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ORIGINAL ARTICLE

On environmental protection, an investigation of a quasilinear evolution system's controllability and optimality

B. Radhakrishnan[1] · **T. Sathya**[1] · **P. Anukokila**[2]

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Abstract

The present study revealed the quasi-linear evolution system's controllability and optimal control outcomes. Starting off with the analytic semi-group theory, controllability results for quasilinear systems were presented. The latter employs fixed-point theorems to arrive at the quasi-linear system's optimum control. Moreover, two concerns related to enviornmental protection are examined. The first one is with optimisation and control theory with an emphasis on minimising environmental harm, while the second one is about numerical models of the dynamics and transformation of air pollutants. Two concrete instances are then provided to demonstrate the application of our key findings.

Keywords Controllability · Optimal control · Quasilinear evolution system · Semi-group theory

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✉ B. Radhakrishnan
radhakrishnanb1985@gmail.com

T. Sathya
sathyathangavel23@gmail.com

P. Anukokila
anuparaman@gmail.com

1 Department of Mathematics, PSG College of Technology, Coimbatore 641004, India

2 Department of Mathematics, PSG College of Arts and Science, Coimbatore 641014, India

1 Introduction

Numerous writers have investigated at the controllability problem for the quasilinear evolution system. The controllability of abstract neutral functional differential systems with indefinite delay has been explained by Wang [25]. The authors Balachandran and Park [6] investigated the local controllability of a quasilinear integrodifferential evolution system

$$x\prime + A(t,x(t))x(t) = Bu(t) + f(t,x(t),\int_0^t g(t,s,x(s))ds),$$

$$x(0) = x_0.$$

The infinitesimal generator of an analytic semi-group is represented by $-A$ in this instance, and the bounded linear operator B maps U into Banach space X. Debbouche [8] explained the controllability of fractional evolution non-local impulsive quasilinear delay integro-differential systems in Banach spaces.

The existence and uniqueness of a classical solution of quasilinear functional differential equation

$$\frac{du(t)}{dt} + A(t,u(t))u(t) = F(t,u_t)t \in [0,T]$$

$$u(0) = \phi \ on \ [\tau,0],$$

where $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau,0]$ was established by Bahuguna [4]. Balachandran et al. [5] examined the existence and uniqueness of the classical solution to the quasilinear integro -differential evolution equations using the C_0 semigroup. By applying Hausdorff's measure of non-compactness, Radhakrishnan [21] explored the existence of a mild solution to the quasilinear integrodifferential impulsive with non-local initial condition in Banach spaces

$$\frac{d}{dt}(x(t) + e(t,x(t),\int_0^t k(t,s,x(s)ds) + A(t,x(t))x(t)$$

$$= f(t,x(t) + \int_0^t g(t,s,x(s))ds$$

$$x(0) + h(x) = x_0, t \neq t_k$$

$$\Delta x(t_k) = I_k(x(t_k)), k = 1,2,...n,$$

where $A : [0,b] \times X \to X$ is a continuous function in Banach space X. Amann [1] and Kato [18] first studied the quasilinear evolution equations with parabolic systems and application to partial differential equations. The existence of classical solutions of abstract quasilinear integrodifferential equations investigated by Oka and Tanaka [19] for the quasilinear evolution equations. The subsequent equation is a prime instance of

$$x_t(t,s) + \Psi(x(t,s))_s = \int_0^t b(t-v)\Psi(x(v,s))_s dv + f(t,s), t \in [0,T],$$

$$x(0,s) = \phi(s)s \in \mathbb{R},$$

arise in a conservation law that is nonlinear and possesses memory [6]. Additionally, [12] investigated the analyticity of quasilinear evolution equation solutions. Examining the optimum control and controllability of these kinds of equations in Banach spaces is fascinating. Optimal control is a branch of mathematics concerned with maximizing or minimizing a system subject to prescribed constraints. It is an outgrowth of a wide range of mathematics. Balders and Flytzanis [10] studied the necessary and sufficient conditions for lower semi continuity of integral functional and the presence of ideal controllers for a class of infinite dimensional nonlinear systems. The occurrence of optimum controls via continuous dependency on parameters was examined by Jakszto [17]. Wei [23] established optimum controllers and nonlinear impulsive integrodifferential equations of mixed type over Banach spaces. The solvability and optimum controls of infinitely delayed fractional integrodifferential evolution systems were examined by Wang [25]. For parabolic equations without beginning conditions and controls in the coefficients, Bokalo [7] investigated the existence of optimum control solutions. The most effective method to regulate a tumor-immune model using immunochemotherapy and time delay was studied **by** [22]. Guechi and colleagues addressed the optimum control issue and approximation controllability of non-local fractional dynamical systems by using sectorial operators and the fixed point theorem [13].

In this paper, we consider the quasilinear evolution system of the form

$$\left.\begin{aligned} x\prime(\eta) &= A(\eta, x(\eta))x(\eta) + f(\eta, x(\eta)) + B(\eta)u(\eta), \eta \in J = [0,T], \\ x(0) &= x_0, \end{aligned}\right\} \tag{1.1}$$

in which the infinitesimal generator for an analytic semi group within a Banach space $\mathbb{X}$ is denoted by $A(\eta, x(\eta))$. **The system** (1.1) **us now used.** To be more precise: The control $u(\cdot)$ in $\mathcal{L}^2(I, \mathcal{U})$, a Banach space containing all admissible controls with $\mathcal{U}$, and the state $x(\cdot)$ take values in Banach space with norm $\|\cdot\|$ and the nonlinear operator $f : J \times \mathbb{X} \to \mathbb{X}$ is uniformly bounded continuous in all arguments and B is the bounded linear operator from $\mathcal{U}$ in to $\mathbb{X}$.

One of the most promising techniques in the literature for solving non-linear differential equations is the homotopy perturbation method (HPM), which may be employed for finding a numerical solution for optimum control problems. He [14] was the first to suggest this approach. He proposed a new perturbation method with homotopy imbedding parameter in [15] and this method is **extended** to the boundary value problems in [16]. The method of solving nonlinear optimal control problem (OCP) leads to a nonlinear two-point boundary value problem (TPBVP) or a hamilton–jacobi-bellman (HJB) equation. By solving a nonlinear TPBVP that is derived from Pontryagin's maximum principle (PMP), the nonlinear OCP may be treated in this manner. The primary benefit of using HPM is that it speeds up the convergence of solutions and produces easily attainable results after a small number of iterations. Using the homotopy perturbation approach, this problem is resolved.

We have emphasized the following are the main contributions of our work:

- The controllability of the quasi-linear evolution system in abstract spaces is investigated in this work, as well as optimal control results are obtained.
- The corresponding system has been optimally controlled by the use of PMP. However relatively few writers apply the semigroup theory for existence, uniqueness, and optimal control results in the majority of the literature that is now available. This demonstrates the apparent innovation of our method.
- In order to clarify our theoretical concepts, this paper also included two numerical examples.

This work is organized as follows: in the third section, the well-posedness and continuous dependency of problem (1.1) is examined. The fourth section will use the Banach fixed point theorem to get the controllability result. In the fifth section, the Mazurs lemma is used to demonstrate optimal control. Lastly, two examples are provided, one of which uses HPM to illustrate a numerical example of a nonlinear optimal control problem. The sixth portion provides a conclusion, whereas the fifth section discusses the governing system's optimal control results. The suggested work in this study on the controllability of quasilinear evolution system with optimal control in abstract space is new in the literature, according to the works. The primary objective of this paper is in establishing such a fact.

2 Preliminaries

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces such that $\mathbb{Y}$ is densely and continuously embedded in $\mathbb{X}$. We recall some definitions and known facts from Pazy [20]. Consider the Cauchy problem for the quasilinear initial problem

$$\left.\begin{aligned} x\prime(t) &= A(t,x(t))x(t)+f(t,x(t)), 0 \leq v \leq t \leq T, \\ x(v) &= y, \end{aligned}\right\} \tag{2.1}$$

where $A(t,x(t))$ is the is the infinitesimal generator of an analytic semi group on a Banach space $\mathbb{X}$. We make the following assumptions.

(P1) The domain $\mathbb{D}(A(t,x(t))) = \mathbb{D}$ of $A(t,x(t)), 0 \leq t \leq T$ is dense in $\mathbb{X}$.

(P2) For $t \in I$, the resolvent $R(\lambda;A(t,x(t))) = (\lambda I - A(t,x(t)))^{-1}$, of $A(t,x(t))$ exists for all λ with $Re\lambda \leq 0$ and there is a constant N_1 such that for $Re\lambda \leq 0, t \in I$,

$$\| R(\lambda;A(t,x(t))) \| \leq N_1[||\lambda|| + 1]^{-1}.$$

(P3) There exist constants N_2 and $0 \leq \alpha \leq 1$ such that for $t, s \in I$,

$$\| A(t, x(t)) - A(s, x(t)) \| \leq N_2 |t - s|^{\alpha}.$$

Let $\mathbb{Y}$ be a Banach space, let $\mathcal{P}$ subset of $\mathbb{X}$ and for every $0 \leq t \leq T$ and $p \in \mathcal{P}$. Let $A(t, p)$ be the infinitesimal generator of analytic semi-group on a Banach space $\mathbb{X}$ and $x \in C([0, T])$ have values in $\mathcal{P}$. Based on this the following theorem gives the uniqueness of the evolution system.

Theorem 2.1 [20] *If the family of operators* $A(t, p), (t, p) \in I \times \mathcal{P}$ *satisfies* $(P1) - (P3)$, *then there is unique evolution system* $U_x(t, s)$ *on* $0 \leq s \leq t \leq T$ ***for which.***

(i) $\| U_x(t, s) \| \leq M_0$(i), for$0 \leq s \leq t \leq T$.
(ii) $U_x(t, s)$(ii) is strongly continuous for $0 \leq s \leq t \leq T$ and

$$U_x(t, r) = U_x(t, s) U_x(s, r), for r \leq s \leq t,$$

$U_x(t, t) = I$.

(iii) For every $w \in \mathbb{D}$, $t \in I$, $U_x(t, s)w$ is differentiable with respect to s on $0 \leq s \leq t \leq T$

$$\frac{\partial}{\partial s} U_x(t, s)w = -U_x(t, s) A(s, x(s))w,$$

(iv) For $0 \leq s \leq t \leq T$, $U_x(t, s) : \mathbb{X} \to \mathbb{D}$ and $t \to U_x(t, s)$ is strongly differentiable in $\mathbb{X}$. The derivative $\frac{\partial}{\partial t} U_x(t, s) \in \mathcal{P}(\mathbb{X})$ and it is strongly continuous on $0 \leq s \leq t \leq T$. Moreover,

$$\frac{\partial U_x(t, s)}{\partial t} + A(t, x) U_x(t, s) = 0, for\, 0 \leq s \leq t \leq T,$$

$$\| \frac{\partial U_x(t, s)}{\partial t} \| = \| A(t, x(s)) U_x(t, s) \| \leq M_0 (t - s)^{-1}$$

$$\| A(t, x) U_x(t, s) A^{-1}(s, x) \| \leq M_0, \quad for\ 0 < s < t < T.$$

From the condition $(A2)$ and the fact that $\mathbb{D}$ is dense in $\mathbb{X}$ imply that for every $t \in [0, T]$, $A(t, x(t))$ is the infinitesimal generator of an analytic semi group.

Define the classical solution of (2.1) as a function $x : [s, T] \to \mathbb{X}$ which is continuous for $s \leq t \leq T$, continuously differentiable for $s < t \leq T$, $x(t) \in \mathbb{D}$ for $s < t \leq T$, $x(s) = y$ and $x\prime(t) = A(t, x(t))x(t) + f(t, x(t))$ holds, for $s < t \leq T$. We will call a function $x(t)$ is a solution of the initial value problem (2.1) if it is a classical solution of the problem.

Theorem 2.2 *Let $A(t, x(t)), 0 \leq t \leq T$ satisfy the conditions $(P1) - (P3)$ and let $U_x(t, s)$ be the evolution system in Theorem 2.1. If f is Holder continuous on $[0, T]$, then the initial value problem (2.1), for each $y \in \mathbb{X}$, has a unique solution $x(t)$ is given by.*

$$x(t) = U_x(t, s)y + \int_s^t U_x(t, r)f(r, x(r))dr.$$

The proof of the above theorem can be found in [11, 20].To establish controllability result the following assumptions are needed:

(H1) (i) $x_0 \in \mathbb{D}(\mathcal{A}_0^{\beta})$, for some $\beta > \alpha$ and $\| \mathcal{A}_0^{\alpha} x_0 \| < \rho$;
(ii) Consider the constant $\mathcal{M}_0$: there iscorresponds to

$$\| U_x(\eta, s) \| \leq \mathcal{M}_0, for 0 \leq s \leq \eta \leq T.$$

(H2) From $\mathcal{L}^2(J_T, U)$ into $\mathbb{X}$, the linear operator W is designated by

$$Wu = \int_0^T U_x(T, s)B(s)u(s)ds,$$

for each fixed $x \in \mathbb{X}$, induces an invertible operator operator W^{-1} defined on $\mathcal{L}^2(J_T, U)/kerW$, and there exists a positive constant $\mathcal{M}_1 > 0$ such that $\| B(s)W^{-1} \| \leq \mathcal{M}_1$.

(H3) (i) There exists a constant $\mathcal{M}_2 > 0$ such that $\| f_x(\eta, y) \| \leq \mathcal{M}_2$;
(ii) For $\eta \in J_T, \mathcal{M}_3 > 0$ such that

$$\| U_x(\eta, s)f_x(\eta, x(\eta)) - U_y(\eta, s)f_y(\eta, y(\eta)) \| \leq \mathcal{M}_3 \| x - y \| .$$

(H4) There exist constants $\mathcal{M}_4, \mathcal{M}_5 > 0$ such that
(i) $\| \mathcal{A}_0^{\alpha} U_x(\tau_1, s) \| \leq \mathcal{M}_4 |\tau_1 - s|^{-\alpha}$.
(ii) $\left\| \mathcal{A}_0^{\alpha} \left[\int_0^{\eta} \left\| U_x(\eta, s)f_x(s, x(s)) - U_y(\eta, s)f_y(s, y(s)) \right\| \right] ds \mathcal{A}_0^{-\beta} \right\| \leq \mathcal{M}_5 |x - y|^{1-\alpha}$.

Definition 2.1 [3] *The system (1.1) is said to be controllable on the interval J_T if and only if for every $x_0, x_T \in \mathbb{X}$, there exists a control $u \in \mathcal{L}^2(J_T, U)$ such that the mild solution $x(\eta)$ of (1.1) satisfies $x(0) = x_0$ and $x(T) = x_T$.*

Definition 2.2 [20] *A function $x(\cdot)$ is said to be a mild solution of the system (1.1)* ***if*** *the following integral equation.*

$x(\eta) = \mathcal{A}_0^\alpha U_x(\eta,0)x_0 + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)f_x(s,x(s))ds + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)B(s)u(s)ds$ is satisfied.

Lemma 2.3 *(Nussbaum fixed point theorem) Let $\mathcal{P}$ be a closed, bounded and convex subset of a Banach space $\mathbb{X}$. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be continuous mapping from $\mathcal{P}$ into $\mathbb{X}$ such that.*

- $(\mathcal{Y}_1 + \mathcal{Y}_2)\mathcal{P} \subset \mathcal{P}$.
- $\| \mathcal{Y}_1 z_1 - \mathcal{Y}_1 z_2 \| \leq \mathcal{Q} \| z_1 - z_2 \|$, for all $z_1, z_2 \in \mathcal{P}$ where $\mathcal{Q} \in [0,1]$ is a constant.
- $\overline{\mathcal{Y}_2(\mathcal{P})}$ is compact. Then the operator $(\mathcal{Y}_1 + \mathcal{Y}_2)$ has a fixed point in $\mathcal{P}$.

3 Controllability result

Theorem 3.1 *If the hypotheses $(H1)-(H4)$ are satisfied, then the system (1.1) is controllable on J_T provided.*

$$\rho M_0 + \mathcal{M}_4|\eta - s|^{-\alpha}\mathcal{M}_1\eta[\| x_T \| - \mathcal{M}_0 \| x_0 \|] < 1.$$

Proof Using $(H2)$ for an arbitrary function $x(\cdot) \in \mathbb{C}(J_T, \mathbb{Y})$, define the control.

$$u(\eta) = W^{-1}\left\{ x_T - U_x(T,0)x_0 - \int_0^T U_x(T,s)f_x(s,x(s))ds \right\}.$$

Define the set $\mathbb{B}_r = \{x \in \mathbb{C}[J_T, \mathbb{X}] : \| x\|_{\mathbb{X}} \leq r\}$ and$\mathcal{Y}_0 = \left\{x(\eta) \in \mathbb{B}_r, \eta \in J_T\right\}$. Clearly $\mathcal{Y}_0$ is a closed and bounded subset of$\mathbb{X}$. Define a mapping $\Psi : \mathcal{Y}_0 \to \mathcal{Y}_0$ by

$$\Psi x(\eta) = \mathcal{A}_0^\alpha U_x(\eta,0)x_0 + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)f_x(s,x(s))ds$$
$$+ \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)B(s) \times W^{-1}\left\{ x_T - U_x(T,0)x_0 - \int_0^T U_x(T,\tau)f_x(s,x(\tau))d\tau \right\}(s)ds$$

has a fixed point and this implies that the system is controllable on J_T. To assert that $\Psi(\mathbb{B}_r) \subset \mathbb{B}_r$, for some positive integer r. If this is not the case, then $\| \Psi x_r \| > r$, that is, for any positive number r, there exists a function $x_r \in \mathbb{B}_r$; nevertheless, $\Psi(x_r)$ does not belong in $\mathbb{B}_r$. However, the equation becomes into $r < \| \Psi x_r(\eta) \|$

$$\leq \|\mathcal{A}_0^\alpha x_0\|\|U_x(\eta,0)\| + \int_0^\eta \|\mathcal{A}_0^\alpha U_x(\eta,s)\|\left\|f_{x_r}\left(s,x_r(s)\right)\right\|ds + \int_0^\eta \|\mathcal{A}_0^\alpha U_x(\eta,s)\|$$

$$\times \left\|B(s)W^{-1}\right\|\left\{\|x_T\| - \|U_x(T,0)x_0\| - \int_0^T \|U_x(T,\tau)\|\left\|f_{x_r}\left(\tau,x_r(\tau)\right)\right\|d\tau\right\}d \leq \rho M_0$$

$$+ \int_0^n , \|A_0^\alpha U_x(\eta,s)\|\left\|f_{x_r}\left(s,x_r(s)\right)\right\|ds + \mathcal{M}_4|\eta - s|^{-\alpha}\mathcal{M}_1\eta\{\|x_T\|$$

$$- \mathcal{M}_0\|x_0\| - \int_0^T \|U_x(T,\tau)\|\left\|f_{x_r}\left(\tau,x_r(\tau)\right)\right\|d\tau\}ds.$$

By dividing both sides by r and using $r \to \infty$ as the limit, one get

$$1 < \rho M_0 + \mathcal{M}_4|\eta - s|^{-\alpha}\mathcal{M}_1\eta[\| x_T \| - \mathcal{M}_0 \| x_0 \|].$$

This runs counter to the requirement. Therefore, $\Psi(\mathbb{B}_r) \subseteq \mathbb{B}_r$, for some positive number $r > 0$. The division of the operator Ψ into Ψ_1 and Ψ_2 is now possible as follows:

$$\left(\Psi_1 x\right)(\eta) = \mathcal{A}_0^\alpha U_x(\eta,0)x_0 + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)f_x(s,x(s))ds$$

$$\left(\Psi_2 x\right)(\eta) = \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)B(s)W^{-1}\{x_T - U_x(T,0)x_0$$

$$- \int_0^T U_x(T,\tau)f_x(T,x(\tau))d\tau\}ds.$$

Next to show that Ψ_1 is a contraction mapping on $\mathbb{B}_r$. For that $x, y \in \mathbb{B}_r$ then, for each $\eta \in J_T$,

$$\begin{aligned}
&\left\|\left(\Psi_1 x\right)(\eta) - \left(\Psi_1 y\right)(\eta)\right\| \\
&\leq \left\|\mathcal{A}_0^\alpha \int_0^\eta \left[\|U_x(\eta,s)f_x(s,x(s)) - U_y(\eta,s)f_y(s,y(s))\right]\right\|ds \\
&\leq \left\|\mathcal{A}_0^\alpha \int_0^\eta \left[\|U_x(\eta,s)f_x(s,x(s)) - U_y(\eta,s)f_y(s,y(s))\right]\mathcal{A}_0^{-\beta}\right\|\left\|\mathcal{A}_0^\beta\right\| \\
&\leq \mathcal{M}_5|x-y|^{1-\alpha}\rho_1,
\end{aligned}$$

where $\Lambda = \mathcal{M}_5|x-y|^{1-\alpha}\rho_1 < 1$. Hence the operator Ψ_1 is a contraction mapping. Now to prove that Ψ_2 is continuous. The operator Ψ_2 is uniformly bounded on $\mathbb{B}_r$. This follows from the inequality

$$\begin{aligned}\left\|\Psi_2 x(\eta)\right\| \le \left\|\mathcal{A}_0^\alpha\right\| \int_0^\eta \left\|U_x(\eta,s)\right\| \left\|B(s)W^{-1}\right\| \Big\{ \left\|x_T\right\| - \left\|U_x(T,0)\right\| \left\|x_0\right\| \\ - \int_0^\eta \left\|U_x(T,\tau)\right\| \left\|f_x(T,x(\tau))\right\| d\tau \Big\} ds \\ \le \left\|\mathcal{A}_0^\alpha\right\| \eta \mathcal{M}_0 \mathcal{M}_1 \left[\left\|x_T\right\| - \mathcal{M}_0\left\|x_0\right\| - T\mathcal{M}_0\mathcal{M}_2\right] \\ \le \sup_{\eta\in J_T} \|\mathcal{P}(\eta)\| \le \mathfrak{S}_0,\end{aligned}$$

where $\mathfrak{S}_0 = \left\|\mathcal{A}_0^\alpha\right\| \eta \mathcal{M}_0 \mathcal{M}_1 \left[\| x_T \| - \mathcal{M}_0 \| x_0 \| - T\mathcal{M}_0\mathcal{M}_2\right]$. Let $\left\{x_q\right\}$ be a sequence in $\mathbb{X}$, such that x_q converges to x. Then

$$\begin{aligned}\left\|\Psi_2 x_q(\eta) - \Psi_2 x(\eta)\right\| \le \left\|\mathcal{A}_0^\alpha\right\| \int_0^\eta \left\|U_{x_q}(\eta,s) - U_x(\eta,s)\right\| \left\|B(s)W^{-1}\right\| \\ \left\{ \left\|x_T\right\| - \left\|U_{x_q}(T,0)x_0 - U_x(T,0)x_0\right\| - \int_0^T \left\|U_{x_q}(T,s) - U_x(T,S)\right\| \times \left\|f_{x_q}(T,x(\tau)) - f_x(T,x(\tau))\right\| d\tau \right\} \\ ds \to 0 as q \to \infty.\end{aligned}$$

Hence theoperator $\Psi_2 x(\eta)$ is continuous. Next to show the operator $\Psi_2 x(\eta)$ is pre compact. Let $\eta > 0$ be fixed, and $\left\{\Psi_2 x_q(\eta) : x_q \in \mathbb{B}_r\right\}$ be a bounded sequence in $\mathbb{B}_r$. Define the operator Ψ_2^μ by

$$\begin{aligned}\Psi_2^\mu x(\eta) = \mathcal{A}_0^\alpha \int_0^{\eta-\mu} U_x(\eta-\mu,s)B(s)W^{-1}\{x_T - U_x(T,0)x_0 \\ - \int_0^T U_x(T,s) f_x(\tau,x(\tau))d\tau\} ds.\end{aligned}$$

It follows that f is completely continuous in $\mathbb{X}$, for every μ, $0<\mu<\eta$ as it is compact at $\eta = 0$ and relatively compact in $\mathbb{X}$, for all $\eta \in \mu$. Furthermore, for every$x \in \mathbb{B}_r$,

$$\| \Psi_2 x(\eta) - \Psi_2^{\mu} x(\eta) \| \leq \int_{\eta}^{\eta-\mu} \| \mathcal{A}_0^{\alpha} U_x(\eta, s) \| \| B(s) W^{-1} \| \{ \| x_T \| - \| U_x(T,0)x_0 \| - \int_0^T \| U_x(T,\tau) \| \| f_x(\tau, x(\tau)) \| d\tau \} ds \to 0 as \mu \to 0.$$

Hence, the set $\{\Psi_2 x(\eta)\}$ is pre-compact in $\mathbb{X}$. Now, let us prove that $\Psi_2(x)$ is equi-continuous. Let $\eta_1, \eta_2 \in J_T, \eta_1 < \eta_2$, and $\mathbb{B}_r$ be a bounded set in $\mathbb{X}$. Let $x \in \mathbb{B}_r$, then

$$\begin{aligned} \| \Psi_2 x(\eta_2) - \Psi_2 x(\eta_1) \| &\leq \mathcal{A}_0^{\alpha} \int_0^{\eta_2-\mu} U_x(\eta_2 - \mu, s) B(s) W^{-1} \{ x_T - U_x(T,0)x_0 \\ &- \int_0^T U_x(T,s) f_x(\tau, x(\tau)) d\tau \} ds \\ &- \mathcal{A}_0^{\alpha} \int_0^{\eta_1-\mu} U_x(\eta_1 - \mu, s) B(s) W^{-1} \{ x_T - U_x(T,0)x_0 \\ &- \int_0^T U_x(T,s) f_x(\tau, x(\tau)) d\tau \} ds \\ &\leq \int_0^{\eta_1} \| A_0^{\alpha} U_x(\eta_1, s) - U_x(\eta_2, s) \| \| B(s) W^{-1} \| \{ \| x_T \| - \| U_x(T,0)x_0 \| \\ &- \int_0^T \| U_x(T,\tau) \| \| f_x(\tau, x(\tau)) \| d\tau \} ds \\ &+ \int_{\eta_1}^{\eta_2} \| A_0^{\alpha} U_x(\eta_2, s) \| \| B(s) W^{-1} \| \{ \| x_T \| - \| U_x(T,0)x_0 \| \\ &- \int_0^T \| U_x(T,\tau) \| \| f_x(\tau, x(\tau)) \| d\tau \} ds \\ &\to 0 as \eta_1 \to \eta_2. \end{aligned}$$

This proves that the set $\{\Psi_2(\mathbb{B}_r)\}$ is equi-continuous. By the bounded convergence theorem, Ψ_2 is a compact operator. Hence by the Nussbaum fixed point theorem there exists a fixed point $x \in \mathbb{B}_r$ such that $(\Psi_1 + \Psi_2)x = x$, which is a solution of the system (1.1). Therefore $(\Psi_1 + \Psi_2)x(T) = x_T$, which implies that the given system is controllable. □

3.1 Optimal controlllability

This section deals the existence of the optimal control for the system (1.1).

$$Problem(P) : \mathcal{J}(x^u, u) = \int_0^T \mathbb{L}(\eta, x^u(\eta), u(\eta))d\eta,$$

where x^u represents the mild solution of (1) that corresponds to the control u in U_{ad}.

Theorem 3.2 [2, 9] *Let $(\mathbb{H}, \beta, \mu)$ be a finite non atomic measure space, $(\mathcal{K}, \| \cdot \|)$ a separable Banach space, and $(\mathcal{V}, |\cdot|)$ a separable reflexive Banach space,and $\mathcal{V}\prime$ its dual. Let $\mathcal{T} : \mathbb{H} \times \mathcal{K} \times \mathcal{V} \to (-\infty, +\infty]$ be a given $\beta \times \mathcal{L}(\mathcal{K} \times \mathcal{V})$ measurable function. The associated integral functional $I_{\mathcal{T}} : \mathcal{L}^1_{\mathcal{K}} \times \mathcal{L}^1_{\mathcal{V}} \to [-\infty, +\infty]$ is defined by.*

$$I_{\mathcal{T}}(y, v) = \int_{\mathbb{H}} \mathcal{T}(\eta, y(\eta), v(\eta))\mu.(d\eta),$$

where $\mathcal{L}^1_{\mathcal{K}}$ denotes the space of all absolutely summable functions from $\mathbb{H}$ to $\mathcal{K}$.

The following three conditions are sufficient for sequential strong–weak lower semi-continuity $I_{\mathcal{T}}$ on $\mathcal{L}^1_{\mathcal{K}} \times \mathcal{L}^1_{V}$.

(e1) $\mathcal{T}(\eta, \cdot, \cdot)$ is sequentially lower semi-continuous on $\mathbb{X} \times \mathcal{V}$, μ.

(e2) There exist $\omega > 0$ and $\phi \in \mathcal{L}^1$ such that.

$\mathcal{T}(\eta, y, d) = \phi(\eta) - \omega(\| y \| + |d|)$, for all $y \in \mathcal{K}, v \in \mathcal{V}$.

(e3) $\mathcal{T}(\eta, x, \cdot)$ is convex on $\mathcal{V}$, for each $x \in \mathcal{K}$ and almost all η.

Moreover, they are also necessary, provided that $I_{\mathcal{T}}(a, v) < \infty$, for some $a \in \mathcal{L}^1_{\mathcal{K}}$, $v \in \mathcal{L}^1_{V}$.

Lemma 3.3 [24] *Let $(u^n)_{n\geq 1} \subset U_{ad}$ and $u^0 \in U_{ad}$ such that $(u^n)_{n\geq 1}$ converges weakly to u^0. Then.*

$$\| Bu^n - Bu^0\|_{\mathcal{L}^2(J_T, U)} \to 0,$$

as $n \to \infty$, if $B \in \mathcal{L}^\infty\left(J_T, \mathcal{U}\right)$.

Theorem 3.1.1 *(Mazur's theorem) ([9]) Let $\mathbb{X}$ be a Banach space and $\mathcal{G}$ be a convex and closed set in $\mathbb{X}$. Then $\mathcal{G}$ is weakly closed in $\mathbb{X}$.*

The following section provides the optimal control result for the quasilinear system (1.1).

3.2 Optimal control result

In order to demonstrate the presence of an optimum control outcome, it is necessary to present the following assumptions:

(A1) (a) The control u takes values from the separable reflexive Banach space $\mathcal{U}$ and assume $B \in \mathcal{L}^{\infty}(J_T : \mathcal{T}(U, \mathbb{X}))$.
(b) The closed, convex, and bounded multivalued map $\mathcal{H} : J_T \to 2^{\mathcal{U}}/\emptyset$ is graph measurable, and $\mathcal{H}$ is a subset of bounded set Θ in U.

(A2) (a) $\mathbb{L} : J_T \times \mathbb{X} \times U \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.
(b) $\mathbb{L}(\eta, \cdot, \cdot)$ is sequentially lower semi-continuous on $\mathbb{X} \times U$ for almost all $\eta \in J_T$.
(c) $\mathbb{L}(\eta, y, \cdot)$ is convex on $\mathcal{U}$ for each $y \in \mathbb{X}$ and almost all $\eta \in J_T$.
(d) There exists a constant $m \in \mathcal{L}^1(J_T)$, $a \geq 0$ and $h_1 > 0$ such that $\mathbb{L}(\eta, x, u) \geq m(\eta) + a \parallel x \parallel + h_1 \parallel u \parallel$.
(e) If $0 < \alpha < h_1$, then for any $0 \leq \tau_1 \leq \eta \leq T$ following inequality holds

$$\parallel A_0^{\alpha} U(\eta, \tau_1) \mathcal{A}_x(\tau_1)^{-h_1} \parallel \leq e_2(\eta - \tau_1)^{h_1 - \alpha}.$$

(A3) There exists a constant e_1 such that $\parallel \mathcal{A}_x(s)^{h_1} \mathcal{A}_0^{-h_1} \parallel \leq e_1$.

Theorem 3.4 *Suppose the assumptions described in (A1) through (A3) are true. The system (1.1) thus has a single optimum pair.*

(x^0, u^0) *on* $\mathbb{C}(J_T, \mathbb{X}) \times U_{ad}$ *that satisfies*

$$\mathcal{J}\big(x^0, u^0\big) \leq \mathcal{J}(x^u, u), for all u^0 \in U_{ad}.$$

Proof. If $\{\mathcal{J}(x^u, u) : u \in \mathcal{U}_{ad}\} = \infty$, then the objective is accomplished. Without loss of generality assume that $\inf\{\mathcal{J}(x^u, u) : u \in \mathcal{U}_{ad}\} = \mathcal{N} < \infty$. It is clear from the assumptions (A1)–(A2) that.

$$\mathcal{J}(x^u, u) \geq \int_0^T m(\eta) d\eta + a \int_0^T \parallel x^u \parallel d\eta + h_1 \int_0^T \parallel u(\eta) \parallel d\eta \geq \mathcal{N} > -\infty.$$

Since $\mathcal{U}_{ad}$ is bounded, which implies that u is bounded and by the definition of an infinimum, there exists a minimizing sequence of the feasible pair $\{x^n, u\} \subset \mathcal{U}_{ad}$, where $\mathcal{U}_{ad}$ is the set of all mild solution of the system (1.1) corresponding to $u \in \mathcal{U}_{ad}$, such that $\mathcal{J}(x^u, u) \to \mathcal{N}$ as $n \to \infty$. Boundedness of $\mathcal{U}_{ad}$ and $\mathcal{L}^2(J_T, \mathcal{U})$ is reflexive which implies that a sub sequence u^i, it converges weakly to $u^0 \in \mathcal{L}^2(J_T, \mathbb{X})$ that is $u^i \to u^0$. But $\mathcal{U}_{ad}$ is closed and convex, hence by Mazurs theorem, it is weakly closed and therefore $u^0 \in \mathcal{U}_{ad}$.

Let x^i be the corresponding sequence of solution which is given by

$$x^i(\eta) = \mathcal{A}_0^\alpha U_x(\eta,0)x_0 + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)[f_x(s,x^i(s))B(s)u^i(s)]ds.$$

Now,

$$\begin{aligned}\| x^i(\eta) \| &\leq \| \mathcal{A}_0^\alpha x_0 \| \| U_x(\eta,0) \| + \int_0^\eta \| \mathcal{A}_0^\alpha U_x(\eta,s) \| \left[\| f_x\left(s,x^i(s)\right) \| \| B(s)u^i(s) \|_{\mathcal{L}^2}\right] ds.\\ &\leq \rho\mathcal{M}_0 + \mathcal{M}_4|\eta-s|^\alpha \eta \mathcal{M}_2 + \mathcal{M}_4|\eta-s|^\alpha \eta \| B(s)u^i(s) \|_{\mathcal{L}^2}\\ &\leq r, i=0,1,2....\end{aligned}$$

Perhaps x^0 is the mild solution of the control $u^0 \in \mathcal{U}_{ad}$, which is provided by

$$x^i(\eta) = \mathcal{A}_0^\alpha U_x(\eta,0)x_0 + \mathcal{A}_0^\alpha \int_0^\eta U_x(\eta,s)\left[f_x\left(s,x^0(s)\right)B(s)u^0(s)\right]ds,$$

then

$$\begin{aligned}\| x^i(\eta) - x^0(\eta) \| &\leq \int_0^\eta \| \mathcal{A}_0^\alpha U_x(\eta,s) \| \| \left[f_x\left(s,x^i(s)\right)B(s)u^i(s)\right.\\ &\quad \left. - f_x\left(s,x^0(s)\right)B(s)u^0(s) \right\| ds\\ &\leq \int_0^\eta \| \mathcal{A}_0^\alpha U_x(\eta,s) \| \| \left\{f_x\left(s,x_s^i(s)\right) - f_x\left(s,x_s^0\right)\right\} - f_x(0) \| ds\\ &\quad + \int_0^\eta \| \mathcal{A}_0^\alpha U_x(\eta,s)\mathcal{A}_x(s)^{-h_1} \| \| \mathcal{A}_x(s)^{h_1}\mathcal{A}_0^{-h_1} \| \| \mathcal{A}_0^h f_x(0) \| ds.\\ &\quad + \int_0^\eta \| \mathcal{A}_0^\alpha U_x(\eta,s) \| \| \left[B(s)u^i(s) - B(s)u^0(s)\right] \| ds.\\ &\leq \mathcal{M}_5 \| x^i(s) - x^0(s) \| + e_1e_2\rho(\eta-s)^{h-\alpha}\eta + \mathcal{M}_4|\eta-s|^{-\alpha}\\ &\quad \times \| \left[B(s)u^i(s) - B(s)u^0(s)\right] \|_{\mathcal{L}^2(J_T,U)}\\ &\leq \mathcal{M}_5 \| x_s^i(s) - x_s^0(s) \|_{\mathcal{L}^2} + S_2 + \mathcal{M}_4|\eta-s|^{-\alpha}\\ &\quad \times \| \left[B(s)u^i(s) - B(s)u^0(s)\right] \|_{\mathcal{L}^2(J_T,U)}.\end{aligned}$$

But $\| x_s^i(\eta) - x_s^0(\eta) \| \leq \sup_{\tau\in[0,s]} \| x_s^i(\eta) - x_s^0(\tau) \|$.

therefore

$$\begin{aligned}\sup_{s\in[0,\eta]} \| x^i(s) - x^0(s) \| &\leq \mathcal{M}_5 \sup_{\tau\in[0,s]} \int_0^\eta \| x^i(s) - x^0(\tau) \|^\mu ds\\ &\quad + \mathcal{M}_4|\eta-s|^{-\alpha}B(s) \| u^i(s) - u^0(s) \|_{\mathcal{L}^2} + S_2\end{aligned}.$$

by Gronwall's inequality $\| x^i(s) - x^0(s) \| \leq S^* \| u^i(s) - u^0(s) \|_{\mathcal{L}^2}$, where $S^* = \mathcal{M}_4|\eta-s|^{-\alpha}e^{\mathcal{M}_5\eta+S_2}, S_2 = e_1e_2\rho(\eta-s)^{h_1-\alpha}\eta$.

It follows from the Lemma 3.3 that,

$$\| x^i - x^0 \| \leq S^* \| u^i - u^0 \|_{L^2(J_T,U)},$$

$$x^i \to x^0.$$

Since the limit is unique, $f\left(s, x^i(s)\right) = f\left(s, x^0(s)\right)$ in $\mathbb{C}\left(J_T, \mathbb{X}\right)$ as $i \to \infty$. Therefore,

$$x(\eta) = U_x(\eta, 0)x_0 + \int_0^\eta U_x(\eta, s)\left[f\left(s, x^0(s)\right) + B(s)u^0(s)\right]ds,$$

which is mild solution of the system (1) corresponding to u^0. Since $\mathbb{C}\left(J_T, \mathbb{X}\right) \to \mathcal{L}^1\left(J_T, \mathbb{X}\right)$, using assumption (A1) – (A3) and Balders theorem,

$$\mathcal{J}(x^u, u) = \lim_{i\to\infty} \int_0^T \mathbb{L}\left(\eta, x^i(\eta), u^i(\eta)\right)d\eta.$$

$$\geq \lim_{i\to\infty} \int_0^T \mathbb{L}\left(\eta, x^0(\eta), u^0(\eta)\right)d\eta$$

$$= \mathcal{J}\left(x^0, u^0\right) \geq \mathcal{N}.$$

This implies that $\mathcal{J}$ attains its minimum at $\left(x^0, u^0\right) \in \mathbb{C}\left(J_T, \mathbb{X}\right) \times U_{ad}$. Hence the proof is completed. □.

3.3 Analysis for Homotopy Perturbation Method

The nonlinear differential equations are solved using the subsequent HPM systematic technique.

Step 1: Consider the nonlinear operator

$$\mathbb{G}(\mathcal{Y}) = \frac{d}{d\eta}\mathcal{Y}(\eta) - \mathcal{E}_1(\mathcal{Y}) - \int_0^1 F_1(x, \eta)\mathcal{E}_2(\mathcal{Y}(\eta))d\eta, \tag{3.1}$$

where $\eta \in (0,1]$, and $\mathcal{Y}$ is a sufficiently smooth function in [0,1].

Step 2: Define the **homotopy** $\mathbb{H}(\mathcal{Y}, P)$, $P \in [0,1]$, such that it satisfies $\mathbb{H}(\mathcal{Y}, 0) = L(\mathcal{Y}) = 0, \mathbb{H}(\mathcal{Y}, 1) = \mathbb{G}(\mathcal{Y}) = 0$. In general $L(\mathcal{Y})$ is considered as a linear operator and nonlinear operator $\mathbb{G}(\mathcal{Y})$ is obtained from the given problem. Now choose a homotopy by using the convex combination of $L(\mathcal{Y})$ and $\mathbb{G}(\mathcal{Y})$ as

$$\mathbb{H}(\mathcal{Y}, P) = (1 - P)L(\mathcal{Y}) + P[\mathbb{G}(\mathcal{Y})] = 0. \tag{3.2}$$

The homotopy perturbation parameter P satisfies $0 \leq P \leq 1$ and it can be increased monotonically from zero to one in such a way that the solution of the problem $L(\mathcal{Y}) = 0$ is continuously deformed to the solution of the original problem $\mathbb{G}(\mathcal{Y}) = 0$.

Step 3: The solution of (3.1) is assumed as a series of P

$$\mathcal{Y} = \mathcal{Y}_0 + P\mathcal{Y}_1 + P^2\mathcal{Y}_2 + ... \tag{3.3}$$

Substituting (3.3) into (3.2) and collecting the same powers of P, the series equations can be obtained in the following form.

$$P^0 : \frac{d}{d\eta}\mathcal{Y}_0(\eta) = 0, P^1 : \frac{d}{d\eta}\mathcal{Y}_1(\eta) = \mathcal{E}_1(\mathcal{Y}) - \int_0^1 F_1(x,\eta)\mathcal{E}_2(\mathcal{Y}(\eta))d\eta \dots,$$

The approximate solution $\mathcal{Y}(\eta)$ is obtained in the form of series

$$\mathcal{Y}(\eta) = \sum_{n=0}^{\infty} \mathcal{Y}_n(\eta).$$

Pontryagin's maximum principle is a set of conditions providing information about solutions to optimal control problems; that is, optimization problems with differential equation constraints. It unifies and extends many classical necessary conditions from the calculus of variations (Martin 2017). It is a powerful tool in optimal control theory. It is utilized to find the best control for carrying a dynamic system from one state to another, particularly if there are constraints on a state or input constraints.

4 Application

The following examples are employed in this section to highlight primary results. Based on Theorem 3.1 and Theorem 3.4, the first example provides the controllability and optimality results. In the second case, the nonlinear optimum control problem is built upon and is solved numerically.

4.1 Example

Consider the following system,

$$\left.\begin{array}{lcl} \frac{\partial}{\partial t}z(t,y) & = & \frac{\partial^2}{\partial y^2}\left(\frac{e^t}{(t+3)^2}\cos y(t)\right)z(t,y) \\ & & + \frac{e^t}{t+5}\left(\frac{\sin|y(t)|}{1+\sin|y(t)|}\right)z(t,y) + \mu(t,y), \\ z(t,0) = z(t,1) & = & 0, t \geq 0 t \in I = [0,b], \\ withcostfunction & & \\ \mathcal{J}(x,u) & = & \int_0^1\int_0^1 u(t,y)|^2 dy + \int_0^1 z(t,y)dy| \end{array}\right\} \tag{4.1}$$

Let us take $\mathbb{X} = \mathcal{U} = \mathcal{L}^2[0,1]$ endowed with usual norm $|\cdot|_{\mathcal{L}^2}$. Put $x(t) = z(t, y)$ and $u(t) = \mu(t, y)$ where $\mu(t, y) : I \times [0,1] \to [0,1]$ is continuous. Consider the function $u \in \mathcal{L}^2[0,1]$ as the control.

$$f(t, y(t)) = \frac{e^t}{t+5}\frac{|y(t)|}{1+|y(t)|}$$

$$A(t, y(t)) = \frac{e^t}{(t+3)^2}\cos y(t).$$

Let $A(t, x(t)) : D(A(t, x(t))) \to \mathbb{X}$ be the operator defined by $Az = z''$, the $A(t, x(t))$ generates a analytic semigroup $U_x(t, s)$ on $\mathbb{X}$. In particular $\mathbb{X} = \mathbb{R}^+, t \in I = [0,1]$.

Let $y \in \mathbb{X}$. Then we have

$$\| A(t, y(t)) \| = \| \frac{e^t}{(t+3)^2}\cos y \| \leq 0.1698.$$

Let us take $x_1, x_2 \in \mathbb{X}$ and $t \in I = [0,1]$.

$$\begin{aligned}
\| f\big(t, x_1(t)\big) - f\big(t, x_2(t)\big) \| &= \| \frac{e^t}{(t+3)^2}\frac{|x_1(t)|}{1+|x_1(t)|} - \frac{e^t}{(t+3)^2}\frac{|x_2(t)|}{1+|x_2(t)|} \| \\
&\leq \| \frac{e^t}{(t+3)^2} \| \| \frac{|x_1(t)|}{1+|x_1(t)|} - \frac{|x_2(t)|}{1+|x_2(t)|} \| \\
&\leq 0.4530 \| x_1 - x_2 \|,
\end{aligned}$$

Therefore the conditions $(H3)$ and $(H4)$ **hold**. More over all conditions stated in Theorem 3.1 are satisfied. With this choice $f, A, and M_2, M_3, M_4$, and $t = 1$, we see that the system (4.1) can be written in the abstract formulation of the system (1.1). Assume that the linear operator W from $\mathcal{L}^2(J, U)/kerW$ into $\mathbb{X}$ defined by

$$Wu = \int_0^T U_x(T, s)C(s)\mu(s)ds$$

has an invertible operator and satisfies condition $(H1)$. In such a way that the constant $\Lambda = 0.5863 < 1$. Hence by Theorem 3.1 the system (6) is controllable on I.

Now consider the system (4.1) for optimal control by using the Theorem 3.4,

$$S_1 = 0.0384, with \alpha = 0.5,$$

$$S_2 = 0 with t, \rho_1, \rho_2, e_1, e_2 = 1$$

and

$$S^* = M_5(1-\alpha)e^{S_1 t + S_2} = 0.0830$$

this implies S^* approaches to zero. Furthermore all conditions expressed in Theorem 3.4 are satisfied. Hence by these Theorems, the system (4.1) is controllable as well as it has at least one optimal pair.

4.2 Example

Consider the partial differential system

$$\left.\begin{array}{ll} \frac{\partial}{\partial t}z(t,y) & = \frac{\partial^2}{\partial y^2}z(t,y) + 0.2z(t,y)^3(t)\csc z(t,y) + 4\mu(t,y), \\ z(t,0) & = 0, z(t,1) = 1.5, t \leq 0, t \in I = [0,b], \\ z(t,y) & = z_0(y) \\ withthecostfunction & \\ \mathcal{J}(x,u) & = \int_0^1 \int_0^1 \mu(t,y)|^2 dy \end{array}\right\} \quad (4.2)$$

where $0 < T \leq 1$. Let us take $\mathbb{X} = \mathcal{U} = \mathcal{L}^2[0,1]$ endowed with usual norm $|\cdot|_{\mathcal{L}^2}$. Put $x(t) = z(t,y)$ and $u(t) = \mu(t,y)$ where $\mu(t,y) : I \times [0,1] \to [0,1]$ is continuous.

Let $A(t,x(t)) : D(A(t,x(t))) \to \mathbb{X}$ be the operator defined by $Az = z''$ and the control $u \in \mathcal{L}^2[0,1]$. In particular $\mathbb{X} = \mathbb{R}^+$.

Now the system (4.2) can be written in the abstract formulation (1.1) with cost function as

$$\left.\begin{array}{ll} \min\mathcal{J} = & \int_0^1 u^2(t)dt \\ \dot{x} & = 0.2x^3(t)\csc x(t) + 4u(t) \\ x(0) & = 0x(1.5) = 0.5. \end{array}\right\} \quad (4.3)$$

Here the aim is to minimize the objective function $\mathcal{J}$.

For that applying the Hamiltonian on (4.3) then the initial value problem,

$$\left.\begin{array}{ll} \dot{x} & = 0.2x^3\csc x(t) - 8\lambda(t) \\ \dot{\lambda} & = -0.2\lambda(t)x^3(t)\csc x(t)\cot x(t) + 0.6\lambda(t)x^2(t)\csc x(t) \\ x(0) & = 0\lambda(1) = \rho, \end{array}\right\} \quad (4.4)$$

has obtained. Where $\rho \in \mathbb{R}$ is an unknown parameter. By using the Pontryagin's maximum principle (PMP), the optimal control law is obtained as,

$$u^*(t) = -2\lambda(t) \quad (4.5)$$

Apply homotopy on (4.4), we get

$$\left.\begin{array}{ll}(1-p)(\dot{v}_1-\dot{x}_0)+p(\dot{v}_1-0.2x^3\csc x(t)+8\lambda(t)) & =0\\ (1-p)(\dot{v}_2-\dot{\lambda}_0)+p(\dot{v}_2+0.2\lambda(t)x^3(t)\csc x(t)\cot x(t)-0.6\lambda(t)x^2(t)\csc x(t)) & =0,\end{array}\right\} \tag{4.6}$$

where $p \in [0,1]$ is an embedding parameter. Then the solution of (4.6) is

$$\left.\begin{array}{ll} v_1 & = x_0+px_1+p^2x_2+...,\\ v_2 & = \lambda_0+p\lambda_1+p^2\lambda_2+...,.\end{array}\right\} \tag{4.7}$$

Substituting (4.7) into (4.6) and equationg the powers of$p$,

$$p^0 : \dot{v}_1(t)-\dot{x}_0(t)=0, v_1(0)=0,$$

$$p^0 : \dot{v}_2(t)-\dot{\lambda}_0(t)=0, v_2(0)=0,$$

$$p^1 : \dot{v}_1(t)+\dot{x}_0(t)-0.2x^3\csc x(t)+8\lambda(t)=0, v_1(0)=0,$$

$$p^1 : \dot{v}_2(t)+\dot{\lambda}_0(t)+0.2\lambda(t)x^3(t)\csc x(t)\cot x(t)-0.6\lambda(t)x^2(t)\csc x(t) \\ =0, v_2(0)=0,$$

$$\vdots$$

$$p^n : \dot{v}_{1n}(t)-0.2\frac{\partial}{\partial p}S_1\left(\sum_{n=0}^{1}p^n v_{1n}, \sum_{n=0}^{1}p^n v_{1n}\right)+8v_{1n}(t)=0, v_{1n}=0,$$

$$p^n : \dot{v}_{2n}(t)+\frac{\partial}{\partial p}S_2\left(\sum_{n=0}^{1}p^n v_{2n}\sum_{n=0}^{1}p^n v_{2n}\right)-0.6\frac{\partial}{\partial p}S_3\left(\sum_{n=0}^{1}p^n v_{2n}\sum_{n=0}^{1}p^n v_{2n}\right) \\ =0, v_{2n}(0)=0.$$

Solving the above equations,the approximate solution has been solved in the form of series,

$$x=1, 0.9497, ..., -2.2414, -2.2338$$

$$\lambda=1.0000, 1.5, ..., 0.1609, 0.1562.$$

From these, the optimal control (u^*) is obtained and it is shown in the Table 1.

Simulation curve of $x(t)$ and the associated optimal control are shown in Figs. 1 and 2 respectively. **The** minimum value ($\mathcal{J}^9$ =)0.000522398 is obtained.

Table 1 Simulation results

i	J(i)	$\| \frac{J(i)-J(i-1)}{J(i)} \|$	i	J(i)	$\| \frac{J(i)-J(i-1)}{J(i)} \|$
1	2.25	0.003325018	14	2.3113121	0.008739365
2	2.2575063	0.0033195	15	2.2912877	0.009715465
3	2.265025	0.003446217	16	2.269241	0.00976289
4	2.2728578	0.003308298	18	2.239213	0.003081146
5	2.280402	0.009206843	19	2.2323348	0.002279507
6	2.3015924	0.00786315	20	2.2272578	0.000938747
7	2.3198336	0.006013058	21	2.2251689	0.001473198
8	2.3338673	0.003915934	22	2.23741764	0.010348189
9	2.3430425	0.001696408	23	2.26081296	0.025185769
10	2.347024	**0.000522398**	24	2.31922441	0.007573752
11	2.3457986	0.002879014	25	2.33692369	–
12	2.3390644	0.004987851	–	–	–
13	2.3274554	0.006984461	–	–	–

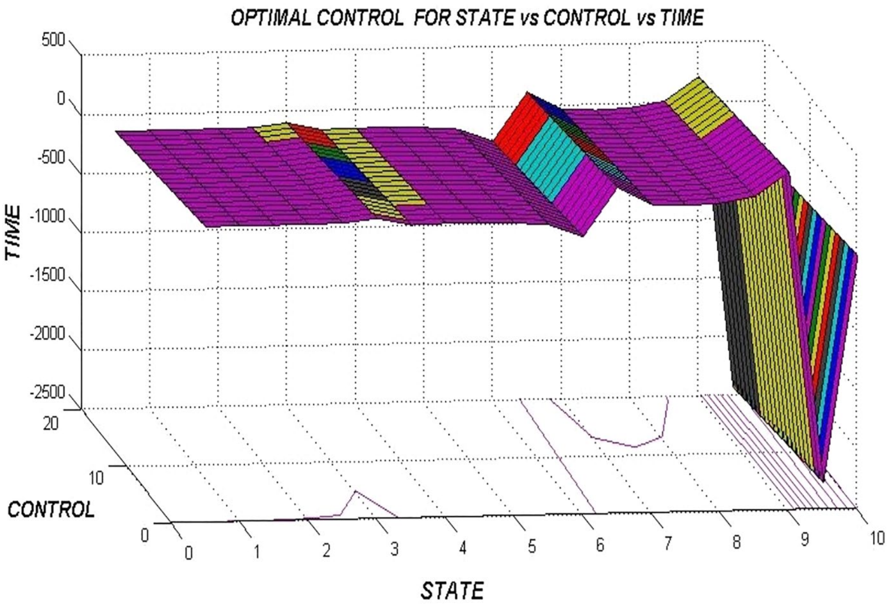

Fig. 1 STATE vs CONTROL versus TIME

5 Results and discussion

The optimal control of the dynamical system is one of the most challenging and hard subjects in control theory. For a linear system, this problem is fully solved and its solution available in many books. For the nonlinear system, the optimal control problem needs some numerical technique to solve. In this article, controllability and optimal

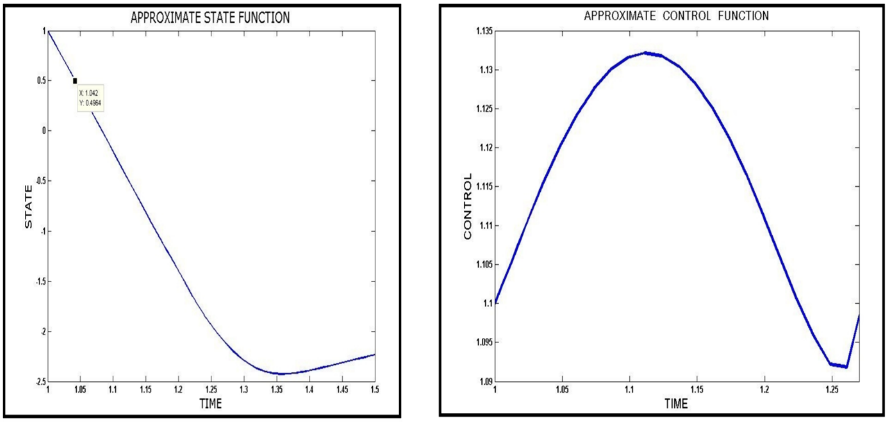

Fig. 2 Approximate state and control function

control has been explained through a particular example (4.1). The nonlinear optimal control problem (4.2) has been investigated numerically and this has been carried out through the use of the HPM. This technique can be applied to a wide class of nonlinear optimal control problems. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions. The initial approximation can be freely chosen with unknown constants. Based on this, Table 1 gives the objective ($\mathcal{J}_i$) values of first 25 controls, similarly the minimum value ($\mathcal{J}^0$ =)0.000522398 is obtained by using the software MATLAB.

6 Conclusion

The existence, uniqueness, and optimal control results for the quasi-linear evolution system in Banach spaces are demonstrated by the authors in the present article. Based on Schauder's fixed-point theorem and the analytic semi group theory, the existence and uniqueness of the quasilinear evolution system are addressed. Additionally, the mild solution for the given system (1.1) has been determined to be the suitable optimum control. Two examples are also provided. Using the abstract results derived from Theorems 3.1 and 3.4, controllability with optimal control was achieved in the first instance. The homotopy perturbation approach was used to solve the system (1.1) in the second example, which is more organized to optimum control. According to this outcome, using MATLAB software to present 2D and 3D graphs is executed successfully. The development of on this notion, the work could ultimately be employed to solve the problem using fractional differential equations and a three-dimensional mesoscale hydrodynamic model to calculate the wind field and turbulence properties in the future.

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Declarations

Conflict of interest Authors don't have a conflict of interest.

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