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Fractional stochastic differential equations and degree theory: a unified perspective

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Abstract

This research focuses on the existence and uniqueness of solutions for a novel class of tempered φ -Caputo stochastic fractional differential equations. The analysis is grounded in foundational concepts from fractional calculus and topological degree theory. We establish the existence of solutions by employing the topological degree approach for condensing maps. Furthermore, the uniqueness of solutions is derived using the Banach fixed point theorem. To validate our theoretical results, illustrative examples are presented.

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1 Introduction

The fundamental concept in the study of infinite-dimensional (*ID*) analysis and its applications to differential equations (*DEs*) is compactness. However, many important operators encountered in functional analysis and *DEs* are not compact, particularly in *ID* Banach spaces (*BSs*). To address this limitation, the concept of measures of noncompactness (*MNC*) has been introduced and extensively developed. These measures quantify the “degree” to which a bounded set fails to be relatively compact, offering a powerful tool for studying fixed point (*FP*) theory, operator equations, and various forms of differential and integral equations.

The relevance of *MNC* is particularly significant in the study of stochastic *DEs*, which model dynamical systems under the influence of randomness. Stochastic *DEs* are widely used in fields ranging from physics and biology to finance and engineering. The analysis of such equations, especially in the *ID* setting or in the presence of memory effects, impulses, or fractional dynamics, often leads to operator equations that are not compact. In such cases, traditional *FP* theorems based on compactness fail to apply, and *MNC* serve as a crucial alternative. Igor Podlubny in [1] and Anatoly A. Kilbas et al., in [2] presents essential ideas that encourage an introductory understanding and real-world use of fractional derivative (*FDs*) and fractional *DEs*. Bernt Oksendal, Lawrence C. Evans, and Mao Xuerong in [3–5] provide core principles that support basic exploration and application

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aspects of stochastic *DEs*. Lakshmikantham V et al., in [6] offer readers the fundamental concepts necessary for a foundational study and practical application of impulsive differential equations. As a result *SFIDEs* can simulate complicated dynamical systems that are sensitive to randomness, memory effects, and abrupt changes in state, their research has recently attracted a lot of attention. These equations are particularly useful in describing real-world processes in physics, engineering, finance, and biology, where both stochastic perturbations and impulsive effects play a crucial role [7–11]. For instance, in [12], the authors investigated a stochastic prey-predator model both numerically and analytically which provided comparative simulations based on physical scenarios that highlights the effectiveness of the approach and in [13], the authors analyzed the Sasa-Satsuma equation using planar dynamical theory and the beta differential operator. They explored chaos and phase transitions, a periodic term $\cos(\omega t)$ was introduced, and investigated the effect of the beta differential operator on various solitons (bright, dark, W-shaped, breather).

Furthermore, the inclusion of tempered φ ($T\varphi$) Caputo derivative (*CD*) in modeling *DEs* has enhanced the accuracy of systems where memory effects exhibit exponential fading, making it highly relevant in fields like anomalous diffusion [14], finance [15], and biological systems [16]. Unlike the standard *CD*, the tempered version incorporates an exponential tempering parameter, which allows for better physical interpretation and avoids the infinite memory drawback inherent in classical fractional operators. Especially, in the study of *SFIDEs* with nonlocal conditions, incorporating $T\varphi$ -*CD* provides a more realistic framework to capture systems influenced by both history and abrupt changes, see [17–21].

To rigorously analyze such complex systems, topological degree theory (*TDT*) offers a significant advantage. It is particularly effective in dealing with the nonlinear, nonlocal, and impulsive nature of the problem without imposing stringent regularity or compactness assumptions [22–24]. The theory is well-suited to fractional differential operators, including tempered ones, as it can handle operators that are non-compact or lack strict continuity, which often arise in *ID* fractional spaces. Moreover, it provides a robust framework for proving existence and under certain contractive conditions, uniqueness of solutions, particularly when dealing with systems perturbed by Gaussian noise. A fundamental challenge in analyzing such equations is the presence of nonlocal conditions, which extend classical initial conditions to incorporate global information about the system.

The synergy between the $T\varphi$ -*CD* and *TDT* enables the treatment of a wide class of fractional stochastic models that would be otherwise intractable using classical methods. This makes it a powerful approach for establishing the existence and uniqueness (*EU*) of solutions in systems characterized by memory, randomness, impulse, and nonlocal dependencies.

In this work, we investigate the *EU* of solution for a class of *SFIDEs* involving the $T\varphi$ -*CD* and nonlocal conditions in a *BS* setting. The presence of Gaussian noise introduces stochastic fluctuations that reflect the inherent uncertainties in practical models. Our approach is grounded in *TDT*, which provides a powerful framework for establishing the existence of solutions to nonlinear *DEs*. In [25], Mahmoud et al. analyze the *EU* of an impulsive fractional stochastic *DE*, based on approximation scheme of Carathéodory type. Omar et al. in [26] explore the φ -*CD* in the stochastic sense. By extending previous results in impulsive and fractional *DEs*, this study enhances the present literature on stochastic fractional dynamics and offers new insights into their mathematical properties and applications.

Motivated by the above works, we consider *SFIDEs* with nonlocal conditions:

$$\begin{cases} {}^{TC}_{0+}D_{\varphi(t)}^{\alpha,\mu}\zeta(t) = J_1(t, \zeta(t)) + J_2(t, \zeta(t))\frac{d\eta(t)}{dt}, t \in \mathfrak{S} \setminus \{t_1, t_2, t_3, \dots, t_m\}, \\ \Delta\zeta(t_i) = I_i(\zeta(t_i)), \quad i = 1, 2, 3, \dots, m, \\ \zeta(0) + \eta(x) = \zeta_0, \end{cases} \quad (1)$$

where $\frac{1}{2} < \alpha < 1$, $\mu > 0$, $\mathfrak{S} := [0, T]$, $J_j : \mathfrak{S} \times \Upsilon \rightarrow \Upsilon$ ($j = 1, 2$), are measurable and continuous, $\zeta_0 \in \Upsilon$, is measurable and $\eta : \chi \rightarrow \Upsilon$ is a continuous map (*CM*), where $\chi = PC(\mathfrak{S}, \Upsilon)$. $I_i : \Upsilon \rightarrow \Upsilon$ is a *CM* and t_i satisfies, $0 = t_{i-1} < t_i < \dots < t_{i+1} = T$. ${}^{TC}_{0+}D_{\varphi(t)}^{\alpha,\mu}$ is the fractional $T\varphi$ -CD and $W(t)$ is a standard Brownian motion (*BM*) on the complete probability space, $(\Omega, \mathcal{S}, \mathbb{P})$.

In our study, we used the *FP* theorem of the topological degree type and the approach related to the *MNC* methodology. Bana and Goebel were the primary initiators of this approach in 1980. Later, in 1986, Szufla Stanislaw used the *MNC* to investigate existence theorems in [27] and then implemented and utilized in numerous studies, see [28, 29]. The modeling and prediction of financial markets have increasingly incorporated tools from fractional calculus and stochastic analysis to better capture memory effects and random fluctuations inherent in real-world data. In [30], the authors, have used the collocation method to analyze a non-linear stochastic *DEs*, driven by fractional *BM*, to demonstrate its applicability in stochastic models such as the stock model. Traditional models fail to account for long-range dependence and abrupt market shifts, which are common in asset pricing, volatility dynamics, and risk assessment. To address this, *SFIDEs* involving $T\varphi$ -CD models fading memory, which aligns well with financial time series where the influence of past events gradually diminishes.

The main contributions of our work are highlighted as follows:

- The fractional $T\varphi$ -CD stochastic model enriches the existing literature with a novel perspective.
- The *TDT* technique for condensing operators is utilized to establish the *EU* of solutions under appropriate Lipschitz conditions.
- The obtained results are illustrated graphically through suitable examples.

Moreover, the advantages of our work are that, $T\varphi$ -CD provides better realism through tempering where the parameter μ helps model fading memory more realistically than standard power-law memory. The function $\varphi(t)$ provides a time deformation, capturing phenomena with nonuniform evolution in time. The system (1) balances both deterministic drift and stochastic fluctuations allowing for more comprehensive modeling. Additionally, handling impulses and nonlocality simultaneously enables modeling of systems with discontinuous events and nonlocal histories, often found in biological, control, or mechanical systems.

The structure of the paper is as follows: Section 2 gives the required preliminaries that includes definitions, propositions, lemmas and theorems. Section 3 proves the *EU* results. The next section provides theoretical examples demonstrating the applicability of the main result. And finally, the last section concludes the paper and outlines the potential future research directions.

2 Preliminaries

The concepts that lay the foundation of our analysis are stated below:

Let $C(\mathfrak{S}, \mathbb{R})$ be the space of all continuous functions, and $AC(\mathfrak{S}, \mathbb{R})$ be the space of all absolutely continuous functions. Let χ be a BS with the norm $\|\zeta\|_{PC} = \sup\{\|\zeta(t)\| : t \in \mathfrak{S}\}$.

Definition 1 [22] Let $\Theta : \mathcal{K} \rightarrow \Upsilon$ be a bounded CM , where $\mathcal{K} \subset \Upsilon$. Then Θ is

- σ -Lipschitz (σ -L) if there exists $k \geq 0$ such that $\sigma(\Theta(U)) \leq k\sigma(U)$ for all bounded subsets $U \subseteq \mathcal{K}$;
- a strict σ -contraction if there exists $0 \leq k < 1$ such that $\sigma(\Theta(U)) \leq k\sigma(U)$;
- σ -condensing if $\sigma(\Theta(U)) < \sigma(U)$ for all bounded subsets $S \subseteq \mathcal{K}$ with $\sigma(U) > 0$. In other words, $\sigma(\Theta(U)) \geq \sigma(U)$ implies $\sigma(U) = 0$,

where σ is the Kuratowski MNC .

Definition 2 [22] Let $\mathcal{K} \subset \Upsilon$ and $J : \mathcal{K} \rightarrow \Upsilon$, we recall that $J : \mathcal{K} \rightarrow \Upsilon$ is Lipschitz if there exists $k > 0$ such that

$$\|Jx - Jy\| \leq k\|x - y\| \quad \forall x, y \in \mathcal{K},$$

and if $k < 1$ then J is strict contraction.

Definition 3 [26] Let $\alpha > 0$. The fractional integral (FI) in the sense of Riemann-Liouville (RL) for $\zeta \in L^1[0, a]$ is defined by

$$I^\alpha \zeta(y) = \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} \zeta(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 4 [21] Let $\alpha > 0$, the real function $\zeta(t)$ be continuous on $[a, b]$ and $\varphi \in C^1[a, b]$ is an increasing differentiable function such that $\varphi'(t) \neq 0$ for all $a \leq t \leq b$. Then the φ - RL - FI of order α is defined by

$$I_a^{\alpha, \varphi} \zeta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t [\varphi(t) - \varphi(s)]^{\alpha-1} \varphi'(s) \zeta(s) ds.$$

Definition 5 [21] The $T\varphi$ -type RL integral of fractional order α , index $\mu \in \mathbb{R}$ of function ζ is

$${}^{TC}_{0+} D_{\varphi(t)}^{\alpha, \mu} \zeta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \ddot{\varphi}(t) \zeta(s) ds,$$

where $\ddot{\varphi}(t) = \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1} e^{-\mu(\varphi(t) - \varphi(s))}$.

Moreover, if $x \in AC^n(\mathfrak{S})$, then the φ -tempered FD of RL and Caputo type, with order $\alpha \in (n-1, n)$ and index $\mu \in \mathbb{R}$, of the function ζ are given respectively by:

$${}^{TR}_{0+} D_{\varphi(t)}^{\alpha, \mu} \zeta(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} + \mu \right)^n {}^T_{0+} I_{\varphi(t)}^{n-\alpha, \mu} \zeta(t), \text{ and}$$

$${}^{TC}_{0+}D_{\varphi(t)}^{\alpha,\mu}\zeta(t) = {}^T_{0+}I_{\varphi(t)}^{n-\alpha,\mu}\left(\frac{1}{\varphi'(t)}\frac{d}{dt} + \mu\right)^n\zeta(t).$$

Definition 6 (Stochastic Process) A Stochastic process is a set $\{\zeta(t)|t \geq 0\}$ of all random variables.

Proposition 1 [23] If $J_1, J_2 : \mathcal{K} \rightarrow \Upsilon$ are σ -L with respective constants a_1 and a_2 , then $J_1 + J_2$ is σ -L with constant $a_1 + a_2$.

Proposition 2 [23] If $J_1 : \mathcal{K} \rightarrow \Upsilon$ is Lipschitz with constant a then J_1 is σ -L with the same constant a .

Proposition 3 [23] If $J_1 : \mathcal{K} \rightarrow \Upsilon$ is compact, then J_1 is σ -L with constant $a = 0$.

Proposition 4 [20] If $\operatorname{Re}(\alpha) \geq 0$ and $l \in \mathbb{C}(\operatorname{Re}(\alpha) > 0)$, then

1. $I_{a^+}^{\alpha}(t-a)^{l-1}(t) = \frac{\Gamma(l)}{\Gamma(l+\alpha)}(t-a)^{l+\alpha-1}, \quad (\operatorname{Re}(\alpha) > 0).$
2. $D_{a^+}^{\alpha}(t-a)^{l-1}(t) = \frac{\Gamma(l)}{\Gamma(l-\alpha)}(t-a)^{l-\alpha-1}, \quad (\operatorname{Re}(\alpha) \geq 0).$

Lemma 1 [20] For $\alpha, \tau > 0$ we have the following semigroup property:

$$I_{a^+}^{\alpha}I_{a^+}^{\tau}h(t) = I_{a^+}^{\alpha+\tau}h(t), \quad t > a. \quad (2)$$

Lemma 2 [20] If $h \in C^n([a, b], \mathbb{R})$, $h_{n-\alpha}(t) \in AC^n[a, b]$, then the equality

$$I_{a^+}^{\alpha} {}^C D_{a^+}^{\alpha} h(t) = h(t) - \sum_{j=1}^n \frac{(h_{n-\alpha}^{(j)}(a))}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j}, \quad (3)$$

holds everywhere on $[a, b]$.

Let $H = \left\{ (I-F, \mathcal{K}, y) : \mathcal{K} \subseteq \Upsilon \text{ open and bounded, } F \in C_{\sigma}(\bar{\mathcal{K}}), y \in \Upsilon \setminus (I-F)(\partial\mathcal{K}) \right\}$ be the family of admissible triplets, where $C_{\sigma}(\bar{\mathcal{K}})$ represents the class of all σ -condensing maps $F : \bar{\mathcal{K}} \rightarrow \Upsilon$, $\bar{\mathcal{K}}$ is the closure of $\mathcal{K}$.

Theorem 1 Let $F : \mathcal{K} \rightarrow \Upsilon$ be σ -condensing and

$$\Xi = \left\{ \varphi \in \Upsilon : \text{there exists } 0 \leq \omega \leq 1 \text{ such that } \varphi = \omega F \varphi \right\}.$$

If Ξ is a bounded set in Υ , then there exists $\epsilon > 0$ such that $\Xi \subset \mathcal{U}_{\epsilon}(0)$ then

$$D(I - \omega F, \mathcal{U}_{\epsilon}(0), 0) = 1 \text{ for all } \omega \in [0, 1],$$

where $D : H \rightarrow \mathbb{Z}$ is a function of degree 1. Thus, F has atleast one FP, and the set of all FPs of F lies in $\mathcal{U}_{\epsilon}(0)$.

Theorem 2 (Cauchy-Schwarz inequality) If $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_m$ are arbitrary real numbers, we have

$$\left(\sum_{\tau=1}^m u_{\tau} v_{\tau} \right)^2 \leq \sum_{\tau=1}^m u_{\tau}^2 \times \sum_{\tau=1}^m v_{\tau}^2.$$

Theorem 3 (*Banach Contraction Mapping Principle*) Let Υ be a complete metric space, and $\Psi : \Upsilon \rightarrow \Upsilon$ is a contraction mapping with a contraction constant k , then Ψ has a unique FP.

3 Main results

This section analyzes the solution to SFIDEs (1) and also contains the needed prerequisites.

Definition 7 The function $\zeta \in \chi$ is said to be the solution of the system (1) if it satisfies the system (1).

Lemma 3 *The FI equation*

$$\zeta(t) = \begin{cases} \zeta_0 e^{-\mu(\varphi(t)-\varphi(0))} - \eta(\zeta) e^{-\mu(\varphi(t)-\varphi(0))} - \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t) J_1(s, \zeta(s)) ds \\ - \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) + \sum_{0 < t_i < t} e^{-\mu(\varphi(t)-\varphi(0))} I_i(\zeta(t_i)), \end{cases} \quad (4)$$

has a solution $\zeta \in \chi$, for $t \in (t_i, t_{i+1})$, $i = 1, 2, 3, \dots, m$ if and only if ζ is the solution to the SFIDEs (1).

Proof Assume that ζ is a solution to the SFIDEs (1). If $t \in [0, t_1)$ then

$$\begin{cases} {}^{TC}_{0+} D_{\varphi(t)}^{\alpha, \mu} \zeta(t) = J_1(t, \zeta(t)) + J_2(t, \zeta(t)) \frac{dW(t)}{dt}, \quad t \in [0, t_1) \\ \text{with } \zeta(0) = \zeta_0 - \eta(\zeta). \end{cases}$$

If $t_1 \leq t < t_2$, then

$$\begin{cases} {}^{TC}_{0+} D_{\varphi(t)}^{\alpha, \mu} \zeta(t) = J_1(t, \zeta(t)) + J_2(t, \zeta(t)) \frac{dW(t)}{dt}, \quad t \in [t_1, t_2) \\ \text{with } \Delta \zeta(t_1) = \zeta(t_1^+) - \zeta(t_1^-) = I_1(\zeta(t_1)). \end{cases} \quad (5)$$

Integrating the expression (5) from t_1 to t , we obtain

$$\zeta(t) = \zeta(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s).$$

It follows that

$$\begin{aligned} \zeta(t) &= \zeta(t_1^-) + I_1(\zeta(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) \\ &= \begin{cases} \zeta(0) + I_1(\zeta(t_1)) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t) J_1(s, \zeta(s)) ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s). \end{cases} \end{aligned}$$

If $t_2 \leq t < t_3$, then

$$\begin{cases} {}^{TC}D_{\varphi(t)}^{\alpha,\mu}\zeta(t) &= J_1(t, \zeta(t)) + J_2(t, \zeta(t)) \frac{d\mathbb{W}(t)}{dt}, \quad t \in [t_2, t_3) \\ \text{with } \Delta\zeta(t_2) &= \zeta(t_2^+) - \zeta(t_2^-) = I_2(\zeta(t_2)). \end{cases} \quad (6)$$

Integrating the equation (6) from t_2 to t , we obtain

$$\zeta(t) = \zeta(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \ddot{\varphi}(t) J_2(s, \zeta(s)) d\mathbb{W}(s).$$

It follows that

$$\begin{aligned} \zeta(t) &= \zeta(t_2^-) + I_2(\zeta(t_2)) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t \ddot{\varphi}(t) J_2(s, \zeta(s)) d\mathbb{W}(s) \\ &= \begin{cases} \zeta(0) + I_1(\zeta(t_1)) + I_2(\zeta(t_2)) \\ -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t_1) J_1(s, \zeta(s)) ds - \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \ddot{\varphi}(t_2) J_1(s, \zeta(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t_1) J_2(s, \zeta(s)) d\mathbb{W}(s) \\ - \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \ddot{\varphi}(t_2) J_2(s, \zeta(s)) d\mathbb{W}(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) d\mathbb{W}(s). \end{cases} \end{aligned}$$

Thus if $t \in (t_i, t_{i+1}]$, we get

$$\begin{aligned} \zeta(t) &= \zeta(0) + \sum_{0 < t_i < t} I_i(\zeta(t_i)) - \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_2(s, \zeta(s)) d\mathbb{W}(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) d\mathbb{W}(s), \quad i = 1, 2, 3, \dots, m. \end{aligned}$$

Hence by condition $\zeta(0) = \zeta_0 - \eta(\zeta)$, we deduce

$$\begin{aligned} \zeta(t) &= \zeta_0 - \eta(\zeta) + \sum_{0 < t_i < t} I_i(\zeta(t_i)) - \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_2(s, \zeta(s)) d\mathbb{W}(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) d\mathbb{W}(s), \quad i = 1, 2, 3, \dots, m. \end{aligned}$$

Conversely, assume that ζ satisfies (4). If $t \in (0, t_1]$ then $\zeta(0) = \zeta_0 - \eta(\zeta)$. If $t_i \in (t_i, t_{i+1}]$, $i = 1, 2, \dots, m$ by using the fact ${}^{TC}D_{\varphi(t)}^{\alpha,\mu}$ is the left inverse of ${}^{TC}I_{\varphi(t)}^{\alpha,\mu}$ and $T\varphi$ -type CD of a constant is equal to zero, we obtain

$$\begin{cases} {}^{TC}D_{\varphi(t)}^{\alpha,\mu}\zeta(t) &= J_1(t, \zeta(t)) + J_2(t, \zeta(t)) \frac{d\mathbb{W}(t)}{dt}, \quad t \in [t_i, t_{i+1}) \\ \text{with } \Delta\zeta(t_i) &= \zeta(t_i^+) - \zeta(t_i^-) = I_i(\zeta(t_i)). \end{cases}$$

This completes the proof. $\square$

Define operators as follows:

$$\begin{aligned} R_1(\zeta(t)) &= \zeta_0 - \eta(\zeta) + \sum_{0 < t_i < t} I_i(\zeta(t_i)) \quad \text{for } t_i \in (t_i, t_{i+1}], \quad i = 1, 2, \dots, m, \\ R_2(\zeta(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s), \quad \text{for } t \in [0, t_1), \\ R_3(\zeta(t)) &= -\frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t_{i-1}} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_1(s, \zeta(s)) ds - \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t_{i-1}} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(t_i) J_2(s, \zeta(s)) dW(s) \\ &\quad \text{for } t \in (t_i, T], \quad i = 1, 2, 3, \dots, m, \end{aligned}$$

where $R_1, R_2, R_3 : \chi \rightarrow \chi$.

From equation (4), we can determine the operator $V : \chi \rightarrow \chi$, $V\zeta = R_1\zeta + R_2\zeta + R_3\zeta$.

It is obvious that V is well defined. Then, the *FI* equation (4) can be written as:

$$\zeta = V\zeta = R_1\zeta + R_2\zeta + R_3\zeta \quad (7)$$

Therefore, the existence of a *FP* for operator V in (7) is equal to the existence of a solution to the *SFIDEs* (1).

The following assumptions are necessary in order to approach the existence of a solution to the *SFIDEs* (1).

Z1: $J_1, J_2 : \mathfrak{S} \times \Upsilon \rightarrow \Upsilon$ are jointly continuous.

Z2: There exist some constants $\delta_{J_1}, \delta_{J_2} > 0$, such that

$$\begin{aligned} \|J_1(t, \zeta_1) - J_1(t, \zeta_2)\|_{PC}^2 &\leq \delta_{J_1}^2 \|\zeta_1 - \zeta_2\|_{PC}^2, \\ \|J_2(t, \zeta_1) - J_2(t, \zeta_2)\|_{PC}^2 &\leq \delta_{J_2}^2 \|\zeta_1 - \zeta_2\|_{PC}^2, \quad \forall \zeta_1, \zeta_2 \in \Upsilon, t \in \mathfrak{S}. \end{aligned}$$

Z3: There exist $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22} > 0$, $\alpha_1 \in [0, 1)$ such that

$$\begin{aligned} \|J_1(t, \zeta_1)\|_{PC}^2 &\leq \delta_{11}^2 \|\zeta_1\|_{PC}^{2\alpha_1} + \delta_{12}^2, \\ \|J_2(t, \zeta_1)\|_{PC}^2 &\leq \delta_{21}^2 \|\zeta_1\|_{PC}^{2\alpha_1} + \delta_{22}^2, \quad \forall (t, \zeta_1) \in \mathfrak{S} \times \Upsilon. \end{aligned}$$

Z4: There are constants $\gamma_1 \in [0, \frac{1}{m})$ and $\delta_\eta > 0$ such that

$$\begin{aligned} \|I_i(\zeta_1) - I_i(\zeta_2)\|^2 &\leq \gamma_1^2 \|\zeta_1 - \zeta_2\|^2, \quad \forall \zeta_1, \zeta_2 \in \Upsilon, i = 1, 2, \dots, m, \\ \|\eta(\zeta_1) - \eta(\zeta_2)\|_{PC}^2 &\leq \delta_\eta^2 \|\zeta_1 - \zeta_2\|_{PC}^2, \quad \forall \zeta_1, \zeta_2 \in \Upsilon. \end{aligned}$$

Z5: There exist $\gamma_1, \gamma_2, \delta_3, \delta_4 > 0, \alpha_2, \alpha_3 \in [0, 1)$ such that

$$\begin{aligned} \|I_i(\zeta_1)\|^2 &\leq \gamma_1^2 \|\zeta_1\|^{2\alpha_2} + \gamma_2^2, \quad \forall \zeta_1 \in \Upsilon, i = 1, 2, \dots, m, \\ \|\eta(\zeta_1)\|_{PC}^2 &\leq \delta_3^2 \|\zeta_1\|_{PC}^{2\alpha_3} + \delta_4^2, \quad \forall \zeta_1 \in \Upsilon. \end{aligned}$$

Lemma 4 *The operator $R_1 : \chi \rightarrow \chi$ is Lipschitz with constant $(\delta_\eta + m\gamma_1)$. Subsequently, R_1 is a σ -L with the same constant $(\delta_\eta + m\gamma_1)$. In addition, R_1 fulfills:*

$$\mathbb{E} \|R_1 \zeta\|_{PC}^2 \leq 3 \left(\mathbb{E} \|\zeta_0\|^2 + \delta_3^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_3} + m\gamma_1^2 \mathbb{E} \|\zeta\|^{2\alpha_2} + \delta_4^2 + m\gamma_2^2 \right). \quad (8)$$

Proof For $[0, t_1]$, by using (Z4), we have

$$\|R_1 \zeta_1 - R_1 \zeta_2\|_{PC}^2 = \|\eta(\zeta_1) - \eta(\zeta_2)\|_{PC}^2 \leq \delta_\eta^2 \|\zeta_1 - \zeta_2\|_{PC}^2, \quad \forall \zeta_1, \zeta_2 \in \chi.$$

With the help of Proposition 4, R_1 is σ -L for $t \in [0, t_1]$.

For $t \in (t_i, t_{i+1}]$, by using (Z4), we get

$$\begin{aligned} \mathbb{E} \|R_1 \zeta_1 - R_1 \zeta_2\|_{PC}^2 &= \mathbb{E} \left\| -\eta(\zeta_1) + \sum_{0 < t_i < t} I_i(\zeta(t_i)) + \eta(\zeta_2) - \sum_{0 < t_i < t} I_i(\zeta_2(t_i)) \right\|_{PC}^2 \\ &\leq 2\mathbb{E} \|\eta(\zeta_1) - \eta(\zeta_2)\|_{PC}^2 + 2 \sum_{0 < t_i < t} \mathbb{E} \|I_i(\zeta_1(t_i)) - I_i(\zeta_2(t_i))\|^2 \\ &\leq \delta_\eta^2 \|\zeta_1 - \zeta_2\|_{PC}^2 + m\gamma_1^2 \|\zeta_1 - \zeta_2\|^2. \end{aligned}$$

Then,

$$\mathbb{E} \|R_1 \zeta_1 - R_1 \zeta_2\|_{PC}^2 \leq (\delta_\eta^2 + m\gamma_1^2) \|\zeta_1 - \zeta_2\|_{PC}^2$$

Hence, R_1 is Lipschitz with constant $(\delta_\eta^2 + m\gamma_1^2) \in [0, 1)$ for $t_i \in (t_i, t_{i+1}]$. Proposition 4 gives that R_1 is a σ -L for $t \in (t_i, t_{i+1}]$.

With the help of Theorem 2, (8) and (Z5),

$$\begin{aligned} \mathbb{E} \|R_1(\zeta(t))\|_{PC}^2 &\leq 3\mathbb{E} \|\zeta_0\|^2 + 3\mathbb{E} \|\eta(\zeta)\|_{PC}^2 + 3 \sum_{0 < t_i < t} \mathbb{E} \|I_i(\zeta(t_i))\|^2 \\ &\leq 3 \left(\mathbb{E} \|\zeta_0\|^2 + (\delta_3^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_3} + \delta_4^2) + m(\gamma_1^2 \mathbb{E} \|\zeta\|^{2\alpha_2} + \gamma_2^2) \right) \\ &\leq 3 \left(\mathbb{E} \|\zeta_0\|^2 + \delta_3^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_3} + m\gamma_1^2 \mathbb{E} \|\zeta\|^{2\alpha_2} + \delta_4^2 + m\gamma_2^2 \right). \quad \square \end{aligned}$$

Lemma 5 *The operators $R_2, R_3 : \chi \rightarrow \chi$ are continuous and so is $R_2 + R_3$. In addition, $R_2 + R_3$ satisfies the following condition:*

$$\mathbb{E} \|R_2 \zeta\|_{PC}^2 + \mathbb{E} \|R_3 \zeta\|_{PC}^2 \leq \frac{(1 + m^2) \widehat{\varphi}(t) (\delta_{11}^2 + \delta_{12}^2) \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + (\delta_{21}^2 + \delta_{22}^2)}{\Gamma(\alpha)} \quad (9)$$

for all $\zeta \in \chi$.

Proof Let the bounded sequence $\{\zeta_n\}$ be a set $U_k (k > 0) \subseteq \chi$ such that $\zeta_n \rightarrow \zeta$ in U_k . Now, we have to show that $\|R_2 \zeta_n - R_2 \zeta\|_{PC} \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that $J_1(s, \zeta_n(s)) \rightarrow J_1(s, \zeta(s))$ and $J_2(s, \zeta_n(s)) \rightarrow J_2(s, \zeta(s))$ as $n \rightarrow \infty$ due to the continuity of J_1 and J_2 . By applying (Z3), we obtain for all $s \in \mathfrak{S}$,

$$\|J_1(s, \zeta_n(s)) - J_1(s, \zeta(s))\|_{PC} \leq \|J_1(s, \zeta_n(s))\|_{PC} + \|J_1(s, \zeta(s))\|_{PC}$$

$$\begin{aligned} &\leq 2(\delta_{11} \|\zeta\|_{PC}^{\alpha_1} + \delta_{12}), \text{ and} \\ \|J_2(s, \zeta_n(s)) - J_2(s, \zeta(s))\|_{PC} &\leq \|J_2(s, \zeta_n(s))\|_{PC} + \|J_2(s, \zeta(s))\|_{PC} \\ &\leq 2(\delta_{21} \|\zeta\|_{PC}^{\alpha_1} + \delta_{22}). \end{aligned}$$

Then,

$$\begin{aligned} \ddot{\varphi}(t) \|J_1(s, \zeta_n(s)) - J_1(s, \zeta(s))\|_{PC} &\leq 2\ddot{\varphi}(t)(\delta_{11} \|\zeta\|_{PC}^{\alpha_1} + \delta_{12}), \\ \ddot{\varphi}(t) \|J_2(s, \zeta_n(s)) - J_2(s, \zeta(s))\|_{PC} &\leq 2\ddot{\varphi}(t)(\delta_{21} \|\zeta\|_{PC}^{\alpha_1} + \delta_{22}), \end{aligned}$$

since $s \rightarrow 2(\ddot{\varphi}(t))^2(\delta_{11}^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + \delta_{21}^2)$ and $s \rightarrow 2(\ddot{\varphi}(t))^2(\delta_{12}^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + \delta_{22}^2)$ are integrable for $s, t \in \mathfrak{S}$ and with the help of Lebesgue Dominated Convergence Theorem and Itô isometry,

$$\begin{aligned} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_1(s, \zeta_n(s)) - J_1(s, \zeta(s))\|_{PC}^2 ds &\rightarrow 0 \text{ as } n \rightarrow \infty, \\ \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_2(s, \zeta_n(s)) - J_2(s, \zeta(s))\|_{PC}^2 ds &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E} \|(R_2 \zeta_n)(t) - (R_2 \zeta)(t)\|_{PC}^2 &\leq \left\{ \begin{aligned} &\frac{1}{(\Gamma(\alpha))^2} \int_0^t \mathbb{E} \|J_1(s, \zeta_n(s)) - J_1(s, \zeta(s))\|_{PC}^2 ds \\ &+ \frac{1}{(\Gamma(\alpha))^2} \int_0^t \mathbb{E} \|J_2(s, \zeta_n(s)) - J_2(s, \zeta(s))\|_{PC}^2 ds \end{aligned} \right\} \rightarrow 0, \\ \forall t \in \mathfrak{S}. \end{aligned}$$

Therefore, as $R_2 \zeta_n \rightarrow R_2 \zeta$, R_2 is continuous for $n \rightarrow \infty$.

Proceeding the similar process leads to the continuity of R_3 on $PC([0, t_1], \Upsilon)$, $PC((t_i, t_{i+1}], \Upsilon)$ and $PC((t_m, T], \Upsilon)$.

By the property of continuous operators, $R_2 + R_3$ is continuous.

With the help of (9) and (Z3),

$$\begin{aligned} \mathbb{E} \|R_2(\zeta(t)) + R_3(\zeta(t))\|_{PC}^2 &\leq 2\mathbb{E} \|R_2(\zeta(t))\|_{PC}^2 + 2\mathbb{E} \|R_3(\zeta(t))\|_{PC}^2 \\ &\leq \left\{ \begin{aligned} &\frac{2}{\Gamma(\alpha)} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_1(s, \zeta(s))\|_{PC}^2 ds \\ &+ \frac{2}{\Gamma(\alpha)} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_2(s, \zeta(s))\|_{PC}^2 ds \\ &+ \frac{2}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \mathbb{E} \|J_1(s, \zeta(s))\|_{PC}^2 ds \\ &+ \frac{2}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \mathbb{E} \|J_2(s, \zeta(s))\|_{PC}^2 ds \end{aligned} \right\} \end{aligned}$$

$$\leq \begin{cases} \frac{2}{\Gamma(\alpha)} \int_0^t (\ddot{\varphi}(t))^2 (\delta_{11} \mathbb{E} \|\zeta\|_{PC}^{\alpha_1} + \delta_{21})^2 ds \\ + \frac{2}{\Gamma(\alpha)} \int_0^t (\ddot{\varphi}(t))^2 (\delta_{12} \mathbb{E} \|\zeta\|_{PC}^{\alpha_1} + \delta_{22})^2 ds \\ + \frac{2}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 (\delta_{11} \mathbb{E} \|\zeta\|_{PC}^{\alpha_1} + \delta_{21})^2 ds \\ + \frac{2}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 (\delta_{12} \mathbb{E} \|\zeta\|_{PC}^{\alpha_1} + \delta_{22})^2 ds. \end{cases}$$

Then,

$$\begin{aligned} & \mathbb{E} \|R_2 \zeta\|_{PC}^2 + \mathbb{E} \|R_3 \zeta\|_{PC}^2 \\ & \leq \frac{(1+m^2) \widehat{\varphi}(t) \delta_{11}^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + \delta_{21}^2}{\Gamma(\alpha)} + \frac{(1+m^2) \widehat{\varphi}(t) \delta_{12}^2 \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + \delta_{22}^2}{\Gamma(\alpha)} \\ & \leq \frac{(1+m^2) \widehat{\varphi}(t) (\delta_{11}^2 + \delta_{12}^2) \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + (\delta_{21}^2 + \delta_{22}^2)}{\Gamma(\alpha)}, \end{aligned}$$

where

$$\widehat{\varphi}(t) = \frac{(\varphi(t) - \varphi(0))^{2\alpha-1}}{2\mu(2\alpha-1)} (1 - e^{-2\mu(\varphi(t) - \varphi(0))}).$$

□

The following assumption is necessary to discuss the compactness of R_2, R_3 .

Z6: For any $k > 0$, there arise a constant $\beta_k > 0$ such that

$$\sigma(\|J_1(s, U)\|^2) \leq \beta_k \sigma(\|U\|^2)$$

for all $s \in \mathfrak{S}$, $U \subset U_k := \{\zeta_1 \in \chi : \|\zeta_1\|_{PC} \leq k\}$, and

$$\frac{4\widehat{\varphi}(T)\beta_k}{(\Gamma(\alpha+1))^2} < 1.$$

Lemma 6 *The operators $R_2, R_3 : \chi \rightarrow \chi$ are compact implying σ -L with zero constant and so is $R_2 + R_3$.*

Proof To prove the compactness and relative compactness of R_2 and R_3 , in χ , we consider a bounded subset $U \subset \chi$. Let $\{\zeta_n\}$ be a sequence on $U \subset U_k, \forall \zeta_n \in U$. By (9), we get

$$\mathbb{E} \|R_2 \zeta\|_{PC}^2 + \mathbb{E} \|R_3 \zeta\|_{PC}^2 \leq \frac{(1+m^2) \widehat{\varphi}(t) (\delta_{11}^2 + \delta_{12}^2) \mathbb{E} \|\zeta\|_{PC}^{2\alpha_1} + (\delta_{21}^2 + \delta_{22}^2)}{\Gamma(\alpha)} := \epsilon,$$

$\forall \zeta_n \in U$, so $R_2(U)$ and $R_3(U)$ are bounded. Now, for $0 \leq t_1 < t_2 < T$, we get

$$\begin{aligned} & \| (R_2 \zeta)(t_2) - (R_2 \zeta)(t_1) \|_{PC} \\ & = \left\| \left\| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \ddot{\varphi}(t_2) J_1(s, \zeta(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t_1) J_1(s, \zeta(s)) ds \right. \right. \\ & \quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \ddot{\varphi}(t_2) J_2(s, \zeta(s)) dW(s) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \ddot{\varphi}(t_1) J_2(s, \zeta(s)) dW(s) \right\|_{PC} \right\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{E} \|(R_2 \zeta)(t_2) - (R_2 \zeta)(t_1)\|_{PC}^2 &\leq \begin{cases} \frac{1}{(\Gamma(\alpha))^2} \int_0^{t_1} ((\ddot{\varphi}(t_2))^2 - (\ddot{\varphi}(t_1))^2) \mathbb{E} \|J_1(s, \zeta(s))\|_{PC}^2 ds \\ + \frac{1}{(\Gamma(\alpha))^2} \int_{t_1}^{t_2} (\ddot{\varphi}(t_2))^2 \mathbb{E} \|J_1(s, \zeta(s))\|_{PC}^2 ds \\ + \frac{1}{(\Gamma(\alpha))^2} \int_0^{t_1} ((\ddot{\varphi}(t_2))^2 - (\ddot{\varphi}(t_1))^2) \mathbb{E} \|J_2(s, \zeta(s))\|_{PC}^2 ds \\ + \frac{1}{(\Gamma(\alpha))^2} \int_{t_1}^{t_2} (\ddot{\varphi}(t_2))^2 \mathbb{E} \|J_2(s, \zeta(s))\|_{PC}^2 ds \end{cases} \\ &\leq \frac{(\delta_{11}^2 + \delta_{21}^2)k^{2\alpha_1} + \delta_{12}^2 + \delta_{22}^2}{(\Gamma(\alpha))^2} \widehat{\varphi}(t_2 - t_1) \end{aligned}$$

Therefore we have,

$$\mathbb{E} \|(R_2 \zeta)(t_2) - (R_2 \zeta)(t_1)\|_{PC}^2 \leq \frac{(\delta_{11}^2 + \delta_{21}^2)k^{2\alpha_1} + \delta_{12}^2 + \delta_{22}^2}{(\Gamma(\alpha))^2} \widehat{\varphi}(t_2 - t_1).$$

When $t_2 \rightarrow t_1$, $\{R_2 \zeta_n\}$ becomes equicontinuous.

Take

$$U(t) := \left\{ \zeta_n(t) : \zeta_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) \right\} \subset U_k,$$

$$U(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_1(s, \zeta(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(t) J_2(s, \zeta(s)) dW(s) \text{ and}$$

$$\mathbb{E} \|U(t)\|^2 \leq \frac{2}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_1(s, \zeta(s))\|^2 ds + \frac{2}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_2(s, \zeta(s))\|^2 ds.$$

As $U(t) \subset \chi$ is equicontinuous, using Proposition 1, $t \rightarrow \sigma(U(t))$ is continuous on $\mathfrak{S}$, and by (Z6), for $s, t \in \mathfrak{S}$ we get

$$\begin{aligned} &\sigma(\mathbb{E} \|U(t)\|^2) \\ &\leq \frac{2}{(\Gamma(\alpha))^2} \sigma \left(\int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_1(s, \zeta(s))\|^2 ds + \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E} \|J_2(s, \zeta(s))\|^2 ds \right) \\ &\leq \frac{2}{(\Gamma(\alpha))^2} \left(\int_0^t (\ddot{\varphi}(t))^2 \sigma(\mathbb{E} \|J_1(s, \zeta(s))\|^2) ds + \int_0^t (\ddot{\varphi}(t))^2 \sigma(\mathbb{E} \|J_2(s, \zeta(s))\|^2) ds \right) \\ &\leq \frac{2\beta_k}{(\Gamma(\alpha))^2} \left(\int_0^t \ddot{\varphi}(t) \sigma(\mathbb{E} \|U\|^2) ds + \int_0^t \ddot{\varphi}(t) \sigma(\mathbb{E} \|U\|^2) ds \right) \\ &\leq \frac{4\widehat{\varphi}(T)\beta_k}{(\Gamma(\alpha))^2} \sigma(\mathbb{E} \|U\|^2) < \sigma(\mathbb{E} \|U\|^2) \end{aligned}$$

The condition $\frac{4\widehat{\varphi}(T)\beta_k}{(\Gamma(\alpha))^2} < 1$ infers $\sigma(\mathbb{E}||U||^2) = 0$. Since $R_2(U) \subset \chi$ is relatively compact, it is a σ -L with zero constant by Proposition 3.

The compactness of the operators R_3^i on $C((t_i, t_{i+1}], \Upsilon)$ and R_3^m on $C((t_m, T], \Upsilon)$ may be obtained by repeating the same procedure of the operator R_2 on $C([0, t_1], \Upsilon)$. By Proposition 3, R_3 is a σ -L with zero constant. $\square$

Theorem 4 *If (Z1) - (Z6) are true, then the collection of SFIDEs (1)'s solution is bounded and there exists at least one solution $\zeta \in \chi$.*

Proof Let us consider the continuous, σ -L and bounded operators, $R_1, R_2, R_3 : \chi \rightarrow \chi$ as defined above. By Proposition 2, $V : \chi \rightarrow \chi$ is a σ -L and strict σ -contraction with constant $(\delta_\eta^2 + m\gamma_1^2) \in [0, 1)$.

Set $M = \{\zeta \in \chi : \exists \lambda \in [0, 1] \text{ such that } \zeta = \lambda V\zeta\}$.

Consider, $\zeta \in M$ and $\lambda \in [0, 1]$ such that $\zeta = \lambda V\zeta$. It follows from (8) and (9)

$$\begin{aligned} \mathbb{E}||\zeta||_{PC}^2 &= \mathbb{E}||\lambda V\zeta||_{PC}^2 \leq \lambda^2(\mathbb{E}||R_1\zeta||_{PC}^2 + \mathbb{E}||R_2\zeta||_{PC}^2 + \mathbb{E}||R_3\zeta||_{PC}^2) \\ &\leq \mathbb{E}||\zeta_0||^2 + \delta_3^2 \mathbb{E}||\zeta||_{PC}^{2\alpha_3} + m\gamma_1^2 \mathbb{E}||\zeta||^{2\alpha_2} + \delta_4^2 + m\gamma_2^2 \\ &\quad + \frac{(1+m)\widehat{\varphi}(T)(\delta_{11}^2 + \delta_{21}^2)\mathbb{E}||\zeta||_{PC}^{2\alpha_1} + (\delta_{12}^2 + \delta_{22}^2)}{(\Gamma(\alpha))^2}. \end{aligned}$$

The above inequality, together with $\alpha_1 < 1, \alpha_2 < 1$ and $\alpha_3 < 1$ demonstrate that M has a bound in χ . The set of the FP s of V is bounded in χ and as a result, we deduce from Theorem 3 that V has at least one FP . $\square$

Remark 1

1. When the assumption (Z3) is defined for $\alpha_1 = 1$, we obtain that the conclusions of Theorem 4 hold, provided $\frac{(1+m)\widehat{\varphi}(T)\delta_{11}^2}{(\Gamma(\alpha))^2} < 1$.
2. When the assumption (Z4) is defined for $\alpha_2 = \alpha_3 = 1$, we obtain that the conclusions of Theorem 4 hold, provided $(\delta_3^2 + m\gamma_1^2) < 1$.
3. When the assumptions (Z3) and (Z4) are defined for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we obtain that the conclusions of Theorem 4 hold, provided $\delta_3^2 + m\gamma_1^2 + \frac{(1+m)\widehat{\varphi}(T)\delta_{11}^2}{(\Gamma(\alpha))^2} < 1$.

Theorem 5 *If the assumptions (Z1) to (Z6) hold, with*

$$\delta_\eta^2 + m\gamma_1^2 + \frac{(1+m)\widehat{\varphi}(T)(\delta_{j_1}^2 + \delta_{j_2}^2)}{(\Gamma(\alpha))^2} < 1 \quad (10)$$

then (1) has a unique solution (US), $\zeta \in \chi$.

Proof For arbitrary $\zeta_1, \zeta_2 \in \chi$, we can easily demonstrate that V is a contraction operator on χ by (Z2) and (Z4) as follows:

$$\mathbb{E}||(V\zeta_1)(t) - (V\zeta_2)(t)||_{PC}^2$$

$$\leq \left\{ \begin{aligned} & 6\mathbb{E}||\eta(\zeta_1) - \eta(\zeta_2)||_{PC}^2 + \sum_{0 < t_i < t} 6\mathbb{E}||I_i(\zeta_1(t_i)) - I_i(\zeta_2(t_i))||^2 \\ & + \frac{6}{(\Gamma(\alpha))^2} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \mathbb{E}||J_1(s, \zeta_1(s) - J_1(s, \zeta_2(s)))||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \mathbb{E}||J_2(s, \zeta_1(s) - J_2(s, \zeta_2(s)))||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E}||J_1(s, \zeta_1(s) - J_1(s, \zeta_2(s)))||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \mathbb{E}||J_2(s, \zeta_1(s) - J_2(s, \zeta_2(s)))||_{PC}^2 ds \end{aligned} \right.$$

$$\leq \left\{ \begin{aligned} & \delta_\eta^2 6\mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2 + \sum_{0 < t_i < t} \gamma_1^2 6\mathbb{E}||\zeta_1 - \zeta_2||^2 \\ & + \frac{6}{(\Gamma(\alpha))^2} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \delta_{J_1}^2 \mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (\ddot{\varphi}(t_i))^2 \delta_{J_2}^2 \mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \delta_{J_1}^2 \mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2 ds \\ & + \frac{6}{(\Gamma(\alpha))^2} \int_0^t (\ddot{\varphi}(t))^2 \delta_{J_2}^2 \mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2 ds, \quad i = 1, 2, \dots, m. \end{aligned} \right.$$

Therefore we have,

$$\mathbb{E}||(V\zeta_1)(t) - (V\zeta_2)(t)||_{PC}^2 \leq \left[\delta_\eta^2 + m\gamma_1^2 + \frac{(1+m)\widehat{\varphi}(T)(\delta_{J_1}^2 + \delta_{J_2}^2)}{(\Gamma(\alpha))^2} \right] \mathbb{E}||\zeta_1 - \zeta_2||_{PC}^2.$$

According to the inequality (10), V is a contraction operator on χ . We conclude that operator V has a unique FP on χ by applying Theorem 3. This suggests that SFIDEs (1) has a US, completing the proof. $\square$

Remark 2 The space $C(\mathcal{T}, \Upsilon)$, where $\mathcal{T} \in \mathbb{R}$ is a compact interval and Υ is a BS, forms a crucial setting for studying the evolution of functions over time. This space, consisting of all continuous Υ -valued functions on $\mathcal{T}$, is itself a BS under supremum norm, and it frequently arises in the context of initial value problems and evolution equations.

4 Examples

In this section, we provide some examples to substantiate our results.

Example 1 Consider the following SFIDEs:

$$\left\{ \begin{aligned} {}^{TC}D_{\varphi(t)}^{\alpha, \mu} \zeta(t) &= \frac{8}{7\Gamma(\frac{4}{5})} (\varphi(t) - \varphi(0))^{\frac{4}{3}} \frac{e^{-\cos^2 \zeta(t)}}{\sqrt{16+t^\alpha}} + \frac{t^\alpha e^{-\alpha}}{15} \sin \zeta(t) \frac{d\eta(t)}{dt}, t \in (0, T)/\{t_1\}, \\ \Delta \zeta(t_1) &= \frac{1}{15} \zeta, \\ \zeta(0) + \sin(t) &= 1. \end{aligned} \right. \quad (11)$$

As $J_1 = \frac{8}{7\Gamma(\frac{4}{5})} (\varphi(t) - \varphi(0))^{\frac{4}{3}} \frac{e^{-\cos^2 \zeta(t)}}{\sqrt{16+t^\alpha}}$, and $J_2 = \frac{t^\alpha e^{-\alpha}}{15} \sin \zeta(t)$, and we have $I(\zeta(t_1)) = \frac{1}{15} \zeta$, and $\eta(\zeta) = \sin(t)$. The assumptions in Theorem 5 are satisfied.

$$||J_1(t, \zeta_1) - J_1(t, \zeta_2)||^2$$

$$\begin{aligned}
 &= \left\| \frac{8}{7\Gamma(\frac{4}{5})} (\varphi(t) - \varphi(0))^{\frac{4}{3}} \frac{e^{-\cos^2 \zeta_1(t)}}{\sqrt{16+t^\alpha}} - \frac{8}{7\Gamma(\frac{4}{5})} (\varphi(t) - \varphi(0))^{\frac{4}{3}} \frac{e^{-\cos^2 \zeta_2(t)}}{\sqrt{16+t^\alpha}} \right\|^2, \\
 &\leq \frac{1.0415}{4.2} \|\zeta_1 - \zeta_2\|^2 \implies \delta_{f_1}^2 = 0.248, \\
 \|J_2(t, \zeta_1) - J_2(t, \zeta_2)\|^2 &= \left\| \frac{t^\alpha e^{-\alpha}}{15} \sin \zeta_1(t) - \frac{t^\alpha e^{-\alpha}}{15} \sin \zeta_2(t) \right\|^2 \\
 &\leq \left(\frac{1}{15}\right)^2 \|\zeta_1 - \zeta_2\|^2 \implies \delta_{f_2}^2 = 0.0044, \\
 \|\eta(\zeta_1) - \eta(\zeta_2)\|^2 &= (\sin(t))^2 \|\zeta_1 - \zeta_2\|^2 \implies \delta_\eta^2 = 0.7081, \\
 \|I(\zeta_1) - I(\zeta_2)\|^2 &= \left(\frac{1}{15}\right)^2 \|\zeta_1 - \zeta_2\|^2 \implies \gamma_1^2 = 0.0044.
 \end{aligned}$$

With $\varphi(t) = t$, $\mu = 0.6$ and $\alpha = 0.8$, it yields

$$\widehat{\varphi}(t) = \frac{(\varphi(t) - \varphi(0))^{2\alpha-1}}{2\mu(2\alpha-1)} (1 - e^{-2\mu(\varphi(t) - \varphi(0))}) = 1.2165.$$

For $T = 1$ and $m = 1$ the condition (10) becomes,

$$\delta_\eta^2 + m\gamma_1^2 + \frac{(1+m)\widehat{\varphi}(T)(\delta_{f_1}^2 + \delta_{f_2}^2)}{(\Gamma(\alpha))^2} = 0.8358 < 1.$$

Now with the help of Theorem 5, Equation (11) possesses a *US*.

Example 2 Let us examine the below *SFIDE*'s overall growth rate when random fluctuations (noise) from unforeseen outside variables are present:

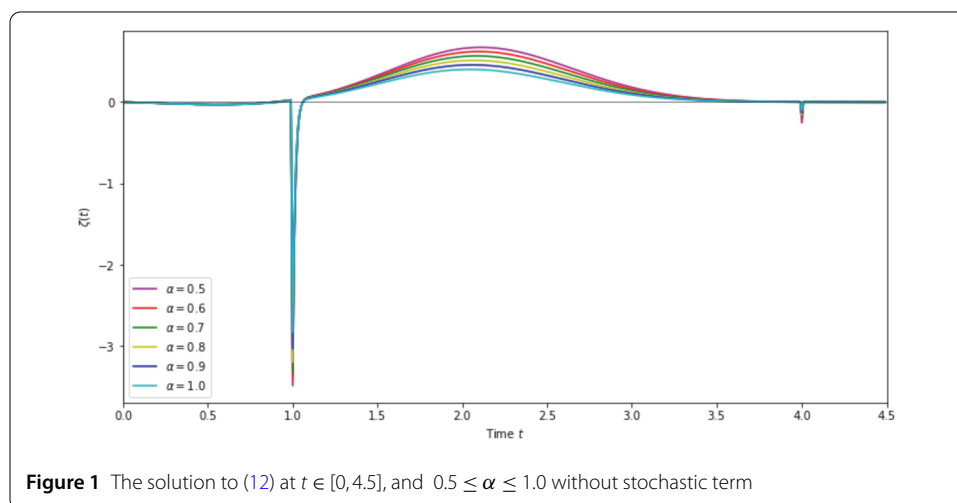
$$\begin{cases} {}^{TC}D_{\varphi(t)}^{\alpha,\mu} \zeta(t) &= \frac{(\varphi(t) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \cos\left(\frac{1}{10}\zeta(t)\right) + \sin\left(\frac{1}{5}\zeta(t)\right) \frac{d\mathbb{W}(t)}{dt}, t \in (0, T)/\{t_1, t_2\}, \\ \Delta \zeta(t_i) &= \frac{4}{11}\zeta, \quad i = 1, 2 \\ \zeta(0) + \frac{t}{25}\zeta &= 0. \end{cases} \quad (12)$$

As $J_1 = \frac{(\varphi(t) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \cos\left(\frac{1}{10}\zeta(t)\right)$, $J_2 = \sin\left(\frac{1}{5}\zeta(t)\right)$, $I_i(\zeta(t_i)) = \frac{4}{11}\zeta$, and $\eta(\zeta) = \frac{t}{25}\zeta$. Then it is easy to observe that Theorem 5's assumptions are all met.

$$\begin{aligned}
 &\|J_1(t, \zeta_1) - J_1(t, \zeta_2)\|^2 \\
 &= \left\| \frac{(\varphi(t) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \cos\left(\frac{1}{10}\zeta_1(t)\right) - \frac{(\varphi(t) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \cos\left(\frac{1}{10}\zeta_2(t)\right) \right\|^2 \\
 &\leq \frac{1}{78.7237} \|\zeta_1 - \zeta_2\|^2 \implies \delta_{f_1}^2 = \frac{1}{78.7237}, \\
 \|J_2(t, \zeta_1) - J_2(t, \zeta_2)\|^2 &= \left\| \sin\left(\frac{1}{5}\zeta_1(t)\right) - \sin\left(\frac{1}{5}\zeta_2(t)\right) \right\|^2 \\
 &\leq \frac{1}{25} \|\zeta_1 - \zeta_2\|^2 \implies \delta_{f_2}^2 = \frac{1}{25}, \\
 \|\eta(\zeta_1) - \eta(\zeta_2)\|^2 &= \frac{1}{625} \|\zeta_1 - \zeta_2\|^2 \implies \delta_\eta^2 = \frac{1}{625}, \\
 \|I(\zeta_1) - I(\zeta_2)\|^2 &= 2 \times \frac{16}{121} \|\zeta_1 - \zeta_2\|^2 \implies \gamma_1^2 = \frac{32}{121}.
 \end{aligned}$$

Table 1 $\zeta(t)$ for $0.5 \leq \alpha \leq 1$ without stochastic term

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.5	-0.029282	-0.029231	-0.029253	-0.029354	-0.029541	-0.029817
1.0	-3.487897	-3.427259	-3.331774	-3.200640	-3.033855	-2.832332
1.5	0.329384	0.313178	0.294404	0.273251	0.249973	0.224883
2.0	0.659305	0.611714	0.561455	0.509161	0.455528	0.401298
2.5	0.520058	0.472065	0.423821	0.375876	0.328787	0.283103
3.0	0.182424	0.162458	0.143037	0.124335	0.106518	0.089739
3.5	0.030408	0.026621	0.023025	0.019643	0.016495	0.013598
4.0	-0.254595	-0.219872	-0.187526	-0.157658	-0.130344	-0.105640
4.5	0.000113	0.000095	0.000079	0.000064	0.000051	0.000039



With $\varphi(t) = t^2$, $\mu = \frac{1}{2}$ and $\alpha = \frac{3}{5}$.

$$\widehat{\varphi}(t) = \frac{(\varphi(t) - \varphi(0))^{2\alpha-1}}{2\mu(2\alpha-1)} (1 - e^{-2\mu(\varphi(t) - \varphi(0))}) = 0.8382.$$

Let us verify whether (10) is satisfied for $T = \frac{1}{2}$ and $m = 2$,

$$\delta_\eta^2 + m\gamma_1^2 + \frac{(1+m)\widehat{\varphi}(T)(\delta_{f_1}^2 + \delta_{f_2}^2)}{(\Gamma(\alpha))^2} = 0.57036 < 1.$$

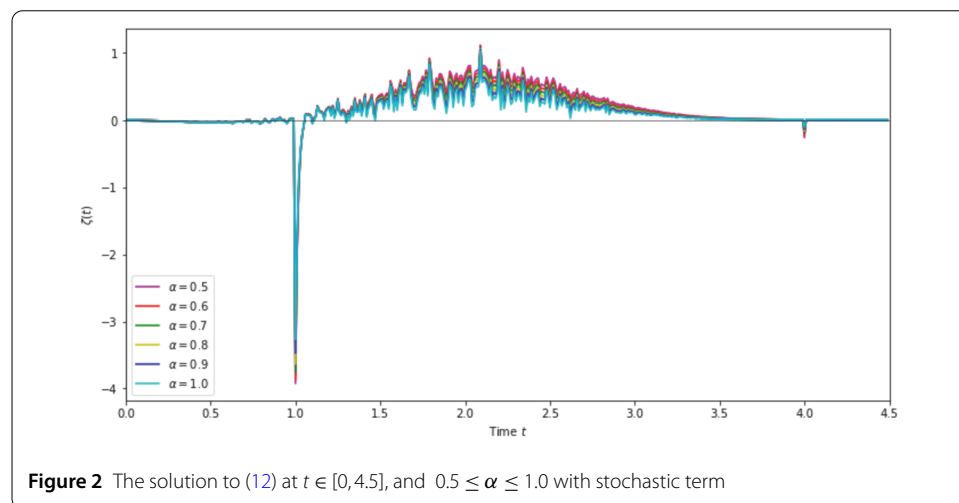
Therefore, our result can be used to solve the SFIDEs (12). □

$$\zeta(t) = \begin{cases} \zeta_0 e^{-\mu(\varphi(t) - \varphi(0))} - \eta(\zeta) e^{-\mu(\varphi(t) - \varphi(0))} \\ - \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(s) \frac{(\varphi(s) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \sin\left(\frac{1}{10}\zeta(s)\right) ds \\ - \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} \ddot{\varphi}(s) \cos\left(\frac{1}{5}\zeta(s)\right) d\mathbb{W}(s) + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(s) \frac{(\varphi(s) - \varphi(0))^{2-\alpha}}{\Gamma(2-\alpha)} \sin\left(\frac{1}{10}\zeta(s)\right) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \ddot{\varphi}(s) \cos\left(\frac{1}{5}\zeta(s)\right) d\mathbb{W}(s) + \sum_{0 < t_i < t} e^{-\mu(\varphi(t) - \varphi(0))} I_i(\zeta(t_i)), \quad i = 1, 2. \end{cases}$$

As we can see, the trajectory of the system (12) of order $\alpha \in (1/2, 1)$, subject to a nonlocal initial condition and deterministic impulses at $t_1 = 1$ and $t_2 = 4$. Between im-

Table 2 $\zeta(t)$ for $0.5 \leq \alpha \leq 1$ with stochastic term

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.5	-0.031958	-0.031718	-0.031564	-0.031502	-0.031537	-0.031672
1.0	-3.924572	-3.863926	-3.768433	-3.637290	-3.470497	-3.268965
1.5	0.367610	0.352949	0.335779	0.316291	0.294741	0.271445
2.0	0.685351	0.639583	0.591265	0.541036	0.489598	0.437697
2.5	0.451315	0.396979	0.341862	0.286490	0.231398	0.177121
3.0	0.172764	0.151731	0.131138	0.111156	0.091950	0.073675
3.5	0.030797	0.027059	0.023517	0.020195	0.017112	0.014284
4.0	-0.255414	-0.220805	-0.188586	-0.158857	-0.131694	-0.107149
4.5	0.000111	0.000093	0.000076	0.000061	0.000047	0.000034

**Figure 2** The solution to (12) at $t \in [0, 4.5]$, and $0.5 \leq \alpha \leq 1.0$ with stochastic term**Table 3** $\zeta(t)$ for $0.5 \leq \alpha \leq 1$ without stochastic term

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.5	-0.029282	-0.029231	-0.029253	-0.029354	-0.029541	-0.029817
1.0	0.030777	0.029619	0.027771	0.025213	0.021947	0.017987
1.5	0.329384	0.313178	0.294404	0.273251	0.249973	0.224883
2.0	0.659305	0.611714	0.561455	0.509161	0.455528	0.401298
2.5	0.520058	0.472065	0.423821	0.375876	0.328787	0.283103
3.0	0.182424	0.162458	0.143037	0.124335	0.106518	0.089739
3.5	0.030408	0.026621	0.023025	0.019643	0.016495	0.013598
4.0	-0.254595	-0.219872	-0.187526	-0.157658	-0.130344	-0.105640
4.5	0.000105	0.000089	0.000073	0.000060	0.000047	0.000036
5.0	0.000002	0.000002	0.000001	0.000001	0.000001	0.000000

pulses, the state $\zeta(t)$ evolves according to (12) with tempering parameter $\varphi(t)$. The nonlocal condition prescribes $\zeta(0) = \zeta_0 + \eta(\zeta)$, so the initial level reflects aggregated pre-history. At the impulse times, $\Delta\zeta(t_i), i \in \{1, 2\}$, producing visible jump discontinuities at $t_1 = 1$ and $t_2 = 4$. For $t \neq t_i$, paths are continuous but unpredictable due to the stochastic forcing; immediately after each impulse, the trajectory relaxes toward the drift-dominated trend with a fractional-order memory kernel, which is shortened by tempering.

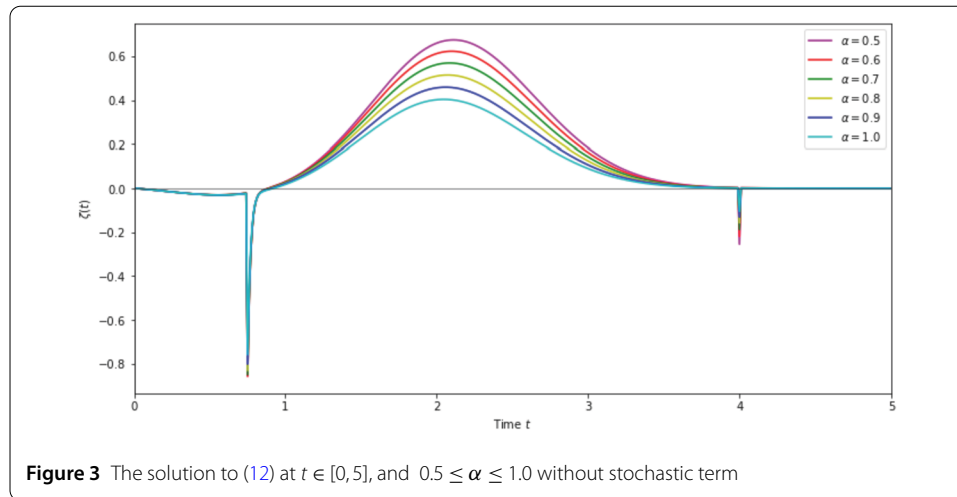
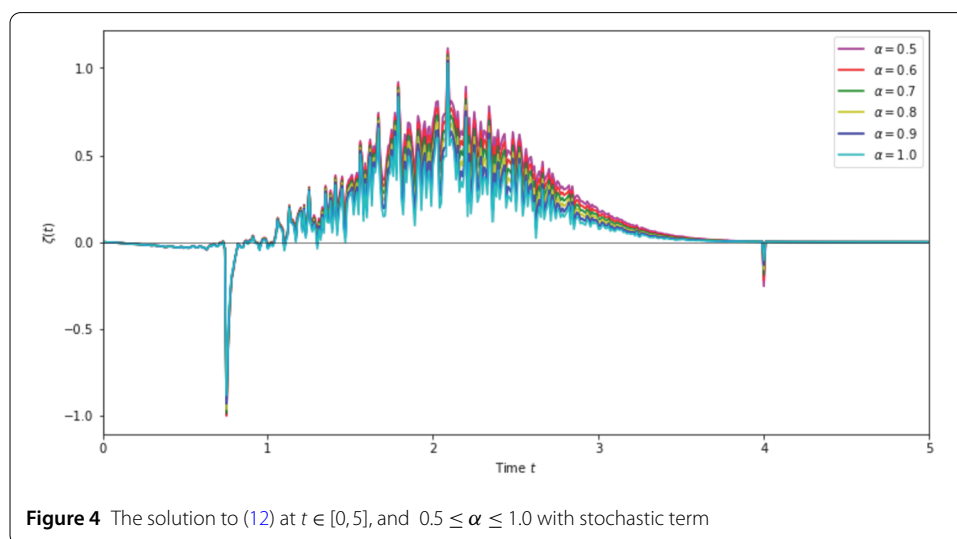


Table 4 $\zeta(t)$ for $0.5 \leq \alpha \leq 1$ with stochastic term

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.5	-0.031958	-0.031718	-0.031564	-0.031502	-0.031537	-0.031672
1.0	-0.027261	-0.028410	-0.030250	-0.032799	-0.036057	-0.040008
1.5	0.367610	0.352949	0.335779	0.316291	0.294741	0.271445
2.0	0.685351	0.639583	0.591265	0.541036	0.489598	0.437697
2.5	0.451315	0.396979	0.341862	0.286490	0.231398	0.177121
3.0	0.172764	0.151731	0.131138	0.111156	0.091950	0.073675
3.5	0.030797	0.027059	0.023517	0.020195	0.017112	0.014284
4.0	-0.255414	-0.220805	-0.188586	-0.157658	-0.131694	-0.107149
4.5	0.000105	0.000089	0.000073	0.000060	0.000047	0.000036
5.0	0.000002	0.000002	0.000001	0.000001	0.000001	0.000000



Tables (1, 2) and Fig. (1, 2) represent the solution $\zeta(t)$ for $t \in [0, 4.5]$ with impulses at $t_1 = 1$ and $t_2 = 4$. Tables (3, 4) and Fig. (3, 4) represent the solution $\zeta(t)$ for $t \in [0, 5]$ with impulses at $t_1 = \frac{3}{4}$ and $t_2 = 4$. With the aid of Theorem 5, Equation (12) has a *US* since this simulation makes it evident where the result is.

5 Conclusion

In this paper, we have established the *EU* of solution for a class of *SFIDEs* incorporating the *T φ -CD* with nonlocal condition in a *BS*. By utilizing *TDT*, we have provided a rigorous mathematical foundation for addressing the challenges posed by stochastic perturbations, memory effects, and impulse influences. The inclusion of Gaussian noise enhances the model's relevance in capturing real-world uncertainties and dynamic fluctuations. Our findings contribute to the growing field of *SFIDEs* and extend existing results by incorporating both impulse effects and tempered fractional derivatives. Future research could focus on numerical methods for approximating solutions, and exploring stability properties under different stochastic perturbations.

Author contributions

Each author examined and approved to the final manuscript, and they all volunteered equally to this effort.

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Data availability

No datasets were generated or analysed during the current study.

Declarations

Competing interests

The authors declare no competing interests.

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