounded. Suppose*

(i) Ψ_1 is a contraction with a contraction constant k and
(ii) Ψ_2 is upper semicontinuous (u.s.c) and compact.

Then either

(i) The operator inclusion $\psi\vartheta \in \Psi_1\vartheta + \Psi_2\vartheta$ has a solution for $\psi = 1$, or
(ii) There is an element $\mathfrak{a} \in \partial\mathscr{U}$ such that $\psi\mathfrak{a} \in \Psi_1\mathfrak{a} + \Psi_2\mathfrak{a}$ for some $\psi > 1$, where $\partial\mathscr{U}$ is the boundary of $\mathscr{U}$ in $\mathscr{X}$.

3 Existence of Mild Solution

Solvability of (1.1) using the fixed point Theorem 2.1 is discussed in this section.

Definition 3.1 An $\Im_t$−adapted stochastic processes $\vartheta(t) \in \mathscr{X} : t \in \mathcal{J}'$ is said to be a mild solution of (1.1), when the following conditions hold:

(i) there exists an $\Im_t$−adapted measurable function $\omega \in \mathcal{L}^2(\Im, \mathscr{H})$ such that $\omega(t) \in \partial\Upsilon(t, \vartheta(t))$ for a.e $t \in \mathcal{J}'$ and

(ii) $\vartheta(\mathfrak{t}) \in \mathscr{H}$ has cadlag path on $\mathfrak{t} \in \mathcal{J}'$ a.s, the following stochastic integral gets satisfied:

$$\vartheta(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s) + \omega(s) + \sigma\,(s, \vartheta(s), \\ \vartheta(\mathfrak{e}(\vartheta(s), s)))]\,ds \\ + \int_0^{\mathfrak{t}} \int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\,(s, \vartheta(s), \delta(s))\,\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta(\mathfrak{b}_i^-)\right) \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi(s) + \omega(s) + \sigma\left(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\right)\right]ds \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\,(s, \vartheta(s), \delta(s))\,\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \; \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases} \tag{3.1}$$

In order to prove our results, the following hypotheses are necessary.

(A1) The linear operator $\mathfrak{A} : \mathscr{H} \to \mathscr{H}$ generates a $\mathcal{C}_0$-semigroup $\mathbb{T}(\cdot)$. Thus, $\exists$ $\mathbb{M} > 0$ being constant such that $\|\mathbb{T}(\mathfrak{t})\| \leq \mathbb{M}$.

(A2) Let $\Upsilon : \mathcal{J} \times \mathscr{H} \to \mathscr{R}$ satisfies the condition:

  (i) $\Upsilon(., \vartheta)$ is measurable, for all $\vartheta \in \mathscr{H}$.
  (ii) $\Upsilon(\mathfrak{t}, .)$ is locally Lipschitz continuous with Lipschitz constant $\mathbb{M}_\omega$, for a.e $\mathfrak{t} \in \mathcal{J}$.
  (iii) There exist a function $\mathfrak{r}_1 \in \mathcal{L}^2(\mathcal{J}, \mathscr{R}^+)$ and a constant $\varsigma_\omega \geq 0$ such that

$$\begin{aligned} \mathbb{E}\,\|\partial\Upsilon(\mathfrak{t}, \vartheta(\mathfrak{t}))\|^2 &= \sup\{\|\omega(\mathfrak{t})\|^2 : \omega(\mathfrak{t}) \in \partial\Upsilon\,(\mathfrak{t}, \vartheta(\mathfrak{t}))\} \\ &\leq \mathfrak{r}_1(\mathfrak{t}) + \varsigma_\omega \mathbb{E}\|\vartheta\|^2, \;\; \forall\;\; \vartheta \in \mathscr{H} \text{ and } a.e.\mathfrak{t} \in \mathcal{J}. \end{aligned}$$

(A3) The nonlinear function $\sigma : \mathcal{J} \times \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is Lipschitz continuous. For $\mathfrak{x}_1, \mathfrak{x}_2, \mathrm{y}_1, \mathrm{y}_2 \in \mathscr{H}$ and $\mathbb{M}_\sigma > 0$.

$$\begin{gathered} \mathbb{E}\,\|\sigma\,(\mathfrak{t}, \mathfrak{x}_1, \mathrm{y}_1) - \sigma\,(\mathfrak{t}, \mathfrak{x}_2, \mathrm{y}_2)\|^2 \leq \mathbb{M}_\sigma\left[\mathbb{E}\,\|\mathfrak{x}_1 - \mathfrak{x}_2\|^2 + \|\mathrm{y}_1 - \mathrm{y}_2\|^2\right], \\ \mathbb{E}\,\|\sigma(\cdot, 0, \vartheta(0))\|^2 \leq \tilde{\sigma_0}. \end{gathered}$$

(A4) Let $\mathfrak{e} : \mathscr{H} \times \mathscr{R}^+ \to \mathscr{R}^+$ satisfies the Lipschitz continuous. For all $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{H}$ and $\mathbb{M}_\mathfrak{e} > 0 \ni$

$$\mathbb{E}|\mathfrak{e}(\vartheta_1, \mathfrak{t}) - \mathfrak{e}(\vartheta_2, \mathfrak{t})|^2_{\mathscr{R}^+} \leq \mathbb{M}_\mathfrak{e}\mathbb{E}\|\vartheta_1 - \vartheta_2\|^2$$

and $\mathfrak{e}(., 0) = 0$.

(A5) $\lambda : \mathcal{J} \times \mathscr{H} \times \mathcal{Z} \setminus \{0\} \to \mathscr{H}$ is Lipschitz constant $\mathbb{M}_\lambda$ a.e. $\mathfrak{t} \in \mathcal{J}$. There exist a function $\mathfrak{r}_2 \in \mathcal{L}^2(\mathcal{J}, \mathscr{R}^+)$ and a positive constant $\varsigma_\lambda \ni$

$$\int_{\mathcal{Z}} \mathbb{E} \|\lambda(\mathfrak{t}, \vartheta, \delta)\mathrm{h}(d\delta)\|^2 \le \mathfrak{r}_2(\mathfrak{t}) + \varsigma_\lambda \mathbb{E}\|\vartheta\|^2.$$

(A6) (i) $\mathfrak{h}_i : [\mathfrak{b}_i, \mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$ such that $\mathfrak{h}_i(\cdot, \vartheta)$ is continuous, $\forall\ \vartheta \in \mathscr{H}$ and $i = 1, 2, \cdots, m$. (ii) $\mathfrak{h}_i : [\mathfrak{b}_i, \mathfrak{a}_i] \times \mathscr{H} \to \mathscr{H}$, $i = 1, 2, \cdots m$ is uniformly continuous on bounded sets and for $\mathfrak{t} \in [\mathfrak{b}_i, \mathfrak{a}_i]$, $\mathfrak{h}_i(\mathfrak{t}, \cdot)$ is a mapping from any bounded subsets of $\mathscr{H}$ into relatively compact subsets of $\mathscr{H}$. Also, there exist positive constants $\mathfrak{f}_i$ with

$$\mathbb{E} \|\mathfrak{h}_i(\mathfrak{t}, \vartheta_1) - \mathfrak{h}_i(\mathfrak{t}, \vartheta_2)\|^2 \le \mathfrak{f}_i \mathbb{E} \|\vartheta_1 - \vartheta_2\|^2, \ \mathfrak{t} \in [\mathfrak{b}_i, \mathfrak{a}_i],\ \vartheta_1, \vartheta_2 \in \mathscr{H}.$$

(A7) Let $\varpi \in \mathscr{W}$ be the control function and the operator $\mathscr{B} \in \mathcal{L}_\infty(\mathcal{J}, \mathcal{L}(\mathscr{H}, \mathcal{K}))$. $\|\mathscr{B}\|$ stands for norm of the operator $\mathscr{B}$.

Define the multivalued operator $\mathfrak{S} : \mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H}) \to \mathscr{P}\left(\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H})\right)$ by

$$\mathfrak{S}(\vartheta) = \left\{\omega \in \mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H}) : \omega(\mathfrak{t}) \in \partial\Upsilon\left(\mathfrak{t}, \vartheta(\mathfrak{t})\right),\ a.e.\mathfrak{t} \in \mathcal{J}\right\},\quad \forall\ \vartheta \in \mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H}).$$

Lemma 3.1 *[28] Suppose (A1), (A2) holds. Then, the operator $\mathfrak{S}$ satisfies the condition: if $\vartheta_n \to \vartheta$ in $\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H})$, $\mathfrak{z}_n \to \mathfrak{z}$ weakly in $\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H})$ and $\mathfrak{z}_n \in \mathfrak{S}(\vartheta_n)$, then $\mathfrak{z} \in \mathfrak{S}(\vartheta)$.*

Lemma 3.2 *[28] If the hypotheses (A1), (A2) get satisfied, for $\vartheta \in \mathcal{L}^2_{\mathfrak{F}}(\mathcal{J}, \mathscr{H})$, $\mathfrak{S}(\vartheta)$ is nonempty, convex and have weakly compact values.*

Theorem 3.1 *Assuming the hypotheses (A1)–(A6) hold, then for every $\vartheta_0 \in \mathscr{H}$, the system (1.1) has a mild solution, if*

$$\mathfrak{C} = \max_{i=1,2,\cdots,m} \left\{\frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha - 1}\left[2\mathbb{M}_\sigma \mathfrak{l}(\mathbb{M}_\mathfrak{e} + 1) + \mathbb{M}_\lambda\right], \mathfrak{f}_i, 3\mathbb{M}^2\mathfrak{f}_i + \frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha - 1}\left[2\mathbb{M}_\sigma \mathfrak{l}(\mathbb{M}_\mathfrak{e} + 1) + \mathbb{M}_\lambda\right]\right\} < 1. \tag{3.2}$$

Proof Define the multivalued operator $\Phi : \mathfrak{X} \to 2^{\mathfrak{X}}$ as $\Phi(\vartheta) = \{\mathrm{y} \in \mathfrak{X} : \mathrm{y}(\mathfrak{t}) = \eta(\mathfrak{t})\}$ such that

$$\eta(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\dfrac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \displaystyle\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\dfrac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right) \\ \left[\mathscr{B}\varpi(s) + \omega(s) + \sigma\left(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\right)\right] ds \\ + \displaystyle\int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\dfrac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, d\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta(\mathfrak{b}_i^-)\right) + \displaystyle\int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\dfrac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right) \\ \left[\mathscr{B}\varpi(s) + \omega(s) + \sigma\left(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\right)\right] ds \\ + \displaystyle\int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\dfrac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, d\delta), \ \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases}$$

Decompose the operator $\Phi(\vartheta)$ as $\Phi(\vartheta) = \Phi_1(\vartheta) + \Phi_2(\vartheta)$, where $\Phi_1(\vartheta) = \{y \in \mathfrak{X} : y(\mathfrak{t}) = \eta_1(\mathfrak{t})\}$ such that

$$\eta_1(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\sigma\left(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\right) ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta(\mathfrak{b}_i^-)\right) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\sigma\big(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\big) ds \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \, \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases}$$

and $\Phi_2(\vartheta) = \{y \in \mathfrak{X} : y(\mathfrak{t}) = \eta_2(\mathfrak{t})\}$ such that

$$\eta_2(\mathfrak{t}) = \begin{cases} \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi(s) + \omega(s)\right], \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ 0, \quad \mathfrak{t} \in (\mathfrak{b}_i, \mathfrak{a}_i] \quad i = 1, 2, \cdots m, \\ \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\big[\mathscr{B}\varpi(s) + \omega(s)\big] ds, \quad \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases}$$

Step 1: To claim Φ_1 is a contraction mapping.

Define $\mathscr{B}_{\mathfrak{l}} = \{\vartheta \in \mathfrak{X} : \mathbb{E}\|\vartheta\|^2 \leq \mathfrak{l}\}$, for $\mathfrak{l} > 0$. Now for $\mathfrak{t} \in (0, \mathfrak{b}_1]$, $\vartheta_1, \vartheta_2 \in \mathscr{B}_{\mathfrak{l}}$ and using (A3)–(A6), we have

$$\begin{aligned} &\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t}) - (\Phi_1\vartheta_2)(\mathfrak{t})\right\|^2_{\mathcal{PC}} \\ &= \sup_{\mathfrak{t}\in\mathcal{J}'} \mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t}) - (\Phi_1\vartheta_2)(\mathfrak{t})\right\|^2 \\ &\leq \sup_{\mathfrak{t}\in\mathcal{J}'}\Bigg[\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\sigma\left(s, \vartheta_1(s), \vartheta_1\left(\mathfrak{e}(\vartheta_1(s), s)\right)\right)\right. \\ &\quad \left.-\sigma\left(s, \vartheta_2(s), \vartheta_2\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right)\right] ds \\ &\quad + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\lambda\left(s, \vartheta_1(s), \delta(s)\right)\right. \\ &\quad \left.-\lambda\left(s, \vartheta_2(s), \delta(s)\right)\right]\tilde{\mathbb{N}}(ds, \mathrm{d}\delta)\Bigg\|^2\Bigg] \\ &\leq 2\sup_{\mathfrak{t}\in\mathcal{J}'}\Bigg[\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\Big[\sigma\left(s, \vartheta_1(s), \vartheta_1\left(\mathfrak{e}(\vartheta_1(s), s)\right)\right) \\ &\quad -\sigma\left(s, \vartheta_1(s), \vartheta_1\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right) \\ &\quad -\sigma\left(s, \vartheta_2(s), \vartheta_2\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right) + \sigma\left(s, \vartheta_1(s), \vartheta_1\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right)\Big] ds\Bigg\|^2 \\ &\quad + \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\lambda\left(s, \vartheta_1(s), \delta(s)\right)\right. \end{aligned}$$

$$
\begin{aligned}
&-\lambda\left(s, \vartheta_2(s), \delta(s)\right)] \tilde{\mathbb{N}}(ds, \mathrm{d}\delta)\Big\|^2\Big] \\
&\leq 2 \sup_{\mathfrak{t}\in\mathcal{J}'}\Big[2\mathfrak{b}_1\mathbb{M}^2\mathbb{M}_\sigma\Big(\int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\left\|\vartheta_1\left(\mathfrak{e}(\vartheta_1(s), s)\right)-\vartheta_1\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right\|^2 ds \\
&+\int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\left\|\vartheta_1-\vartheta_2\right\|^2 \times \left\|\mathfrak{e}\left(\vartheta_2(s), s\right)\right\|^2 ds\Big) \\
&+\mathfrak{b}_1\mathbb{M}^2\mathbb{M}_\lambda \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\left\|\vartheta_1(s)-\vartheta_2(s)\right\|^2 ds\Big] \\
&\leq \frac{2\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\left[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_{\mathfrak{e}}+1)+\mathbb{M}_\lambda\right] \sup_{s\in(0,\mathfrak{b}_1)} \mathbb{E}\left\|\vartheta_1(s)-\vartheta_2(s)\right\|^2 .
\end{aligned}
$$

For $\mathfrak{t}\in(\mathfrak{b}_i, \mathfrak{a}_i]$, we have

$$
\begin{aligned}
&\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t})-(\Phi_1\vartheta_2)(\mathfrak{t})\right\|_{\mathcal{PC}}^2=\sup_{\mathfrak{t}\in\mathcal{J}'}\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t})-(\Phi_1\vartheta_2)(\mathfrak{t})\right\|^2 \\
&\quad\leq \sup_{\mathfrak{t}\in\mathcal{J}'}\mathbb{E}\left\|\mathfrak{h}_i\left(\mathfrak{t}, \vartheta_1(\mathfrak{b}_i^-)\right)-\mathfrak{h}_i\left(\mathfrak{t}, \vartheta_2(\mathfrak{b}_i^-)\right)\right\|^2 \\
&\quad\leq \mathfrak{f}_i \sup_{\mathfrak{t}\in(\mathfrak{b}_i,\mathfrak{a}_i]}\mathbb{E}\left\|\vartheta_1(\mathfrak{t})-\vartheta_2(\mathfrak{t})\right\|^2, \quad i=1,2,\cdots,m.
\end{aligned}
$$

For $\mathfrak{t}\in[\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i=1,2,\cdots,m$.

$$
\begin{aligned}
&\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t})-(\Phi_1\vartheta_2)(\mathfrak{t})\right\|_{\mathcal{PC}}^2 \\
&\quad=\sup_{\mathfrak{t}\in\mathcal{J}'}\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t})-(\Phi_1\vartheta_2)(\mathfrak{t})\right\|^2 \\
&\quad\leq \sup_{\mathfrak{t}\in\mathcal{J}'}\mathbb{E}\Big\|\mathbb{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\mathfrak{h}_i\left(s, \vartheta_1(\mathfrak{b}_i^-)\right)-\mathfrak{h}_i\left(s, \vartheta_2(\mathfrak{b}_i^-)\right)\right] \\
&\quad+\int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\sigma\left(s, \vartheta_1(s), \vartheta_1\left(\mathfrak{e}(\vartheta_1(s), s)\right)\right)\right. \\
&\quad\left.-\sigma\left(s, \vartheta_2(s), \vartheta_2\left(\mathfrak{e}(\vartheta_2(s), s)\right)\right)\right] ds \\
&\quad+\int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)\left[\lambda\left(s, \vartheta_1(s), \delta(s)\right)\right. \\
&\quad\left.-\lambda\left(s, \vartheta_2(s), \delta(s)\right)\right] \tilde{\mathbb{N}}(ds, \mathrm{d}\delta)\Big\|^2 \\
&\quad\leq 3\mathbb{M}^2\mathfrak{f}_i+3\frac{\mathrm{T}^{2\alpha}\mathbb{M}^2}{2\alpha-1}\left[2\mathbb{M}_\sigma\mathfrak{l}(2\mathbb{M}_{\mathfrak{e}}+1)\mathbb{M}_\lambda\right] \\
&\qquad\sup_{s\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}]}\mathbb{E}\left\|\vartheta_1(s)-\vartheta_2(s)\right\|^2 .
\end{aligned}
$$

Therefore, for $\mathfrak{t} \in \mathcal{J}$,

$$\mathbb{E}\left\|(\Phi_1\vartheta_1)(\mathfrak{t}) - (\Phi_1\vartheta_2)(\mathfrak{t})\right\|^2_{\mathcal{PC}} \leq \mathfrak{C}_1 \sup_{\mathfrak{t}\in\mathcal{J}'} \mathbb{E}\left\|\vartheta_1(\mathfrak{t}) - \vartheta_2(\mathfrak{t})\right\|^2.$$

where $\mathfrak{C}_1 = \max_{i\in\{1,2,\cdots,m\}}\left[3\mathbb{M}^2\mathfrak{f}_i + \frac{3\mathrm{T}^{2\alpha}\mathbb{M}^2}{2\alpha-1}\left(2\mathbb{M}_\sigma\mathfrak{l}(2\mathbb{M}_\mathfrak{e}+1) + \mathbb{M}_\lambda\right)\right]$. Using (3.2), we have Φ_1 as a contraction.

Step 2: Claim: Φ_2 is convex for $\vartheta \in \mathfrak{X}$.

If $y_1, y_2 \in \mathcal{F}_2(\vartheta)$, then there exist $\omega_1, \omega_2 \in \mathfrak{S}(\vartheta)$ such that $\ni \mathfrak{t} \in (0, \mathfrak{b}_1]$, we have

$$\begin{aligned} y_1(\mathfrak{t}) &= \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s) + \omega_1(s, \vartheta(s))]\,ds \\ y_2(\mathfrak{t}) &= \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s) + \omega_2(s, \vartheta(s))]\,ds. \end{aligned}$$

We may consider $0 \leq \kappa \leq 1$, and then, for $\mathfrak{t} \in (0, \mathfrak{b}_1]$, we have

$$\begin{aligned} (\kappa y_1 + (1-\kappa)y_2)(\mathfrak{t}) = \int_0^{\mathfrak{t}} &\mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s) \\ &+ (\kappa\omega_1 + (1-\kappa)\omega_2)(s, \vartheta(s))]\,ds. \end{aligned}$$

Using Lemma 3.2, we can obtain the convexity of $\mathfrak{S}(\vartheta)$ and $(\kappa\omega_1 + (1-\kappa)\omega_2) \in \mathfrak{S}(\mathfrak{t})$ for $\mathfrak{t} \in \mathfrak{S}(\mathfrak{t})$. Hence, $(\kappa y_1 + (1-\kappa)y_2) \in \mathfrak{S}(\mathfrak{t})$ for $\mathfrak{t} \in (0, \mathfrak{b}_1]$. The result holds for $\mathfrak{t} \in (\mathfrak{a}_i, \mathfrak{b}_{i+1}]$ also. Thus, $\Phi_2(\mathfrak{t})$ is convex. By Lemma 3.2, obviously $\Phi_2(\mathfrak{t})$ is nonempty and has weakly compact values for all $\vartheta \in \mathfrak{X}$.

Step 3: Claim: $\Phi_2(\vartheta)(\vartheta)$ maps $\mathscr{B}_\mathfrak{l} \to \mathscr{B}_\mathfrak{l}$ in $\mathfrak{X}$.

For $\vartheta \in \mathscr{B}_\mathfrak{l}$ and $\mathfrak{t} \in (0, \mathfrak{b}_1]$ and by using Holder's inequality,

$$\begin{aligned} \mathbb{E}\left\|\Phi_2(\vartheta)\right\|^2_{\mathcal{PC}} &= \mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s) + \omega(s)]\,ds\right\|^2_{\mathcal{PC}} \\ &\leq \mathbb{M}^2\mathfrak{b}_1\left[\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}\right. \\ &\quad \left. + \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}[\mathfrak{r}_2(s) + \varsigma_\omega\mathfrak{l}]ds\right] \\ &\leq \mathbb{M}^2\mathfrak{b}_1\left[\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}\right. \\ &\quad \left. + \frac{\mathfrak{b}_1^{2\alpha-1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)} + \frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\varsigma_\omega\mathfrak{l}\right] \\ &\leq \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})} + \varsigma_\omega\mathfrak{l}\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\\
&\leq \tilde{\mathfrak{C}}_1.
\end{aligned}
$$

For $\mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i = 1, 2, \cdots, m$,

$$
\begin{aligned}
\mathbb{E}\|\Phi_2(\vartheta)\|^2_{\mathcal{PC}} &= \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s)+\omega(s)]\,ds\right\|^2_{\mathcal{PC}}\\
&\leq \frac{\mathbb{M}^2\mathrm{T}^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+\frac{\mathbb{M}^2\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\\
&\leq \tilde{\mathfrak{C}}_2.
\end{aligned}
$$

Hence, $\mathbb{E}\|\Phi_2(\vartheta)\|_{\mathcal{PC}} \leq \tilde{\mathfrak{C}}_3$ where $\tilde{\mathfrak{C}}_3 = \max\{\tilde{\mathfrak{C}}_1, \tilde{\mathfrak{C}}_2\}$.

Step 4: Claim: $\{\Phi_2(\vartheta) : \vartheta \in \mathscr{B}_{\mathfrak{l}}\}$ is equicontinuous.
For every $\vartheta \in \mathscr{B}_{\mathfrak{l}}$ and $\mathfrak{t} \in (0, \mathfrak{b}_1]$, when $\mathfrak{t}_1 = 0$, $0 < \mathfrak{t}_2 < \epsilon_0$ and ϵ_0 being sufficiently small, it follows that

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}} \leq \mathbb{E}\left\|\int_0^{\mathfrak{t}_2}\mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha-s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s)+\omega(s)]\,ds\right\|^2\\
&\leq 2\mathbb{M}^2\frac{\mathfrak{t}_2^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+2\mathbb{M}^2\frac{\mathfrak{t}_2^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\\
&\leq 2\mathbb{M}^2\frac{\epsilon_0^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\right]+2\mathbb{M}^2\frac{\epsilon_0^{2\alpha+1/2}}{2\alpha-1/2}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\\
&\to 0 \text{ as } \epsilon_0 \to 0 \text{ for } \vartheta \in \mathscr{B}_{\mathfrak{l}}.
\end{aligned}
$$

In a similar manner,
$\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}} \to 0$ as $\epsilon_0 \to 0$ for $\mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i = 1, 2, \cdots, m$. Likewise, for $\mathfrak{t} \in \mathscr{B}_{\mathfrak{l}}$. Let $0 < \mathfrak{t}_1 < \mathfrak{t}_2 < \mathfrak{b}_1$, there exist $\omega \in \mathfrak{S}(\vartheta)$ and for all $\vartheta \in (0, \mathfrak{b}_1]$.

$$
\begin{aligned}
&\mathbb{E}\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}\\
&= \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}_2}\mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s)+\omega(s)]\,ds\\
&\quad -\int_0^{\mathfrak{t}_1}\mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)[\mathscr{B}\varpi(s)+\omega(s)]\,ds\Bigg\|^2\\
&\leq \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}_1-\epsilon}\mathfrak{s}^{\alpha-1}\left[\mathbb{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathbb{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathscr{B}\varpi(s)+\omega(s)]\,ds\\
&\quad +\int_{\mathfrak{t}_1-\epsilon}^{\mathfrak{t}_1}\mathfrak{s}^{\alpha-1}\left[\mathbb{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathbb{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathscr{B}\varpi(s)+\omega(s)]\,ds\\
&\quad +\int_{\mathfrak{t}_2}^{\mathfrak{t}_1}\mathfrak{s}^{\alpha-1}\left[\mathbb{T}\left(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\right)-\mathbb{T}\left(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\right)\right][\mathscr{B}\varpi(s)+\omega(s)]\,ds\Bigg\|^2
\end{aligned}
$$

$$\leq 6\Bigg[\Bigg(\frac{(\mathfrak{t}_1-\epsilon)^{2\alpha}}{2\alpha-1}\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]+\frac{(\mathfrak{t}_1-\epsilon)^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg)$$
$$\times\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Big\|\mathbb{T}\Big(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Big)-\mathbb{T}\Big(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Big)\Big\|^2$$
$$+\frac{\epsilon^{2\alpha}}{2\alpha-1}\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]+\frac{\epsilon^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg)$$
$$\times\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Big\|\mathbb{T}\Big(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Big)-\mathbb{T}\Big(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Big)\Big\|^2+$$
$$\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha}}{2\alpha-1}\mathbb{M}^2\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]$$
$$+\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha-1/2}}{2\alpha-1/2}\mathbb{M}^2\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg].$$

In a similar way, for $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i=1,2,\cdots,m$ there exist $\omega\in\mathfrak{S}(\vartheta)$,

$$\mathbb{E}\,\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2_{\mathcal{PC}}$$
$$\leq 6\Bigg[\Bigg(\frac{(\mathfrak{t}_1-\epsilon-\mathfrak{a}_i)^{2\alpha}}{2\alpha-1}\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]$$
$$+\frac{(\mathfrak{t}_1-\epsilon-\mathfrak{a}_i)^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg)$$
$$\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Big\|\mathbb{T}\Big(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Big)-\mathbb{T}\Big(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Big)\Big\|^2+\frac{\epsilon^{2\alpha}}{2\alpha-1}\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}$$
$$+\varsigma_\omega\mathfrak{l}]+\frac{\epsilon^{2\alpha+1/2}}{2\alpha-1}\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg)$$
$$\times\sup_{s\in(0,\mathfrak{b}_1]}\mathbb{E}\Big\|\mathbb{T}\Big(\frac{\mathfrak{t}_2^\alpha-s^\alpha}{\alpha}\Big)-\mathbb{T}\Big(\frac{\mathfrak{t}_1^\alpha-s^\alpha}{\alpha}\Big)\Big\|^2$$
$$+\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha}}{2\alpha-1}\mathbb{M}^2\Big[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})}+\varsigma_\omega\mathfrak{l}\Big]$$
$$+\frac{(\mathfrak{t}_2-\mathfrak{t}_1)^{2\alpha-1/2}}{2\alpha-1/2}\mathbb{M}^2\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\Bigg].$$

For $\mathfrak{t}\in(0,\mathfrak{b}_1]$ and $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$ are independent of ϑ and approaches zero as $\mathfrak{t}_2\to\mathfrak{t}_1$ and $\epsilon\to 0$. Thus, $\mathbb{E}\,\|(\Phi_2\vartheta)(\mathfrak{t}_2)-(\Phi_2\vartheta)(\mathfrak{t}_1)\|^2\to 0$ as $\epsilon\to 0$ independent of $\vartheta\in\mathscr{B}_\mathfrak{l}$ which follows $\{\Phi_2(\vartheta),\vartheta\in\mathscr{B}_\mathfrak{l}\}$ is equicontinuous.

Step 5: Claim: $\Phi_2(\vartheta)$ is completely continuous.

Let ξ be a real number and $\mathfrak{t} \in \mathcal{J}$ be fixed with $0 < \xi < \mathfrak{t}$. The set $\pi(\mathfrak{t}) = \{\Phi_2(\mathfrak{t})\}$ is relatively compact. We may define,

$$(\Phi_2^{\xi}\vartheta)(\mathfrak{t}) = \begin{cases} \displaystyle\int_0^{\mathfrak{t}-\xi} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\left[\mathscr{B}\varpi(s)+\omega(s,\vartheta^{\xi}(s))\right]ds, & \mathfrak{t}\in[0,\mathfrak{b}_1], \\ 0, \qquad \mathfrak{t}\in(\mathfrak{b}_i,\mathfrak{a}_i], \quad i=1,2,\cdots,m, & \\ \displaystyle\int_{\mathfrak{a}_i}^{\mathfrak{t}-\xi} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\left[\mathscr{B}\varpi(s)+\omega(s,\vartheta^{\xi}(s))\right]ds, & \mathfrak{t}\in(\mathfrak{a}_i,\mathfrak{b}_{i+1}]. \end{cases}$$

Since $\mathbb{T}(\cdot)$ is compact, the set $\pi^{\epsilon}(\mathfrak{t}) = \{\Phi_2^{\epsilon}(\mathfrak{t})\}$ is relatively compact. Now, for each $0 < \xi < \mathfrak{t}$ and $\mathfrak{t} \in (\mathfrak{a}_i, \mathfrak{b}_{i+1}]$, $i = 1, 2, \cdots$, we obtain

$$\begin{aligned} \mathbb{E}\left\|\Phi_2(\mathfrak{t})-\phi_2^{\xi}(\mathfrak{t})\right\|^2 &\le 2\mathbb{E}\left\|\int_{\mathfrak{t}-\xi}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\mathscr{B}\varpi(s)ds\right\|^2 \\ &\quad +2\mathbb{E}\left\|\int_{\mathfrak{t}-\xi}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\left[\omega(s)-\omega(s,\vartheta^{\xi}(s))\right]ds\right\|^2 \end{aligned}$$

Thus, when $\xi \to 0$, the above inequality tends to zero. Thus, the set $\pi(\mathfrak{t})$ is relatively compact. Thus, from Step 4 and Arzela Ascoli theorem, Φ_2 is completely continuous.

Step 6: Claim: $\Phi_2(\vartheta)$ has a closed graph.

Let $\tilde{\vartheta}_n \to \tilde{\vartheta}_*$ in $\mathscr{X}$ and $\tilde{\mathrm{y}}_n \to \tilde{\mathrm{y}}_*$ in $\mathfrak{X}$. Our aim is to show that $\tilde{\mathrm{y}}_* \in \Phi_2(\tilde{\vartheta}_*)$. Let $\tilde{\mathrm{y}}_n \in \Phi_2(\tilde{\vartheta}_n)$ then $\exists\ \omega(s, \tilde{\vartheta}_n) \in \mathfrak{S}(\tilde{\vartheta}_n)$ with

$$\tilde{\mathrm{y}}_n(\mathfrak{t}) = \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)[\mathscr{B}\varpi(s)+\omega(s)]\,ds \tag{3.3}$$

By the use of (A2)(iii),

$$\{(\omega(s, \vartheta_n(s)))\}_{n\ge 1} \subseteq \mathcal{L}^2_{\mathfrak{I}}(\mathcal{J}, \mathscr{H}) \text{ is bounded.} \tag{3.4}$$

Let us consider the subsequence, $\omega(s, \tilde{\vartheta}_n(s)) \to \omega(s, \tilde{\vartheta}_*(s))$ weakly in $\mathcal{L}^2_{\mathfrak{I}}(\mathcal{J}, \mathscr{H})$.

By the condition of compactness of $\mathbb{T}(\mathfrak{t})$, using (A2) and equating (3.3) and (3.4),

$$\tilde{\vartheta}_n(\mathfrak{t}) \to \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^{\alpha}-s^{\alpha}}{\alpha}\right)\left[\mathscr{B}\varpi(s)+\omega(s,\tilde{\vartheta}_*(s))\right]ds \tag{3.5}$$

We may note that $\tilde{\mathrm{y}}_n \to \tilde{\mathrm{y}}_*$ in $\mathfrak{X}$ and $\omega(s, \tilde{\vartheta}_n) \in \mathfrak{S}(\tilde{\vartheta}_n)$. By lemma 3.2 and (3.5) gives $\omega(s, \tilde{\vartheta}_*) \in \mathfrak{S}(\tilde{\vartheta}_*)$. Hence, $\tilde{\vartheta}_* \in \Phi_2(\tilde{\vartheta}_*)$. Thus $\Phi_2(\vartheta)$ has a closed graph. In the view of Proposition 3.3 12(2) in [33], Φ_2 is upper semicontinuous.

Thus, $\Phi_2(\vartheta)$ has a closed graph.

Step 7: The operator inclusion $\pi\vartheta \in \Phi_1(\vartheta) + \Phi_2(\vartheta)$ has a solution for $\pi = 1$. By Theorem 2.1, it is sufficient to claim that there is no $\vartheta \in \mathfrak{X}$ that fulfils $\pi\vartheta \in$

$\Phi_1(\vartheta) + \Phi_2(\vartheta)$ for $\pi > 1$ and there exists $\omega \in \mathfrak{S}(\vartheta)$ with

$$\vartheta(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right) \\ [\mathscr{B}\varpi(s) + \omega(s) + \sigma\,(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s)))]\,ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\,(s, \vartheta(s), \delta(s))\,\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta(\mathfrak{b}_i^-)\right) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right) \\ \left[\mathscr{B}\varpi(s) + \omega(s) + \sigma\big(s, \vartheta(s), \vartheta(\mathfrak{e}(\vartheta(s), s))\big)\right]ds + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right) \\ \lambda\,(s, \vartheta(s), \delta(s))\,\tilde{\mathbb{N}}(ds, \mathrm{d}\delta),\ \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases}$$

For $\mathfrak{t} \in (0, \mathfrak{b}_1]$, $\exists\, \omega \in \mathfrak{S}(\vartheta)$,

$$\begin{aligned} &\mathbb{E}\|\vartheta(\mathfrak{t})\|^2_{\mathcal{PC}} \\ &= 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\|\mathscr{B}\|^2\mathfrak{b}_1 \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\varpi(s)\|^2 ds + \mathfrak{b}_1\mathbb{M}^2 \\ &\quad \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\left[\mathfrak{r}_1(s) + \varsigma_\omega\mathbb{E}\|\vartheta(s)\|^2\right] ds \\ &\quad + 2\mathfrak{b}_1\mathbb{M}^2 \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\,\|\sigma\,(s, \vartheta(s), \vartheta\,(\mathfrak{e}(\vartheta(s), s))) \\ &\quad -\sigma\,(s, 0, \vartheta\,(\mathfrak{e}(\vartheta(0), 0)))\|^2\, ds \\ &\quad + 2\mathfrak{b}_1\mathbb{M}^2 \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\,\|\sigma\,(s, 0, \vartheta\,(\mathfrak{e}(\vartheta(0), 0)))\|^2\, ds + \mathfrak{b}_1\mathbb{M}^2 \\ &\quad \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\left[\mathfrak{r}_2(s) + \varsigma_\omega\mathbb{E}\|\vartheta(s)\|^2\right] ds\Big] \\ &\le 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\left(\|\mathscr{B}\|^2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})} + 2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\sigma_0\right) \\ &\quad + \frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\left[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)} + \|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{R}^+)}\right] \\ &\quad + \mathfrak{b}_1\mathbb{M}^2\,[\varsigma_\omega + 2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_{\mathfrak{e}} + 1) + \varsigma_\lambda] \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2 ds\Big] \\ &\le \mathbb{S}_1 + \mathbb{S}_2 \int_0^{\mathfrak{t}} \mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2 ds. \end{aligned}$$

where

$$\mathbb{S}_1 = 5\Big[\mathbb{M}^2\mathbb{E}\|\vartheta_0\|^2 + \mathbb{M}^2\left(\|\mathscr{B}\|^2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})} + 2\frac{\mathfrak{b}_1^{2\alpha}}{2\alpha-1}\sigma_0\right)$$

$$+\frac{\mathbb{M}^2\mathfrak{b}_1^{2\alpha+1/2}}{2\alpha-1/2}\left[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}+\|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}\right]$$
$$\mathbb{S}_2=5\mathfrak{b}_1\mathbb{M}^2\left[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda\right].$$

By generalized Gronwall's inequality,

$$\mathbb{E}\|\vartheta(\mathfrak{t})\|^2_{\mathcal{PC}}\leq\mathbb{S}_1e^{\mathbb{S}_2\mathfrak{t}}=\mathfrak{d}_1. \tag{3.6}$$

For $\mathfrak{t}\in[\mathfrak{a}_i,\mathfrak{b}_{i+1}]$, $i=1,2,\cdots,m$ $\exists\,\omega\in\mathfrak{S}(\vartheta)$ such that

$$\begin{aligned}\mathbb{E}\,\|\vartheta(\mathfrak{t})\|^2 \ &\leq 5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i\mathbb{E}\|\vartheta(\mathfrak{t})\|^2+5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{W})}+2\sigma_0\right]\\&\quad+5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\left[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}+\|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}\right]\\&\quad+5\mathrm{T}\mathbb{M}^2\left[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda\right]\int_{\mathfrak{a}_i}^{\mathfrak{t}}\mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2ds.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\,\|\vartheta(\mathfrak{t})\|^2 \ &\leq \frac{1}{1-5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i}\Bigg[\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{W})}+2\sigma_0\right]\\&\quad+5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\left[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}+\|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}\right]\\&\quad+5\mathrm{T}\mathbb{M}^2\left[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda\right]\int_{\mathfrak{a}_i}^{\mathfrak{t}}\mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2ds\Bigg].\end{aligned}$$

Therefore,

$$\mathbb{E}\|\vartheta(\mathfrak{t})\|^2_{\mathcal{PC}}\leq\mathbb{S}_3+\mathbb{S}_4\int_{\mathfrak{a}_i}^{\mathfrak{t}}\mathfrak{s}^{2\alpha-2}\mathbb{E}\|\vartheta(s)\|^2ds.$$

where

$$\begin{aligned}\mathbb{S}_3&=\max_{i=1,2,\cdots,m}\frac{1}{1-5\mathbb{M}^2\mathrm{T}\mathfrak{f}_i}\Bigg[\mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\left[\|\mathscr{B}\|^2\|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{W})}+2\sigma_0\right]\\&\quad+5\mathbb{M}^2\frac{\mathrm{T}^{2\alpha+1/2}}{2\alpha-1/2}\left[\|\mathfrak{r}_1\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}+\|\mathfrak{r}_2\|_{\mathcal{L}^2_{\mathfrak{J}}(\mathcal{J},\mathscr{R}^+)}\right]\\\mathbb{S}_4&=5\mathrm{T}\mathbb{M}^2\left[\varsigma_\omega+2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)+\varsigma_\lambda\right]\end{aligned}$$

Using Gronwall's inequality,

$$\mathbb{E}\|\vartheta(\mathfrak{t})\|^2_{\mathcal{PC}}\leq\mathbb{S}_3e^{\mathbb{S}_4\mathfrak{t}}=\mathfrak{d}_2. \tag{3.7}$$

By summarizing (3.6) and (3.7),

$$\mathbb{E}\|\vartheta(\mathfrak{t})\|^2_{\mathcal{PC}} \leq \mathfrak{d} \text{ where } \mathfrak{d} = \max\{\mathfrak{d}_1, \mathfrak{d}_2\}.$$

The set $\chi_{\mathfrak{d}} = \{\vartheta \in \mathfrak{X}, \mathbb{E}\|\vartheta\|^2_{\mathcal{PC}} < \mathfrak{d} + 1\}$. Clearly, $\chi_{\mathfrak{d}}$ is an open subset of $\mathfrak{X}$. From $\chi_{\mathfrak{d}}$, there is no $\vartheta \in \mathfrak{X}$ that satisfies $\pi\vartheta \in \Phi_1(\vartheta) + \Phi_2(\vartheta)$ for $\pi > 1$. Hence, we conclude that the operator inclusion $\vartheta \in \Phi(\vartheta)$ has a mild solution in $\mathcal{J}$. □

4 Optimal Control

The existence of optimal control for the system (1.1) is investigated in this section using the Balder's theorem.

The Lagrange optimal control problem can be considered as follows: Considering $(\vartheta^0, \varpi^0) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad}$ such that

$$\mathfrak{I}(\vartheta^0, \varpi^0) \leq \mathfrak{I}(\vartheta^{\varpi}, \varpi), \text{ for all } (\vartheta, \varpi) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad},$$

where

$$\mathfrak{I}(\varpi) = \mathbb{E}\left\{\int_0^{\mathfrak{b}} L(t, \vartheta^{\varpi}(\mathfrak{t}), \varpi(\mathfrak{t}))dt\right\}$$

and ϑ^{ϖ} denotes the mild solution of (1.1) corresponding to the control $\varpi \in \mathscr{W}_{ad}$. Let us presume the following assumptions for the existence of solutions of the problem.

(A9) (i) The functional $L : \mathcal{J} \times \mathscr{H} \times \mathscr{W} \to \mathscr{R} \cup \{\infty\}$ is Borel measurable.
(ii) $L(\mathfrak{t}, ., .)$ is sequentially l.s.c on $\mathscr{H} \times \mathscr{W}$ $\forall$ $\mathfrak{t} \in \mathcal{J}$ a.e.
(iii) $L(\mathfrak{t}, \vartheta, .)$ is convex on $\mathscr{W}$ for each $\vartheta \in \mathscr{H}$ and almost all $\mathfrak{t} \in \mathcal{J}$.
(iv) There exist constants $\mathfrak{d} \geq 0$, $\mathfrak{f} > 0$ being constant, μ being non-negative and $\mu \in \mathcal{L}'(\mathcal{J}, \mathscr{R})$ with

$$L(\mathfrak{t}, \vartheta, \varpi) \geq \mu(\mathfrak{t}) + \mathfrak{d}\mathbb{E}\,\|\vartheta\|^p_{\mathscr{H}} + \mathfrak{f}\,\|\varpi\|^p_{\mathscr{W}}.$$

Theorem 4.1 *The Lagrange problem permits at least one optimum pair if assumptions (A1)–(A8) are true and $\mathscr{B}$ is a strongly continuous operator, (i.e.) $\exists$ an admissible control pair $(\vartheta^0, \varpi^0) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad} \ni$*

$$\begin{aligned}\mathfrak{I}(\vartheta^0, \varpi^0) &= \mathbb{E}\left(\int_0^{\mathfrak{b}} \mathfrak{I}(t, \vartheta^0(t), \varpi^0(t))dt\right)\\ &\leq \mathfrak{I}(\vartheta^{\varpi}, \varpi), \quad \forall\ (\vartheta^{\varpi}, \varpi) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad}.\end{aligned}$$

Proof If $\inf\{\mathfrak{I}(\vartheta^{\varpi}, \varpi) \mid (\vartheta^{\varpi}, \varpi) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad}\} = +\infty$, there is nothing to demonstrate. Without loss of generality, one can assume that $\inf\{\mathfrak{I}(\vartheta^{\varpi}, \varpi) \mid (\vartheta^{\varpi}, \varpi) \in \mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H})) \times \mathscr{W}_{ad}\} = \rho < +\infty$. By using (A8), we have $\rho > -\infty$. A minimizing sequence of feasible pair $\{(\vartheta^{\hat{m}}, \varpi^{\hat{m}})\} \subset \mathcal{P}_{ad}$

exists according to the definition of infimum, where $\mathcal{P}_{ad} = \{(\vartheta, \varpi) : \vartheta$ is a mild solution of system (1.1) corresponding to $\varpi \in \mathscr{W}_{ad}\} \ni \mathfrak{J}(\vartheta^{\hat{m}}, \varpi^{\hat{m}}) \to \rho$ as $\hat{m} \to +\infty$. Since $\{\varpi^{\hat{m}}\} \subseteq \mathscr{W}_{ad}, \hat{m} = 1, 2, \cdots \varpi^{\hat{m}}$ is a bounded subset of the separable reflexive Banach space $\mathcal{L}^p(\mathcal{J}, \mathscr{W})$, $\exists$ a subsequence, relabelled as $\varpi^{\hat{m}}$ and $\varpi^0 \in \mathcal{L}^p(\mathcal{J}, \mathscr{W})$ $\ni \varpi^{\hat{m}} \to \varpi^0$ weakly in $\mathcal{L}^p(\mathcal{J}, \mathscr{W})$. Since $\mathscr{W}_{ad}$ is closed and convex by Marzur lemma [34], $\varpi^0 \in \mathscr{W}_{ad}$. Let $\{\varpi^{\hat{m}}\}$ denotes the sequence of solutions of the system corresponding to $\varpi^{\hat{m}}$, ϑ^0 is the mild solution corresponding to ϖ^0. $\vartheta^{\hat{m}}, \vartheta^0$ satisfy the following integral equations:

$$\vartheta^{\hat{m}}(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{\hat{m}}(s) + \omega(s, \vartheta^{\hat{m}}(s))\right. \\ \left. +\sigma\left(s, \vartheta^{\hat{m}}(s), \vartheta^{\hat{m}}(\mathfrak{e}(\vartheta^{\hat{m}}(s), s))\right)\right]ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta^{\hat{m}}(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta^{\hat{m}}(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta^{\hat{m}}(\mathfrak{b}_i^-)\right) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{\hat{m}}(s) + \omega(s, \vartheta^{\hat{m}}(s))\right. \\ \left. +\sigma\left(s, \vartheta^{\hat{m}}(s), \vartheta^{\hat{m}}(\mathfrak{e}(\vartheta^{\hat{m}}(s), s))\right)\right]ds \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta^{\hat{m}}(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases} \tag{4.1}$$

and

$$\vartheta^{0}(\mathfrak{t}) = \begin{cases} \mathbb{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\vartheta_0 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{0}(s) + \omega(s, \vartheta^{0}(s))\right. \\ \left. +\sigma\left(s, \vartheta^{0}(s), \vartheta^{0}(\mathfrak{e}(\vartheta^{0}(s), s))\right)\right]ds \\ + \int_0^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta^{0}(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \quad \mathfrak{t} \in (0, \mathfrak{b}_1], \\ \mathfrak{h}_i\left(\mathfrak{t}, \vartheta^{0}(\mathfrak{b}_i^-)\right), \quad i = 1, 2, \cdots m, \\ \mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\mathfrak{h}_i\left(s, \vartheta^{0}(\mathfrak{b}_i^-)\right) + \int_{\mathfrak{a}_i}^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{0}(s) + \omega(s, \vartheta^{0}(s))\right. \\ \left. +\sigma\left(s, \vartheta^{0}(s), \vartheta^{0}(\mathfrak{e}(\vartheta^{0}(s), s))\right)\right]ds \\ + \int_{\mathfrak{a}_i}^{\mathfrak{t}}\int_{\mathcal{Z}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\lambda\left(s, \vartheta^{0}(s), \delta(s)\right)\tilde{\mathbb{N}}(ds, \mathrm{d}\delta), \mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]. \end{cases} \tag{4.2}$$

From the boundedness of $\{\varpi^{\hat{m}}\}$, $\{\varpi^0\}$ and theorem 3.1, it follows that there exist a positive number δ such that $\mathbb{E}\|\vartheta^{\hat{m}}\|^2 \leq \delta$, $\mathbb{E}\|\vartheta^0\|^2 \leq \delta$. For $\mathfrak{t} \in [0, \mathfrak{b}_1]$,

$$\mathbb{E}\left\|\vartheta^{\hat{m}}(\mathfrak{t}) - \vartheta^0(\mathfrak{t})\right\|^2 \leq 4\left[\mathscr{G}_1 + \mathscr{G}_2 + \mathscr{G}_3 + \mathscr{G}_4\right]$$

where

$$
\begin{aligned}
\mathscr{G}_1 &= \mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{\hat{m}}(s) - \mathscr{B}\varpi^0(s)\right]\mathrm{d}s\right\|^2 \\
&\le \mathbb{M}^2\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\int_0^{\mathfrak{t}}\mathbb{E}\left\|\mathscr{B}\varpi^{\hat{m}}(s) - \mathscr{B}\varpi^0(s)\right\|^2 \\
\mathscr{G}_2 &= \mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\omega\left(s, \vartheta^{\hat{m}}(s)\right) - \omega\left(s, \vartheta^0(s)\right)\right]\mathrm{d}s\right\|^2 \\
&\le \mathbb{M}^2\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\int_0^{\mathfrak{t}}\mathbb{E}\left\|\omega\left(s, \vartheta^{\hat{m}}(s)\right) - \omega\left(s, \vartheta^0(s)\right)\right\|^2\mathrm{d}s \\
&\le \mathbb{M}^2\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\mathbb{M}_\omega\int_0^{\mathfrak{t}}\mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2\mathrm{d}s \\
\mathscr{G}_3 &= \mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\sigma\left(s, \vartheta^{\hat{m}}(s), \vartheta^{\hat{m}}(\mathfrak{e}(\vartheta^{\hat{m}}(s), s))\right)\right.\right. \\
&\qquad \left.\left. -\sigma\left(s, \vartheta^0(s), \vartheta^0(\mathfrak{e}(\vartheta^0(s), s))\right)\right]\mathrm{d}s\right\|^2 \\
&\le \mathbb{M}^2\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\left[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)\right]\int_0^{\mathfrak{t}}\mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2\mathrm{d}s \\
\mathscr{G}_4 &\le \mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\lambda\left(s, \vartheta^{\hat{m}}(s), \delta(s)\right)\right.\right. \\
&\qquad \left.\left. -\lambda\left(s, \vartheta^0(s), \delta(s)\right)\right]\tilde{\mathbb{N}}(\mathrm{d}s, \mathrm{d}\delta)\right\|^2 \\
&\le \mathbb{M}^2\frac{\mathfrak{b}_1^{2\alpha-1}}{2\alpha-1}\mathbb{M}_\lambda\int_0^{\mathfrak{t}}\mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2\mathrm{d}s.
\end{aligned}
$$

For $\mathfrak{t} \in (\mathfrak{b}_i, \mathfrak{a}_i]$,

$$
\begin{aligned}
\mathbb{E}\left\|\vartheta^{\hat{m}}(\mathfrak{t}) - \vartheta^0(\mathfrak{t})\right\|^2 &\le \mathbb{E}\left\|\mathfrak{h}_i\left(\mathfrak{t}, \vartheta^{\hat{m}}(\mathfrak{b}_i^-)\right) - \mathfrak{h}_i\left(\mathfrak{t}, \vartheta^0(\mathfrak{b}_i^-)\right)\right\|^2 \\
&\le \mathfrak{f}_i\mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2.
\end{aligned}
$$

For $\mathfrak{t} \in [\mathfrak{a}_i, \mathfrak{b}_{i+1}]$,

$$
\mathbb{E}\left\|\vartheta^{\hat{m}}(\mathfrak{t}) - \vartheta^0(\mathfrak{t})\right\|^2 \le 4\left[\mathscr{G}_1 + \mathscr{G}_2 + \mathscr{G}_3 + \mathscr{G}_4 + \mathscr{G}_5\right],
$$

where

$$
\begin{aligned}
\mathscr{G}_1 &\le \mathbb{E}\left\|\mathcal{T}\left(\frac{\mathfrak{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathfrak{h}_i(s, \vartheta^{\hat{m}}(\mathfrak{b}_i^-)) - \mathfrak{h}_i(s, \vartheta^0(\mathfrak{b}_i^-))\right]\right\|^2 \\
&\le \mathbb{M}^2\mathfrak{f}_i\mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{G}_2 &\leq \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\left[\mathscr{B}\varpi^{\hat{m}}(s) - \mathscr{B}\varpi^0(s)\right]ds\right\|^2 \\
&\leq \mathbb{M}^2\frac{\mathrm{T}^{2\alpha}}{2\alpha-1}\int_0^{\mathrm{t}} \mathbb{E}\left\|\mathscr{B}\varpi^{\hat{m}}(s) - \mathscr{B}\varpi^0(s)\right\|^2 ds \\
\mathscr{G}_3 &\leq \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\left[\omega\left(s, \vartheta^{\hat{m}}(s)\right) - \omega\left(s, \vartheta^0(s)\right)\right]ds\right\|^2 \\
&\leq \mathbb{M}^2\frac{\mathrm{T}^{2\alpha-1}}{2\alpha-1}\mathbb{M}_\omega\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2 ds \\
\mathscr{G}_4 &\leq \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\left[\sigma\left(s, \vartheta^{\hat{m}}(s), \vartheta^{\hat{m}}(\mathfrak{e}(\vartheta^{\hat{m}}(s), s))\right)\right.\right. \\
&\qquad \left.\left. -\sigma\left(s, \vartheta^0(s), \vartheta^0(\mathfrak{e}(\vartheta^0(s), s))\right)\right]ds\right\|^2 \\
&\leq \mathbb{M}^2\frac{\mathrm{T}^{2\alpha-1}}{2\alpha-1}\left[2\mathbb{M}_\sigma\mathfrak{l}(\mathbb{M}_\mathfrak{e}+1)\right]\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2 ds \\
\mathscr{G}_5 &\leq \mathbb{E}\left\|\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathrm{t}^\alpha - s^\alpha}{\alpha}\right)\left[\lambda\left(s, \vartheta^{\hat{m}}(s), \delta(s)\right)\right.\right. \\
&\qquad \left.\left. -\lambda\left(s, \vartheta^0(s), \delta(s)\right)\right]\tilde{\mathbb{N}}(ds, \mathrm{d}\delta)\right\|^2 \\
&\leq \mathbb{M}^2\frac{\mathrm{T}^{2\alpha-1}}{2\alpha-1}\mathbb{M}_\lambda\int_{\mathfrak{a}_i}^{\mathrm{t}} \mathbb{E}\left\|\vartheta^{\hat{m}}(s) - \vartheta^0(s)\right\|^2 ds.
\end{aligned}
$$

which implies,

$$\sup_{\mathrm{t}\in\mathcal{J}} \mathbb{E}\left\|\vartheta^{\hat{m}}(\mathrm{t}) - \vartheta^0(\mathrm{t})\right\|^2 \leq \mathbb{N}^*\|\mathscr{B}\varpi^{\hat{m}} - \mathscr{B}\varpi^0\|^2_{\mathcal{L}_\mathfrak{p}(\mathcal{J},\mathscr{W})}.$$

Since $\mathscr{B}$ is strongly continuous, we have

$$\|\mathscr{B}\varpi^{\hat{m}} - \mathscr{B}\varpi^0\|^2_{\mathcal{L}_\mathfrak{p}(\mathcal{J},\mathscr{W})} \to 0 \ \ as \ \ \hat{m} \to \infty.$$

Thus, we have

$$\mathbb{E}\left\|\vartheta^{\hat{m}} - \vartheta^0\right\|^2 \to 0 \ \ as \ \ \hat{m} \to \infty.$$

This yields $\vartheta^{\hat{m}} \xrightarrow{w} \vartheta^0$ in $\mathcal{PC}(\mathcal{J}, \mathcal{L}_2(\Omega, \mathscr{H}))$ as $\hat{m} \to \infty$. (A8) implies the assumptions of Balder [35]. In the light of the above, $(\vartheta, \varpi) \to \mathbb{E}\left(\int_0^{\mathfrak{b}} L(t, \vartheta(t), \varpi(t))\right)$ is sequentially lower semicontinuous in the weak topology of $\mathcal{L}^p(\mathcal{J}, \mathscr{W}) \subset \mathcal{L}^1(\mathcal{J}, \mathscr{W})$ and strong topology of $\mathcal{L}^1(\mathcal{J}, \mathscr{W})$. Consequently, $\mathfrak{I}$ is weakly lower semicontinuous on $\mathcal{L}^p(\mathcal{J}, \mathscr{W})$ and with (A8)(iv), $\mathfrak{J} > -\infty$, $\mathfrak{J}$ attains its infimum at $\varpi^0 \in \mathscr{W}_{ad}$, (i.e.)

$$\rho = \lim_{\hat{m}\to\infty} \mathbb{E}\left(\int_0^{\mathfrak{b}} L(t, \vartheta^{\hat{m}}(t), \varpi^{\hat{m}}(t))dt\right)$$
$$\geq \mathbb{E}\left(\int_0^{\mathfrak{b}} L(t, \vartheta^{\hat{m}}(t), \varpi^{\hat{m}}(t))dt\right) = \mathfrak{J}(\vartheta^0, \varpi^0) \geq \rho$$

□

5 Illustration

Consider the conformable stochastic differential inclusion with impulses as follows:

$$\begin{aligned}
&\mathscr{D}^\alpha \vartheta(\mathrm{t}, s) \in \frac{\partial^2}{\partial s^2}\vartheta(\mathrm{t}, s) + \int_0^1 \zeta(\mathrm{t}, \eta)\varpi(\eta, s)\mathrm{d}\eta \\
&\quad +\partial\Upsilon\left(\frac{\mathrm{t}^2 + |\vartheta(\mathrm{t}, s)|^2}{9}\right) + \frac{1}{15}\left[\frac{\mathrm{t}\vartheta(\mathrm{t}, s)}{2} + \vartheta\left(\mathrm{t}, \sqrt{7}\sin\mathrm{t}|\vartheta(\mathrm{t}, s)|/3\right)\right] \\
&\quad + \int_{\mathcal{Z}}\left[\frac{\cos|s\delta|}{5} + \frac{(\mathrm{t}^2+1)\,|\vartheta(\mathrm{t}, s)|}{5 + |\vartheta(\mathrm{t}, s)|}\right]\tilde{\mathbb{N}}(\mathrm{dt}, vdelta),\ \ \mathrm{t} \in (0, 0.25] \cup (0.5, 1] \\
&\quad \text{and } s \in [0, \pi], \\
&\quad \vartheta(\mathrm{t}, \omega) = \frac{(\sin|s| + \mathrm{t}^2)\left|\vartheta^-_{0.25}(\mathrm{t}, s)\right|}{12} \quad \mathrm{t} \in (0.25, 0.5), \\
&\quad \vartheta(\mathrm{t}, 0) = \vartheta(\mathrm{t}, \pi) = 0, \quad \mathrm{t} \in (0, 1) \times [0, \pi] \\
&\quad \vartheta(0, s) = \vartheta_0(s)\ s \in [0, \pi].
\end{aligned} \tag{5.1}$$

Let $\alpha = 0.9$, $\mathscr{B} : [0, \pi] \times [0, 1] \to \mathbb{R}$ is a continuous function. Let $\mathscr{H} = \mathscr{U} = \mathcal{L}^2([0, \pi])$, $\omega_0(s) \in \mathscr{H}$ and let $\mathfrak{A} : \mathcal{L}^2([0, \pi]), \omega_0(s) \in \mathscr{H}$ and let $\mathfrak{A} : \mathcal{L}^2([0, \pi]) \to \mathcal{L}^2([0, \pi])$. $\mathfrak{A}\omega = \Delta\omega$, $\omega \in D(\mathfrak{A})$, we define $D(\mathfrak{A}) = \{\mathrm{y} \in \mathscr{H} : \mathrm{y}, \frac{dy}{ds}$ are absolutely continuous and $\frac{d^2\mathrm{y}}{ds^2} \in \mathscr{H},\ \mathrm{y}(0) = y(\pi) = 0\}$. $\mathfrak{A}$ generates a compact semigroup $\{\mathbb{T}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ is analytic and self-adjoint. Let us take $\mathscr{B}\varpi(\mathrm{t}, s) = \int_0^1 \zeta(\mathrm{t}, \eta)\varpi(\eta, s)\mathrm{d}\eta$ where ζ is continuous. Moreover, $\mathfrak{A}$ has the discrete spectrum, and there exist eigenvalues $-n^2$, $n \in \mathscr{R}$ with the orthogonal eigenvectors $\Psi_n(\mathrm{y}) = \sqrt{\frac{2}{\pi}}\sin(n\mathrm{y})$ then $\mathfrak{A}\mathrm{y} = \sum_{n=1}^{\infty} -n^2\langle \mathrm{y}, \Psi_n\rangle\Psi_n$. Also, $\mathbb{T}(\mathrm{t})\mathrm{y} = \sum_{n=1}^{\infty} e^{-n^2\mathrm{t}}\langle \mathrm{y}, \Psi_n\rangle\Psi_n, \mathrm{y} \in \mathscr{H}$ and for all $\mathrm{t} > 0$. In addition, $\|\mathbb{T}(\mathrm{t})\| \leq 1 = \mathbb{M}$, $\{\mathbb{T}(\mathrm{t})\}_{\mathrm{t}\geq 0}$ is uniformly bounded compact semigroup, (A1) holds and $\mathcal{R}(\lambda, \mathfrak{A}) = (\lambda - \mathfrak{A})^{-1}$ is a compact operator, for every $\lambda \in \eta(\mathfrak{A})$. Define the nonlinear functions as follows:

$$\omega(\mathrm{t}, \vartheta(\mathrm{t}, s)) \in \partial\Upsilon(\mathrm{t}, \vartheta(\mathrm{t}, s)) \text{ and } \omega(\mathrm{t}, \vartheta(\mathrm{t}, s)) = \frac{\mathrm{t}^2 + |\vartheta(\mathrm{t}, s)|^2}{9},$$

$$\sigma\left(\mathfrak{t}, \vartheta(\mathfrak{t}, s), \vartheta\left(\mathfrak{e}(\vartheta(\mathfrak{t}, s))\right), \mathfrak{t}\right) = \frac{1}{15}\left[\frac{\mathfrak{t}\vartheta(\mathfrak{t}, s)}{2} + \vartheta\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta(\mathfrak{t}, s)|/3\right)\right],$$

$$\int_{\mathcal{Z}} \lambda\left(\mathfrak{t}, \vartheta(\mathfrak{t}, s), \delta\right) \tilde{\mathbb{N}}(d\mathfrak{t}, d\delta) = \int_{\mathcal{Z}} \left[\frac{\cos|s\delta|}{5} + \frac{(\mathfrak{t}^2+1)\,|\vartheta(\mathfrak{t}, s)|}{5+|\vartheta(\mathfrak{t}, s)|}\right] \tilde{\mathbb{N}}(d\mathfrak{t}, d\delta),$$

$$\mathfrak{e}\left(\vartheta(\mathfrak{t}, s), \mathfrak{t}\right) = \sqrt{7}\sin\mathfrak{t}|\vartheta(\mathfrak{t}, s)|/3,$$

$$\mathfrak{h}\left(\mathfrak{t}, \vartheta(\mathfrak{b}_i^-, s)\right) = \frac{(\sin|s| + \mathfrak{t}^2)\left|\vartheta^-_{0.25}(\mathfrak{t}, s)\right|}{12}.$$

Describe the ball for $\tilde{\mathfrak{r}} > 0$, $\mathscr{B}_{\tilde{\mathfrak{r}}} = \{\vartheta \in \mathscr{H} : \mathbb{E}\|\vartheta\|^2 \le \tilde{\mathfrak{r}}\}$. Consider for $\vartheta_1, \vartheta_2 \in \mathscr{B}_{\tilde{\mathfrak{r}}}$,

$$\begin{aligned}\mathbb{E}|\mathfrak{e}\left(\vartheta_1(\mathfrak{t}, s), \mathfrak{t}\right) - \mathfrak{e}\left(\vartheta_2(\mathfrak{t}, s), \mathfrak{t}\right)|^2 &= \mathbb{E}\left\|\frac{\sqrt{7}\sin\mathfrak{t}\mid\vartheta_1(\mathfrak{t}, s)}{3} - \frac{\sqrt{7}\sin\mathfrak{t}\mid\vartheta_2(\mathfrak{t}, s)}{3}\right\| \\ &\le \frac{7}{9}\mathbb{E}\left\|\vartheta_1(\mathfrak{t}, s) - \vartheta_2(\mathfrak{t}, s)\right\|^2.\end{aligned}$$

Also, $\mathfrak{e}(\cdot, 0) = 0$ and hence, the nonlinear function $\mathfrak{e}$ satisfies (A4) with $\mathbb{M}_{\mathfrak{e}} = 7/9$. Now,

$$\begin{aligned}&\mathbb{E}\left\|\sigma\left(\mathfrak{t}, \vartheta_1(\mathfrak{t}, s), \vartheta_1\left(\mathfrak{e}(\vartheta_1(\mathfrak{t}, s))\right), \mathfrak{t}\right) - \sigma\left(\mathfrak{t}, \vartheta_2(\mathfrak{t}, s), \vartheta_2\left(\mathfrak{e}(\vartheta_2(\mathfrak{t}, s))\right), \mathfrak{t}\right)\right\|^2 \\ &\quad\le \frac{1}{75}\mathbb{E}\left\|\frac{\mathfrak{t}\vartheta_1(\mathfrak{t}, s)}{2} + \vartheta_1\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_1(\mathfrak{t}, s)|/3\right) - \frac{\mathfrak{t}\vartheta_2(\mathfrak{t}, s)}{2}\right. \\ &\qquad+\vartheta_2\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_2(\mathfrak{t}, s)|/3\right) \\ &\qquad\left.+\vartheta_1\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_2(\mathfrak{t}, s)|/3\right) - \vartheta_1\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_2(\mathfrak{t}, s)|/3\right)\right\|^2 \\ &\quad\le \frac{1}{75}\left[\mathbb{E}\left\|\vartheta_1(\mathfrak{t}, s) - \vartheta_2(\mathfrak{t}, s)\right\|^2 + \mathbb{E}\left\|\vartheta_1\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_2(\mathfrak{t}, s)|/3\right)\right.\right. \\ &\qquad\left.\left.-\vartheta_2\left(\mathfrak{t}, \sqrt{7}\sin\mathfrak{t}|\vartheta_2(\mathfrak{t}, s)|/3\right)\right\|^2\right].\end{aligned}$$

Hence, the nonlinear function σ satisfies (A3) with $\mathbb{M}_\sigma = \frac{1}{75}$. Also,

$$\mathbb{E}\left\|\omega(\mathfrak{t}, \vartheta(\mathfrak{t}, s))\right\|^2 \le \mathbb{E}\left\|\frac{\mathfrak{t}^2 + |\vartheta(\mathfrak{t}, s)|}{9}\right\|^2 \le \frac{1}{81}\left[1 + \mathbb{E}\|\vartheta(\mathfrak{t}, s)\|^2\right]$$

and

$$\begin{aligned}\mathbb{E}\left\|\omega(\mathfrak{t}, \vartheta_1(\mathfrak{t}, s)) - \omega(\mathfrak{t}, \vartheta_2(\mathfrak{t}, s))\right\|^2 &\le \mathbb{E}\left\|\frac{\mathfrak{t}^2 + |\vartheta_1(\mathfrak{t}, s)|}{9} - \frac{\mathfrak{t}^2 + |\vartheta_2(\mathfrak{t}, s)|}{9}\right\|^2 \\ &\le \frac{2\tilde{\mathfrak{r}}}{81}\mathbb{E}\left\|\vartheta_1(\mathfrak{t}, s) - \vartheta_2(\mathfrak{t}, s)\right\|^2.\end{aligned}$$

Hence, ω satisfies (A2) with $\varsigma_\omega = \frac{2}{81}$.

$$\begin{aligned}
&\int_{\mathcal{Z}} \mathbb{E}\left\|\lambda\left(\mathfrak{t}, \vartheta_1(\mathfrak{t}, s), \delta\right)-\lambda\left(\mathfrak{t}, \vartheta_2(\mathfrak{t}, s), \delta\right)\right\|^2 \mathrm{h}(d\delta) \\
&\leq \int_{\mathcal{Z}} \mathbb{E}\left\|\frac{\cos|s\delta|}{5}+\frac{(\mathfrak{t}^2+1)\,|\vartheta(\mathfrak{t}, s)|}{5+|\vartheta(\mathfrak{t}, s)|}-\frac{\cos|s\delta|}{5}+\frac{(\mathfrak{t}^2+1)\,|\vartheta(\mathfrak{t}, s)|}{5+|\vartheta(\mathfrak{t}, s)|}\right\|^2 \mathrm{h}(d\delta) \\
&\leq \frac{2}{25}\mathbb{E}\left\|\vartheta_1(\mathfrak{t}, s)-\vartheta_2(\mathfrak{t}, s)\right\|^2 .
\end{aligned}$$

Hence, λ satisfies (A5) with $\mathbb{M}_\lambda = \frac{2}{25}$. and

$$\begin{aligned}
\int_{\mathcal{Z}} \mathbb{E}\left\|\lambda\left(\mathfrak{t}, \vartheta_1(\mathfrak{t}, s), \delta\right)\right\|^2 \mathrm{h}(d\delta) &\leq \int_{\mathcal{Z}} \mathbb{E}\left\|\frac{\cos|s\delta|}{5}+\frac{(\mathfrak{t}^2+1)\,|\vartheta(\mathfrak{t}, s)|}{5+|\vartheta(\mathfrak{t}, s)|}\right\|^2 \mathrm{h}(d\delta) \\
&\leq \frac{2}{25}\int_{\mathcal{Z}} \mathbb{E}(1+4\|\vartheta\|^2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}\left\|\mathfrak{h}\left(\mathfrak{t}, \vartheta_1(\mathfrak{b}_i^-, s)\right)-\mathfrak{h}\left(\mathfrak{t}, \vartheta_2(\mathfrak{b}_i^-, s)\right)\right\|^2 \\
&\quad=\mathbb{E}\left\|\frac{(\sin|s|+\mathfrak{t}^2)\left|\vartheta_{0.25}^-(\mathfrak{t}, s)\right|}{12}\right. \\
&\qquad\left.-\frac{(\sin|s|+\mathfrak{t}^2)\left|\vartheta_{0.25}^-(\mathfrak{t}, s)\right|}{12}\right\|^2 \\
&\quad\leq \frac{1}{12}\mathbb{E}\left\|\vartheta_1(0.25, s)-\vartheta_2(0.25, s)\right\|^2 ,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left\|\mathfrak{h}\left(\mathfrak{t}, \vartheta_1(\mathfrak{b}_i^-, s)\right)\right\|^2 &= \mathbb{E}\left\|\frac{(\sin|s|+\mathfrak{t}^2)\left|\vartheta_{0.25}^-(\mathfrak{t}, s)\right|}{12}\right\|^2 \\
&\leq \frac{1}{12}\left\|\mathbb{E}(1+1/4)\vartheta_{0.25}^-(\mathfrak{t}, s)\right\|^2 \\
&\leq 0.1\left\|\vartheta\right\|^2 .
\end{aligned}$$

Hence, $\mathfrak{f}_i = \frac{1}{12}$, $\mathbb{M}_\lambda = \frac{2}{25}$, $\mathbb{M}_\lambda = \frac{2}{25}$ and $\mathbb{M}_{\mathfrak{e}} = 7/9$. Now substituting the values in (3.2), we get

$$\begin{aligned}
\mathfrak{C} &= 0.25+\frac{2}{0.8}\left[0.2+0.08\right] \\
&= 0.25+0.7 \\
&= 0.95 < 1,
\end{aligned}$$

one can get $\mathfrak{C} < 1$. Hence, it satisfies the conditions of Theorem 3.1, and hence, there is at least one mild solution for the system (5.1).

Next, define the admissible controls set $\mathscr{W}_{ad} = \left\{\varpi \in \mathscr{W} : \|\varpi\|^2_{\mathcal{L}^2_{\mathfrak{F}}(\mathcal{J},\mathscr{W})} < 1\right\}$. Consider the following cost

$$\mathfrak{I}(\varpi) = \mathbb{E}\left\{\int_0^{\mathfrak{b}} L(t, \vartheta^{\varpi}(\mathfrak{t}), \varpi(\mathfrak{t}))\mathrm{d}t\right\}$$

where

$$\mathfrak{I}(\mathfrak{t}, \vartheta(\mathfrak{t}), \varpi(\mathfrak{t}))(w) = \int_0^1\int_0^1 |\vartheta(\mathfrak{t}, w)|^2 \,\mathrm{d}w\mathrm{d}\mathfrak{t} + \int_0^1\int_0^1 |\varpi(\mathfrak{t}, w)|^2 \,\mathrm{d}w\mathrm{d}\mathfrak{t},$$

and it is easy to see that the hypotheses of Theorem 4.1 are satisfied. Therefore, there exists at least one optimal pair for the problem (5.1).

Remark: Comparisons:

(i) The CFD behaves well in the product rule and chain rule while complicated formulas appear in case of usual fractional calculus.
(ii) The CFD of a constant function is zero while it is not the case for Riemann fractional derivatives.
(iii) Mittag–Leffler functions play an important rule in fractional calculus as a generalization to exponential functions while the fractional exponential function $f(t) = e^{\frac{t^\alpha}{\alpha}}$ appears in case of conformable fractional calculus.
(iv) Conformable fractional derivatives, conformable chain rule, conformable integration by parts, conformable Gronwall's inequality, conformable exponential function, conformable Laplace transform and so forth, all tend to the corresponding ones in usual calculus.
(v) In case of usual calculus, there are some functions that do not have Taylor power series representations about certain points but in the theory of conformable fractional they do have.

6 Conclusion

This manuscript presents a mild solution for the solvability and optimal control of a conformable fractional stochastic differential inclusion with Clarke subdifferential and deviated argument. The proposed conformable fractional impulsive inclusion systems solvability in Hilbert space is established by employing fractional calculus, multivalued analysis, stochastic analysis, semigroup theory and a multivalued fixed point theorem. Furthermore, under some suitable assumptions, the existence of optimal control is derived by employing Balder's theorem. Results for the well-posedness of conformable fractional stochastic differential equations with Levy noise and the averaging principle can be extended to the obtained findings.

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