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Solvability of nonlocal boundary value problems for fractional delay integro-differential equations by using fixed-point technique

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Abstract

In this work, we investigate a class of fractional delay integro-differential equations of Caputo type subject to integral boundary conditions. Using the Burton-Kirk fixed point theorem, we establish sufficient conditions ensuring the existence of solutions, while uniqueness is verified via the Banach contraction principle. The framework involves fractional-order derivatives with delays and a Riemann-Liouville type integral term under nonlocal boundary constraints. To demonstrate the applicability of our results, a concrete example is provided, supported by numerical analysis and graphical representations for enhanced understanding.

Keywords Fractional differential equations · Boundary value problem · Integro-differential equations · Nonlocal condition

Mathematics Subject Classification 26A33 · 34B10 · 34B37 · 45J05

1 Introduction

Fractional differential equations (FDEs) have emerged as powerful tools for modeling complex dynamical systems, particularly in areas such as anomalous diffusion and viscoelasticity. Many of these phenomena cannot be adequately described by classical

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integer-order differential equations, which motivates the use of fractional calculus to capture memory effects and hereditary properties with greater accuracy [12, 15]. In this context, Kilbas [11] investigated Cauchy problems for nonlinear ordinary FDEs, establishing fundamental results on existence and uniqueness. This framework has subsequently been extended to fractional integro-differential equations, which involve both fractional derivatives and fractional integrals, thereby unifying and generalizing ordinary as well as partial differential equations [1]. Furthermore, the application of multidimensional Laplace transforms has facilitated the derivation of closed-form solutions for fractional ODEs and PDEs [10, 17].

In recent years, significant progress has been made in the qualitative theory of functional and fractional-order equations [13, 22], with extensive research addressing stability, boundedness, asymptotic behavior, and almost periodicity. For foundational developments in fractional integrals and derivatives, one may refer to [5, 16, 18]. Among these, delay differential systems are particularly notable for their rich and intricate dynamics, often giving rise to stability issues, oscillatory patterns, and even chaotic behavior [4, 9]. In this context, fractional operators provide a natural and flexible framework for capturing memory-dependent effects and hereditary characteristics.

Boundary conditions (BCs) play a crucial role in determining the qualitative behavior of differential systems under spatial or temporal constraints. Among these, nonlocal boundary conditions are particularly significant, as they incorporate the influence of the entire domain into the boundary behavior of the system [3, 14]. An important subclass is that of integral boundary conditions (IBCs), which encode global solution properties directly within the boundary formulation [20]. Such conditions naturally arise in a variety of applied contexts, including heat and mass transfer, fluid dynamics, population models, and quantum mechanics. For further detailed studies on IBCs, the reader is referred to [23, 24].

The study in [19] addressed initial value and boundary value problems for higher-dimensional nonlinear time-fractional partial differential equations. In contrast, [21] investigated boundary value problems for nonlinear fractional differential equations of order $2 < \beta \leq 3$, establishing existence and uniqueness results by employing b-comparison functions within the framework of complete b-metric spaces.

$$
\begin{aligned}
D^{\beta} u(y) + u(y, u(y)) &= 0, \qquad 0 \leq y \leq 1, \\
u(0) = u'(0) &= 0, \\
u(1) &= \lambda \int_0^1 u(p)dp.
\end{aligned}
$$

Using fixed point techniques, [8] investigated the existence and uniqueness of solutions to fractional differential equations with nonlocal boundary conditions defined on the unit interval.

$$
\begin{aligned}
D^{\beta} u(y) &= f(y, u), \quad 0 \leq y \leq 1, \\
u(0) - au'(0) &= g_1(u), \\
u(1) + bu'(1) &= g_2(u).
\end{aligned}
$$

In a related direction, [7] examined the existence and uniqueness of solutions to fractional differential equations subject to nonlocal and fractional integral boundary conditions on the unit interval. Similarly, [6] established the necessary conditions for the existence of a unique solution to Caputo fractional delay integral boundary value problems defined on [0, 1].

Motivated by the aforementioned contributions, we propose a novel approach to establish the existence of solutions for a system of Caputo-type fractional differential equations. Specifically, we consider a class of complex dynamical systems that integrates fractional delay integro-differential equations with nonlocal boundary conditions of Riemann-Liouville (R-L) integral type, formulated over an arbitrary finite interval.

$$\begin{aligned} {}^{C}D_{0+}^{q_1} v(x) &= A(x)v(x) + f_1\left(x, v_x, \int_0^x f_2\big(x, s, v(s)\big)ds\right), \quad x \in \mathcal{J} := [0, b], \\ v(0) &= \varphi, \\ v'(0) &= a_1 I_{0+}^{q_2} v(\eta), \\ a_2 v(b) + a_3 v'(b) &= f_3(v), \\ v(x) &= f_4(x), \quad -\tau \le x \le 0. \end{aligned} \tag{1.1}$$

Let ${}^{C}D_{0+}^{q_1}$ denote the Caputo fractional derivative of order $q_1 \in (2, 3]$, and let $I_{0+}^{q_2}$ represent the Riemann–Liouville fractional integral of order $q_2 > 0$, defined for $x \in [0, b]$ and $\eta \in [0, 1]$. Consider $A(x)$ to be a bounded linear operator on a Banach space Y. We assume the nonlinear mappings

$$f_1 : \mathcal{J} \times \mathbb{C}[-\tau, 0] \times Y \to Y, \quad f_2 : \Delta \times Y \to Y, \quad f_3 : \mathcal{C}[0, b] \to \mathbb{R},$$

are continuous, with the condition $f_3(0) = 0$, where $\Delta = \{(x, s) : 0 \le s \le x \le b\}$. Furthermore, let $\varphi, a_1, a_2, a_3 \in R$, and $f_4 \in \mathcal{C}[-\tau, 0]$ with $f_4(0) = 0$. If $v : [-\tau, b] \to Y$, then for every $x \in [0, b]$, we define the history segment

$$v_x(\theta) = v(x + \theta), \quad \theta \in [-\tau, 0].$$

For brevity, we introduce the operators

$$F_2 v(x) = \int_0^x f_2\big(x, s, v(s)\big)\, ds, \qquad F(x) = f_1\big(x, v_x, F_2 v(x)\big).$$

One of the key advantages of the proposed system is its ability to model complex processes with memory and delay, such as heat transfer with aftereffects, viscoelastic behavior, and control systems. By establishing existence and uniqueness results for fractional delay integro-differential equations with integral boundary conditions, our work provides a rigorous mathematical foundation for the analysis of such systems. This not only guarantees mathematical consistency but also ensures the stability and predictability of solutions-an essential requirement for the design, control, and optimization of practical processes in science and engineering.

The remainder of this paper is structured as follows. Section 2 reviews the fundamental concepts of Caputo fractional derivatives and Riemann-Liouville integrals, highlighting their essential properties and behavior under different conditions. Section 3 is devoted to the analytical techniques employed to establish the existence and uniqueness of solutions. In Section 4, illustrative examples, together with graphical interpretations, are presented to demonstrate and validate the effectiveness of the proposed framework.

2 Preliminaries and Notation

In this section, we recall some basic definitions and lemmas required for proving the main results.

Here Y denotes the Banach space, $\mathcal{C}(\mathcal{J}, Y)$ represent the Banach space of continuous functions $v(x)$ satisfying the condition $v(x) \in Y$ for $x \in \mathcal{J} = [0, b]$ and $\|v\|_{\mathcal{C}(\mathcal{J},Y)} = \max_{x \in \mathcal{J}} \|v(x)\|$. Let $B(Y)$ denote the Banach space of bounded linear operators from Y into Y with the norm $\|A\|_{B(Y)} = \sup\{\|A(y)\|; \|y\| = 1\}$.

Definition 2.1 [11](*Riemann-Liouville fractional integral*) The Riemann-Liouville fractional integral $I_{0+}^{q_2} v$ of order $q_2 > 0, n-1 < q_2 < n, n \in \mathbb{N}$ of a function $v \in L^1(\mathcal{J})$ is defined by

$$I_{0+}^{q_2} v(s)[x] = I_{0+}^{q_2} v[x] := \frac{1}{\Gamma(q_1)} \int_0^x (x-s)^{q_1-1} v(s) ds,$$

where $\Gamma(q_1)$ is the Euler Gamma function.

Moreover, for $q_1 = 0$, we set $I_{0+}^{q_1} v := v$. For our convenience, let us take $I_{0+}^{q_1} v$ as $I^{q_1} v$.

Definition 2.2 [11](*Caputo fractional derivative*) The Caputo fractional derivative ${}^C D_{0+}^{q_1} v$ of order q_1 of a function $v \in AC^n(\mathcal{J})$ is defined by

$${}^C D_{0+}^{q_1} v(x) = \begin{cases} \frac{1}{\Gamma(m-q_1)} \int\limits_0^x \frac{v^{(m)}(s) ds}{(x-s)^{q_1-m+1}}, & \text{if } q_1 \notin \mathbb{N}, \\ v^{(n)}(x), & \text{if } q_1 \in \mathbb{N}, \end{cases}$$

where $v^{(m)}(x) = \frac{d^m v(x)}{dt^m}, q_1 > 0 (q_1 \notin \mathbb{N}), m = [q_1] + 1$ and $[q_1]$ is the smallest integer greater than or equal to q_1. For our convenience, let us take ${}^C D_{0+}^{q_1} v$ as ${}^C D^{q_1} v$.

Lemma 2.1 [7] *Let $p, q_1 > 0, m = [q_1] + 1$, then the following holds:*

$${}^C D^{q_1} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-q_1+1)} x^{p-q_1}, & (p \in \mathbb{N} \text{ and } p \geq m \text{ or } p \notin \mathbb{N} \text{ and } p > m-1), \\ 0, & p \in \{0, 1, \dots, m-1\}. \end{cases}$$

Lemma 2.2 [2, 7] *Let $p > q > 0$, and $f, g \in L^1(\mathcal{J})$. Then the following holds:*

(i) $I^p I^q f(x) = I^{p+q} f(x)$.
(ii) $I^p I^q f(x) = I^q I^p f(x)$.
(iii) $I^p\{f(x) + g(x)\} = I^p f(x) + I^p g(x)$.
(iv) $I^p\, {}^C D^p f(x) = f(x) - f(0), 0 < p < 1$.
(v) ${}^C D^p I^p f(x) = f(x)$.
(vi) ${}^C D^p f(x) = I^{1-p} D f(x) = I^{1-p} f'(x), 0 < p < 1$.
(vii) ${}^C D^p\, {}^C D^q f(x) \neq {}^C D^{p+q} f(x)$.
(viii) ${}^C D^q\, {}^C D^p f(x) \neq {}^C D^p\, {}^C D^q f(x)$.
(ix) ${}^C D^p I^q f(x) = I^{q-p} f(x)$.

From the above observation, it is clear that the Riemann-Liouville and Caputo fractional differential operators do not possess the semigroup or commutative properties that are inherent to classical integer-order derivatives. For fundamental results and detailed discussions on fractional integrals and derivatives, the reader is referred to [11, 12, 16, 18].

Lemma 2.3 [7] *Let $q_1 > 0$. Then the following holds:*

$$I^{q_1}({}^C D^{q_1} v(x)) = v(x) + \sum_{i=0}^{m-1} b_i x^i,$$
$$\textit{for some } b_i \in \mathbb{R}, i = 0, 1, \cdots, m-1, \textit{ where } m = [q_1] + 1.$$

Lemma 2.4 [7] *Let $q_1 > 0$, $v \in L^1(\mathcal{J}, \mathbb{R})$ Then $\forall x \in \mathcal{J}$, the following holds:*

$$|I^{q_1+1} v(x)| \leq \|I^{q_1} v(x)\|.$$

Lemma 2.5 [7] *The fractional integral I^{q_1}, $q_1 > 0$ is bounded in $L^1(\mathcal{J}, \mathbb{R})$ with*

$$\|I^{q_1} v(x)\| \leq \frac{\|v\|_L}{\Gamma(q_1 + 1)}.$$

Lemma 2.6 [7] (Burton and Kirk fixed-point theorem) *Let $A_1, A_2 : Y \to Y$ be two operators on a Banach space Y, such that A_1 is a contraction and A_2 is completely continuous. Then, either*

(a) *the operator equation $v = A_1(v) + A_2(v)$ has a solution, or*
(b) *the set $\Omega = \{v \in Y : \lambda A_1(\frac{v}{\lambda}) + \lambda A_2(v) = v\}$ is unbounded for $\lambda \in (0, 1)$*

holds.

3 Main Results

In this section, we first present several preliminary lemmas and then establish the existence and uniqueness of solutions to the boundary value problem (BVP) (1.1) by employing fixed point theorems.

Now, we assume the following conditions to prove the required results for the system of equations (1.1).

(H1) $L_1 = b\big(a_2b+2a_3\big)\big(\frac{a_1\eta^{q_2+1}}{\Gamma(q_2+2)}-1\big)-\big(a_2b+a_3\big)\frac{2a_1\eta^{q_2+2}}{\Gamma(q_2+3)} \neq 0, a_2b \neq -2a_3, a_2 \neq 0.$

(H2) There exists $l \in L^1(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|f_1(x,u_1,v_1) - f_1(x,u_2,v_2)\|_Y \leq l(x)\bigg(\|u_1-u_2\|_\infty + \|v_1-v_2\|\bigg),$$

where $v_1, v_2 \in Y, u_1, u_2 \in \mathcal{C}_x = \mathcal{C}[-x,0], x \in \mathcal{J}$.

(H3) Let L_2 be a positive constant such that

$$\|f_2(x,s,u) - f_2(x,s,v)\|_Y \leq L_2\|u-v\|_Y, \text{ where } u, v \in Y.$$

(H4) There exists a positive constant L_3 such that

$$\|f_3(u) - f_3(v)\| \leq L_3\|u-v\|_{\mathcal{J}}, \text{ where } u, v \in \mathcal{C}(\mathcal{J}).$$

(H5) $A : \mathcal{J} \to B(Y)$ is a continuous bounded linear operator and there exists a constant $L_4 > 0$ such that

$$\|A(x)\|_{B(Y)} \leq L_4.$$

(H6) There exists a non-negative function $\bar{p} \in L^1(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|f_1(x,u,v)\|_Y \leq \bar{p}(x)\bigg(1 + \|u\|_\infty + \|v\|\bigg), \text{ where } (x,u,v) \in \mathcal{J} \times \mathcal{C}_x \times Y.$$

For our convenience, let us assume

$$\begin{cases} m_1 = \dfrac{a_1\big(a_2b+2a_3\big)b}{L_1}, & m_2 = -\dfrac{2a_2}{L_1}\left(\dfrac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}\right), & m_3 = \dfrac{m_2a_3}{a_2}, \\ m_4 = -\dfrac{m_2}{a_2}, & m_5 = m_2 - m_1\left(\dfrac{\eta^{q_2}}{\Gamma(q_2+1)}\right), & m_6 = \dfrac{a_1\big(a_2b+a_3\big)}{L_1}, \\ m_7 = \dfrac{m_2(a_2b+a_3)-a_2}{b(a_2b+2a_3)}, & m_8 = \dfrac{m_3(a_2b+a_3)-a_3}{b(a_2b+2a_3)}, & m_9 = \dfrac{-m_4(a_2b+a_3)-1}{b(a_2b+2a_3)}, \\ m_{10} = m_9b - \dfrac{a_1\big(a_2b+a_3\big)\eta^{q_2}}{L_1\Gamma(q_2+1)}. \end{cases} \tag{3.1}$$

Lemma 3.1 *Consider the following system for $y \in \mathcal{C}(\mathcal{J}, Y), q_1 \in (2,3]$,*

$$^{C}D^{q_1}v(x) = y(x), \tag{3.2}$$

with the boundary conditions

$$v(0) = \varphi, \quad v'(0) = a_1I^{q_2}v(\eta), \quad a_2v(b) + a_3v'(b) = f_3(v), \tag{3.3}$$

then the unique solution of the system is given by

$$\begin{aligned}v(x) = {} & I^{q_1}y(x) + (m_6x^2 - m_1x)I^{q_2+q_1}y(\eta) + (m_7x^2 - m_2x)I^{q_1}y(b)\\ & +(m_8x^2 - m_3x)I^{q_1-1}y(b)\\ & +(m_9x^2 - m_4x)f_3(v) + (m_{10}x^2 - m_5x - 1)\varphi.\end{aligned}$$

Proof Applying I^{q_1} to both sides of ${}^CD^{q_1}v(x) = y(x)$, we obtain

$$\begin{aligned}I^{q_1}\left({}^CD^{q_1}v(x)\right) &= I^{q_1}y(x).\\ v(x) &= I^{q_1}y(x) - b_0 - b_1x - b_2x^2, \quad b_0, b_1, b_2 \in \mathbb{R}.\end{aligned}$$

From $v(0) = \varphi$,we know that $-b_0 = \varphi$. Hence we obtain,

$$v(x) = I^{q_1}y(x) + \varphi - b_1x - b_2x^2. \tag{3.4}$$

Also from $v'(0) = a_1I^{q_2}v(\eta)$, we have

$$\left(\frac{a_1\eta^{q_2+1}}{\Gamma(q_2+2)} - 1\right)b_1 + 2\frac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}b_2 = a_1I^{q_2+q_1}y(\eta) - a_1\frac{\eta^{q_2}}{\Gamma(q_2+1)}\varphi.$$

Then by $a_2v(b) + a_3v'(b) = f_3(v)$, we get

$$\left(a_2b + a_3\right)b_1 + \left(a_2b + 2a_3\right)bb_2 = a_2I^{q_1}y(b) + a_3I^{q_1-1}y(b) - f_3(v) + a_2\varphi.$$

Hence,

$$\begin{aligned}b_2 = {} & \frac{-a_1\left(a_2b + a_3\right)}{L_1}I^{q_2+q_1}y(\eta)\\ & +\left(\frac{a_2}{b\left(a_2b + 2a_3\right)} + \frac{2a_1a_2\left(a_2b + a_3\right)\eta^{q_2+2}}{L_1b\left(a_2b + 2a_3\right)\Gamma(q_2+3)}\right)I^{q_2}y(b)\\ & +\left\{\frac{a_1\left(a_2b + a_3\right)\eta^{q_2}}{L_1\Gamma(q_2+1)} - \left(\frac{a_2}{b\left(a_2b + 2a_3\right)} + \frac{2a_1a_2\left(a_2b + a_3\right)\eta^{q_2+2}}{L_1b\left(a_2b + 2a_3\right)\Gamma(q_2+3)}\right)\right\}\varphi\\ & +\left(\frac{a_3}{b\left(a_2b + 2a_3\right)} + \frac{2a_1a_3\left(a_2b + a_3\right)\eta^{q_2+2}}{L_1b\left(a_2b + 2a_3\right)\Gamma(q_2+3)}\right)I^{q_2-1}y(b)\\ & -\left(\frac{1}{b\left(a_2b + 2a_3\right)} + \frac{2a_1\left(a_2b + a_3\right)\eta^{q_2+2}}{L_1b\left(a_2b + 2a_3\right)\Gamma(q_2+3)}\right)f_3(v)\\ = {} & -m_6I^{q_2+q_1}y(\eta) - m_{10}\varphi - m_7I^{q_2}y(b) - m_8I^{q_2-1}y(b) - m_9f_3(v).\\ b_1 = {} & \frac{a_1\left(a_2b + 2a_3\right)b}{L_1}I^{q_2+q_1}y(\eta) - \frac{2a_2}{L_1}\left(\frac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}\right)I^{q_1}y(b)\\ & -\frac{2a_3}{L_1}\left(\frac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}\right)I^{q_1-1}y(b)\\ & +\frac{2}{L_1}\left(\frac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}\right)f_3(v) - \left\{\frac{2a_2}{L_1}\left(\frac{a_1\eta^{q_2+2}}{\Gamma(q_2+3)}\right) - \frac{\left(a_2b + 2a_3\right)b}{L_1}\left(\frac{a_1\eta^{q_2}}{\Gamma(q_2+1)}\right)\right\}\varphi\end{aligned}$$

$$= m_1 I^{q_2+q_1} y(\eta) + m_2 I^{q_1} y(b) + m_3 I^{q_1-1} y(b) + m_4 f_3(v) + m_5 \varphi.$$

Substituting in equation (3.4),

$$\begin{aligned} v(x) = {} & I^{q_1} y(x) + (m_6 x^2 - m_1 x) I^{q_2+q_1} y(\eta) + (m_7 x^2 - m_2 x) I^{q_1} y(b) \\ & + (m_8 x^2 - m_3 x) I^{q_1-1} y(b) \\ & + (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi. \end{aligned}$$

□

According to Lemma 3.1, v solves BVP (1.1) if and only if it satisfies

$$v(x) = \begin{cases} I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] & \\ \quad +(m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] + I^{q_2+q_1} F(s)[\eta]] & \\ \quad +(m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] & \\ \quad +(m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] + I^{q_1-1} F(s)[b]] & \\ \quad +(m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi, & \text{if } x \in \mathcal{J}, \\ f_4(x), & \text{if } x \in [-\tau, 0]. \end{cases}$$

Define an integral operator T:Y $\rightarrow$ Y by

$$(Tv)(x) = \begin{cases} I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] & \\ \quad +(m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] + I^{q_2+q_1} F(s)[\eta]] & \\ \quad +(m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] & \\ \quad +(m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] + I^{q_1-1} F(s)[b]] & \\ \quad +(m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi, & \text{if } x \in \mathcal{J}, \\ f_4(x), & \text{if } x \in [-\tau, 0]. \end{cases} \tag{3.5}$$

Define the operators T_1, T_2 :

$$(T_1 v)(x) = \begin{cases} I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] & \\ \quad +(m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] + I^{q_2+q_1} F(s)[\eta]] & \\ \quad +(m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] & \\ \quad +(m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] + I^{q_1-1} F(s)[b]], & \text{if } x \in \mathcal{J}, \\ f_4(x), & \text{if } x \in [-\tau, 0], \end{cases}$$

$$(T_2 v)(x) = \begin{cases} (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi, & \text{if } x \in \mathcal{J}, \\ 0, & \text{if } x \in [-\tau, 0]. \end{cases}$$

It is obvious that $Tv = T_1 v + T_2 v$.

Lemma 3.2 *Suppose $A(x)$ be a linear operator on Y which is bounded, $f_1 \in C(\mathcal{J} \times \mathbb{C}[-\tau, 0] \times Y, Y)$. Then $v \in Y$ solves BVP (1.1) if and only if $Tv = v$.*

Proof Consider v as a solution of BVP (1.1), satisfying all the equations in (1.1). From Lemma 3.1, we deduce that

$$\begin{aligned} v(x) = {} & I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] + (m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] \\ & + I^{q_2+q_1} F(s)[\eta]] \\ & + (m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] \\ & + (m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] \\ & + I^{q_1-1} F(s)[b]] + (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi = Tv(x). \end{aligned}$$

Conversely, v satisfies

$$\begin{aligned} v(x) &= Tv(x) \\ &= I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] + (m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] \\ &\quad + I^{q_2+q_1} F(s)[\eta]] \\ &\quad + (m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] \\ &\quad + (m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] \\ &\quad + I^{q_1-1} F(s)[b]] + (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi. \\ v(x) &= I^{q_1}\Big[A(s)v(s) + F(s)\Big][x] + (m_6 x^2 - m_1 x) I^{q_2+q_1}\Big[A(s)v(s) + F(s)\Big][\eta] \\ &\quad + (m_7 x^2 - m_2 x) I^{q_1}\Big[A(s)v(s) + F(s)\Big][b] \\ &\quad + (m_8 x^2 - m_3 x) I^{q_1-1}\Big[A(s)v(s) + F(s)\Big][b] \\ &\quad + (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi. \end{aligned}$$

and denote $y(s)[x] = \big[A(s)v(s) + F(s)\big][x]$. Accordingly, the preceding equation may be expressed in the form,

$$\begin{aligned} v(x) = {} & I^{q_1} y(s)[x] + (m_6 x^2 - m_1 x) I^{q_1+q_2} y(s)[\eta] + (m_7 x^2 - m_2 x) I^{q_1} y(s)[b] \\ & + (m_8 x^2 - m_3 x) I^{q_1-1} y(s)[b] + (m_9 x^2 - m_4 x) f_3(v) + (m_{10} x^2 - m_5 x - 1)\varphi. \end{aligned}$$

According to Lemma 3.1, it is clear that $v(x)$ satisfies (3.2) and (3.3). Hence, $v(x)$ satisfies

$$\begin{aligned} {}^{C}D^{q_1} v(x) &= y(s)[x] = A(s)v(s)[x] + F(s)[x], \quad x \in \mathcal{J}, \\ v(0) &= \varphi, \\ v'(0) &= a_1 I^{q_2} v(\eta), \\ a_2 u(b) + a_3 u'(b) &= f_3(v), \\ v(x) &= f_4(x), \quad x \in [-\tau, 0]. \end{aligned}$$

Hence, v satisfies the given BVP (1.1).
Therefore, v is a solution of BVP (1.1) if and only if $Tv = v$. □

Before presenting the main results, we establish some notations that will be used throughout the work.

$$Q_0 = \Big(L_4 + (1 + L_2)\|l\|\Big),$$
$$Q_1 = \frac{1 + |m_7|b^2 + |m_2|b}{\Gamma(q_1)}b + \frac{|m_6|b^2 + |m_1|b}{\Gamma(q_1 + q_2)} + \frac{|m_8|b^2 + |m_3|b}{\Gamma(q_1 - 1)}b,$$
$$Q_2 = (|m_9|b^2 + |m_4|b), \qquad Q_3 = (|m_{10}|b^2 + |m_5|b + 1)|f_4|.$$

We proceed to state our main results. A uniqueness result is obtained by applying the Banach contraction principle.

Theorem 3.1 *Suppose that (H1) - (H5) are satisfied. If*

$$Z_u = Q_0 Q_1 + Q_2 L_3 < 1,$$

then the BVP (1.1) possesses a unique solution on the interval $[-\tau, b]$.

Proof Let $u, v \in Y$ and T be the operator defined in (3.5). Then for each $x \in \mathcal{J}$, we have

$$\begin{aligned}
\|(Tu)(x) - (Tv)(x)\| \leq\; & I^{q_1}\|\big[A(s)u(s) - A(s)v(s)\big][x]\| \\
& + I^{q_1}\|\big[f_1\big(s, u_s, F_2u(s)\big) - f_1\big(s, v_s, F_2v(s)\big)\big][x]\| \\
& + \|m_6x^2 - m_1x\|\Big\{I^{q_2+q_1}\|\big[A(s)u(s) - A(s)v(s)\big][\eta]\| \\
& + I^{q_2+q_1}\|\big[f_1\big(s, u_s, F_2u(s)\big) - f_1\big(s, v_s, F_2v(s)\big)\big][\eta]\|\Big\} \\
& + \|m_7x^2 - m_2x\|\Big\{I^{q_1}\|\big[A(s)u(s) - A(s)v(s)\big][b]\| \\
& + I^{q_1}\|\big[f_1\big(s, u_s, F_2u(s)\big) - f_1\big(s, v_s, F_2v(s)\big)\big][b]\|\Big\} \\
& + \|m_8x^2 - m_3x\|\Big\{I^{q_1-1}\|\big[A(s)u(s) - A(s)v(s)\big][b]\| \\
& + I^{q_1-1}\|\big[f_1\big(s, u_s, F_2u(s)\big) - f_1\big(s, v_s, F_2v(s)\big)\big][b]\|\Big\} \\
& + \|m_9x^2 - m_4x\|\big\{\|f_3(u) - f_3(v)\|\big\}.
\end{aligned}$$

It can be seen from (H2) that,

$$\|f_1(x, u_1, v_1) - f_1(x, u_2, v_2)\|_Y \leq l(x)\Bigg(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|\Bigg).$$

Hence,

$$\|f_1\big(s, u_s, F_2u(s)\big) - f_1\big(s, v_s, F_2v(s)\big)\|_Y \leq l(x)\Bigg(\|u_s - v_s\| + \|F_2u(s) - F_2v(s)\|\Bigg).$$

It can be seen from (H3) that,

$$\|f_2(x,s,u) - f_2(x,s,v)\|_Y \le L_2\|u-v\|_Y.$$

On substituting, we get

$$\begin{aligned}\|f_1\big(s,u_s,F_2u(s)\big) - f_1\big(s,v_s,F_2v(s)\big)\| &\le l(x)\Bigg(\|u_s - v_s\|_\infty + L_2\|u-v\|_Y\Bigg).\\ &\le l(x)(1+L_2)\|u-v\|.\end{aligned}$$

Similarly, $\|A(s)u(s)[x] - A(s)v(s)[x]\| \le \|A(s)\|\|u-v\|[x]$.
Since $A(x)$ is a bounded linear operator and from hypothesis (H5) we have, $\|A(s)\| \le L_4$. Hence $\|A(s)u(s)[x] - A(s)v(s)[x]\| \le L_4\|u-v\|[x]$.
Similarly from (H4),

$$\|f_3(u) - f_3(v)\| \le L_3\|u-v\|.$$

Applying the above, we get

$$\begin{aligned}\|(Tu)(x) - (Tv)(x)\| &\le I^{q_1}\Big(L_4\|u-v\|\Big)[x] + I^{q_1}\Big(l(s)(1+L_2)\|u-v\|\Big)[x]\\ &\quad + \|m_6x^2 - m_1x\|\Big\{I^{q_2+q_1}\Big(L_4\|u-v\|\Big)[\eta] + I^{q_2+q_1}\Big(l(s)(1+L_2)\|u-v\|\Big)[\eta]\Big\}\\ &\quad + \|m_7x^2 - m_2x\|\Big\{I^{q_1}\Big(L_4\|u-v\|\Big)[b] + I^{q_1}\Big(l(s)(1+L_2)\|u-v\|\Big)[b]\Big\}\\ &\quad + \|m_8x^2 - m_3x\|\Big\{I^{q_1-1}\Big(L_4\|u-v\|\Big)[b] + I^{q_1-1}\Big(l(s)(1+L_2)\|u-v\|\Big)[b]\Big\}\\ &\quad + \|m_9x^2 - m_4x\|\big\{L_3\|u-v\|\big\}\\ &\le \Big\{L_4I^{q_1}[x] + (1+L_2)I^{q_1}(l(s))[x]\\ &\quad + \|m_6x^2 - m_1x\|\Big(L_4I^{q_2+q_1}[\eta] + (1+L_2)I^{q_2+q_1}(l(s))[\eta]\Big)\\ &\quad + \|m_7x^2 - m_2x\|\Big(L_4I^{q_1}[b] + (1+L_2)I^{q_1}(l(s))[b]\Big)\\ &\quad + \|m_8x^2 - m_3x\|\Big(L_4I^{q_1-1}[b] + (1+L_2)I^{q_1-1}(l(s))[b]\Big)\Big\}\|u-v\|\\ &\quad + \|m_9x^2 - m_4x\|\big(L_3\|u-v\|\big)\\ &\le \Big\{L_4I^{q_1}[x] + (1+L_2)I^{q_1}(l(s))[x] + (|m_6|b^2 + |m_1|b)\\ &\quad \Big(L_4I^{q_2+q_1}[\eta] + (1+L_2)I^{q_2+q_1}(l(s))[\eta]\Big)\\ &\quad + (|m_7|b^2 + |m_2|b)\Big(L_4I^{q_1}[b] + (1+L_2)I^{q_1}(l(s))[b]\Big)\\ &\quad + (|m_8|b^2 + |m_3|b)\Big(L_4I^{q_1-1}[b] + (1+L_2)I^{q_1-1}(l(s))[b]\Big)\Big\}\|u-v\|\\ &\quad + (|m_9|b^2 + |m_4|b)(L_3\|u-v\|)\end{aligned}$$

$$
\begin{aligned}
&\leq \Bigg\{(L_4+(1+L_2)\|l\|)\Bigg[\frac{b}{\Gamma(q_1)}+\frac{|m_6|b^2+|m_1|b}{\Gamma(q_2+q_1)}\\
&\quad+\frac{|m_7|b^3+|m_2|b^2}{\Gamma(q_1)}+\frac{|m_8|b^3+|m_3|b^2}{\Gamma(q_1-1)}\Bigg]\\
&\quad+(|m_9|b^2+|m_4|b)L_3\Bigg\}\|u-v\|\\
&\leq \Bigg\{Q_0Q_1+Q_2L_3\Bigg\}\|u-v\|.
\end{aligned}
$$

For every $x \in [-\tau, 0]$, we have $\|(Tu)(x)-(Tv)(x)\| = 0$.
Hence for every $x \in [-\tau, b]$, we have

$$\|(Tu)(x)-(Tv)(x)\| \leq \Bigg\{Q_0Q_1+Q_2L_3\Bigg\}\|u-v\|.$$

Owing to $Q_0Q_1+Q_2L_3<1$, it follows that T is a contraction mapping. Consequently, the Banach contraction principle ensures that BVP (1.1) has a unique solution. This concludes the proof. □

We now present some existence results, derived using the Burton and Kirk fixed-point theorem.

Theorem 3.2 *Assume that (H1) - (H6) holds. If*

$$Z_e = Q_1(L_4+\|p\|(1+L_2))+Q_2L_3 < 1 \tag{3.6}$$

then BVP(1.1) possesses at least one solution on the interval $[-\tau, b]$.

Proof Let $T: Y \to Y$ be defined according to (3.5).

Step 1: To prove that $T_1 : Y \to Y$ is continuous. Let $u, v \in Y$. When $u \to v$, namely $\|u-v\| \to 0$, we have

$$
\begin{aligned}
&\sup_{x\in\mathcal{J}} I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[x]\to 0,\\
&\sup_{x\in\mathcal{J}} I^{q_1+q_2}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[\eta]\to 0,\\
&\sup_{x\in\mathcal{J}} I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\to 0,\\
&\sup_{x\in\mathcal{J}} I^{q_1-1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\to 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|T_1u(x)-T_1v(x)| \leq\ & I^{q_1}|A(s)u(s)-A(s)v(s)|[x]+I^{q_1}|f_1(s,u_s,F_2u(s))\\
&-f_1(s,v_s,F_2v(s))|[x]\\
&+|m_6x^2-m_1x|\Big(I^{q_1+x}|A(s)u(s)-A(s)v(s)|[\eta]
\end{aligned}
$$

$$
\begin{aligned}
&+I^{q_1+q_2}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[\eta]\Big)\\
&+|m_7x^2-m_2x|\Big(I^{q_1}|A(s)u(s)-A(s)v(s)|[b]\\
&+I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\Big)\\
&+|m_8x^2-m_3x|\Big(I^{q_1-1}|A(s)u(s)-A(s)v(s)|[b]\\
&+I^{q_1-1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b].
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|T_1u(x)-T_1v(x)\| &= \sup_{x\in\mathcal{J}}|T_1u(x)-T_1v(x)|\\
&\leq \sup_{x\in\mathcal{J}} I^{q_1}|A(s)u(s)-A(s)v(s)|[x]\\
&\quad+\sup_{x\in\mathcal{J}} I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[x]\\
&\quad+\sup_{x\in\mathcal{J}}|m_6x^2-m_1x|\Big(I^{q_1+q_2}|A(s)u(s)-A(s)v(s)|[\eta]\\
&\quad+I^{q_1+q_2}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[\eta]\Big)\\
&\quad+\sup_{x\in\mathcal{J}}|m_7x^2-m_2x|\Big(I^{q_1}|A(s)u(s)-A(s)v(s)|[b]\\
&\quad+I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\Big)\\
&\quad+\sup_{x\in\mathcal{J}}|m_8x^2-m_3x|\Big(I^{q_1-1}|A(s)u(s)-A(s)v(s)|[b]\\
&\quad+I^{q_1-1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\Big)\\
&\leq \sup_{x\in\mathcal{J}} I^{q_1}|A(s)u(s)-A(s)v(s)|[x]\\
&\quad+\sup_{x\in\mathcal{J}} I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[x]\\
&\quad+\sup_{x\in\mathcal{J}}|m_6x^2-m_1x|\Big(I^{q_1+x}|A(s)u(s)-A(s)v(s)|[\eta]\\
&\quad+I^{q_1+q_2}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[\eta]\Big)\\
&\quad+\sup_{x\in\mathcal{J}}|m_7x^2-m_2x|\Big(I^{q_1}|A(s)u(s)-A(s)v(s)|[b]\\
&\quad+I^{q_1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\Big)\\
&\quad+\sup_{x\in\mathcal{J}}|m_8x^2-m_3x|\Big(I^{q_1-1}|A(s)u(s)-A(s)v(s)|[b]\\
&\quad+I^{q_1-1}|f_1(s,u_s,F_2u(s))-f_1(s,v_s,F_2v(s))|[b]\Big).
\end{aligned}
$$

From the above inequality, when $u \to v, \|T_1u(x) - T_1v(x)\| \to 0$, implies the operator T_1 is continuous.

Step 2: Define $B_r = \{v \in Y : \|v\|_Y \le r, r > 0\}$. Let there exist nonzero real constant P_1 such that $P_1 = \max_{(x,s)\in\Delta} \|f_2(x,s,0)\|$. We will prove $T_1(B_r)$ is bounded and equicontinuous. For every $v \in B_r$ and for every $x \in \mathcal{J}$, we have

$$\begin{aligned}|(T_1v)(x)| \le{}& |I^{q_1}A(s)v(s)[x]| + |I^{q_1}F(s)[x]|\\&+|(m_6x^2 - m_1x)|\big[|I^{q_2+q_1}A(s)v(s)[\eta]| + |I^{q_2+q_1}F(s)[\eta]|\big]\\&+|(m_7x^2 - m_2x)|\big[|I^{q_1}A(s)v(s)[b]| + |I^{q_1}F(s)[b]|\big]\\&+|(m_8x^2 - m_3x)|\big[|I^{q_1-1}A(s)v(s)[b]| + |I^{q_1-1}F_1(s)[b]|\big].\end{aligned}$$

By (H3), (H5) and (H6), we get

$$\begin{aligned}|(T_1u)(x)| \le{}& L_4\|v\||I^{q_1}[x]| + |I^{q_1}\bar{p}(s)[x](1 + \|v\| + \|F_2v(s)\|)|\\&+|(m_6x^2 - m_1x)|\big[L_4\|v\||I^{q_2+q_1}[\eta]|\\&+|I^{q_2+q_1}\bar{p}(s)[\eta](1 + \|v\| + \|F_2v(s)\|)|\big]\\&+|(m_7x^2 - m_2x)|\big[L_4\|v\||I^{q_1}[b]| + |I^{q_1}\bar{p}(s)[b](1 + \|v\| + \|F_2v(s)\|)|\big]\\&+|(m_8x^2 - m_3x)|\big[L_4\|v\||I^{q_1-1}[b]|\\&+|I^{q_1-1}\bar{p}(s)[b](1 + \|v\| + \|F_2v(s)\|)|\big].\end{aligned}$$

From (H2), Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned}|(T_1u)(x)| \le{}& L_4\|v\||I^{q_1}[x]| + (1 + \|v\|(1 + L_2) + P_1)|I^{q_1}\bar{p}(s)[x]|\\&+|(m_6x^2 - m_1x)|\big[L_4\|v\||I^{q_2+q_1}[x]|[\eta]|\\&+(1 + \|v\|(1 + L_2) + P_1)|I^{q_2+q_1}\bar{p}(s)[\eta]|\big]\\&+|(m_7x^2 - m_2x)|\big[\|v\||I^{q_1}[b]| + (1 + \|v\|(1 + L_2) + P_1)|I^{q_1}\bar{p}(s)[b]|\big]\\&+|(m_8x^2 - m_3x)|\big[\|v\||I^{q_1-1}[b]|\\&+(1 + \|v\|(1 + L_2) + P_1)|I^{q_1-1}\bar{p}(s)[b]|\big]\\ \le{}& L_4r|I^{q_1}[x]| + (1 + r(1 + L_2) + P_1)|I^{q_1}\bar{p}(s)[x]|\\&+(|m_6|b^2 + |m_1|b)\big[L_4r|I^{q_2+q_1}[x]|[\eta]|\\&+(1 + r(1 + L_2) + P_1)|I^{q_2+q_1}\bar{p}(s)[\eta]|\big]\\&+(|m_7|b^2 + |m_2|b)\big[L_4r|I^{q_1}[b]| + (1 + r(1 + L_2) + P_1)|I^{q_1}\bar{p}(s)[b]|\big]\\&+(|m_8|b^2 + |m_3|b)\big[L_4r|I^{q_1-1}[b]|\\&+(1 + r(1 + L_2) + P_1)|I^{q_1-1}\bar{p}(s)[b]|\big]\\|(T_1u)(x)| \le{}& L_4rQ_1 + \Big(1 + r(1 + L_2) + P_1\Big)\|\bar{p}\|Q_1.\\\|(T_1u)(x)\| \le{}& \sup_{x\in\mathcal{J}}|(T_1u)(x)| \le Q_1\Big(L_4r + \Big(1 + r(1 + L_2) + P_1\Big)\|\bar{p}\|\Big).\end{aligned}$$

Hence $T_1(B_r)$ is bounded.
Now, we will prove that $T_1(B_r)$ is equicontinuous.
Let,

$$\widetilde{M} = \sup_{x\in\mathcal{J}}\{|f_1(x, v, u)| : \|v\| \le r, \|u\| \le L_2 r + P_1\}.$$

Let $x_1, x_2 \in [-\tau, b]$ with $x_1 < x_2$, $v \in B_r$,
Case (i): If $0 \le x_1 < x_2 \le b$, then

$$\begin{aligned}
|(T_1v)(x_2) - (T_1v)(x_1)| \le{}& \frac{1}{\Gamma(q_1)}\int_0^{x_2}(x_2 - s)^{q_1-1}|A(s)v(s)|ds \\
&-\frac{1}{\Gamma(q_1)}\int_0^{x_1}(x_1 - s)^{q_1-1}|A(s)v(s)|ds \\
&+\frac{1}{\Gamma(q_1)}\int_0^{x_2}(x_2 - s)^{q_1-1}|f_1\big(s, v_s, F_2v(s)\big)|ds \\
&-\frac{1}{\Gamma(q_1)}\int_0^{x_1}(x_1 - s)^{q_1-1}|f_1\big(s, v_s, F_2v(s)\big)|ds \\
&+\Big[|m_6|(x_2^2 - x_1^2) + |m_1|(x_2 - x_1)\Big]\Big(I^{q_1+q_2}|A(s)v(s)|[\eta] + I^{q_1+q_2}|F(s)|[\eta]\Big) \\
&+\Big[|m_7|(x_2^2 - x_1^2) + |m_2|(x_2 - x_1)\Big]\Big(I^{q_1}|A(s)v(s)|[b] + I^{q_1}|F(s)|[b]\Big) \\
&+\Big[|m_8|(x_2^2 - x_1^2) + |m_3|(x_2 - x_1)\Big]\Big(I^{q_1-1}|A(s)v(s)|[b] + I^{q_1-1}|F(s)|[b]\Big) \\
\le{}& \frac{1}{\Gamma(q_1)}\int_0^{x_1}\Big((x_2 - s)^{q_1-1} - (x_1 - s)^{q_1-1}\Big)|A(s)v(s)|ds \\
&+\frac{1}{\Gamma(q_1)}\int_{x_1}^{x_2}(x_2 - s)^{q_1-1}|A(s)v(s)|ds \\
&+\frac{1}{\Gamma(q_1)}\int_0^{x_1}\Big((x_2 - s)^{q_1-1} - (x_1 - s)^{q_1-1}\Big)|f_1(s, v_s, F_2v(s))|ds \\
&+\frac{1}{\Gamma(q_1)}\int_{x_1}^{x_2}(x_2 - s)^{q_1-1}|f_1(s, v_s, F_2v(s))|ds \\
&+\big(x_2 - x_1\big)\Big\{\Big[2b|m_6| + |m_1|\Big]\Big(I^{q_1+q_2}|A(s)v(s)|[\eta] + I^{q_1+q_2}|F(s)|[\eta]\Big) \\
&+\Big[2b|m_7| + |m_2|\Big]\Big(I^{q_1}|A(s)v(s)|[b] + I^{q_1}|F(s)|[b]\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big[2b|m_8|+|m_3|\Big]\Big(I^{q_1-1}|A(s)v(s)|[b]+I^{q_1-1}|F(s)|[b]\Big)\Big\}\\
&\le (x_2-x_1)2(L_4r+\widetilde{M})\Big[\frac{1}{\Gamma(q_1)}b^{q_1}+\frac{2b|m_6|+|m_1|}{\Gamma(q_1+q_2)}\eta^{q_1+q_2}\\
&+\frac{2b|m_7|+|m_2|}{\Gamma(q_1)}b^{q_1}+\frac{2b|m_8|+|m_3|}{\Gamma(q_1-1)}b^{q_1-1}\Big].
\end{aligned}
$$

Case (ii): If $-\tau \le x_1 < 0 < x_2 \le b$, then

$$
\begin{aligned}
&|(T_1v)(x_2)-(T_1v)(x_1)| \le |(T_1v)(x_2)-(T_1v)(0)|+|(T_1v)(0)-(T_1v)(x_1)|\\
&\le \frac{1}{\Gamma(q_1)}\int_0^{x_2}(x_2-s)^{q_1-1}|A(s)v(s)|ds\\
&+\frac{1}{\Gamma(q_1)}\int_0^{x_2}(x_2-s)^{q_1-1}|f_1\big(s,v_s,F_2v(s)\big)|ds\\
&+\Big[|m_6|x_2^2+|m_1|x_2\Big]\Big(I^{q_1+q_2}|A(s)v(s)|[\eta]+I^{q_1+q_2}|f_1\big(s,v_s,F_2v(s)\big)|[\eta]\Big)\\
&+\Big[|m_7|x_2^2+|m_2|x_2\Big]\Big(I^{q_1}|A(s)v(s)|[b]+I^{q_1}|f_1\big(s,v_s,F_2v(s)\big)|[b]\Big)\\
&+\Big[|m_8|x_2^2+|m_3|x_2\Big]\Big(I^{q_1-1}|A(s)v(s)|[b]+I^{q_1-1}|f_1\big(s,v_s,F_2v(s)\big)|[b]\Big)\\
&+|f_4(x_1)|\\
&\le (L_4r+\widetilde{M})x_2\Big[\frac{1}{\Gamma(q_1)}b^{q_1}+\frac{b|m_6|+|m_1|}{\Gamma(q_1+q_2)}\eta^{q_1+q_2}+\frac{b|m_7|+|m_2|}{\Gamma(q_1)}b^{q_1}\\
&+\frac{b|m_8|+|m_3|}{\Gamma(q_1-1)}b^{q_1-1}\Big]+|f_4(x_1)|.
\end{aligned}
$$

Case (iii): If $-\tau \le x_1 < x_2 \le 0$, it can be seen from the definition of f_4 that

$$|(T_1v)(x_2)-(T_1v)(x_1)| = |f_4(x_2)-f_4(x_1)|.$$

From Case (i) - Case (iii), it follows that $T_1(B_r)$ is equicontinuous.

Step 3: We now show that T_2 is contraction. Consider $u, v \in Y$. From (H4), we have

$$
\begin{aligned}
|(T_2u)(x)-(T_2v)(x)| &\le |(m_9x^2-m_4x)(f_3(u)-f_3(v))|\\
&\le \Big(|m_9|b^2+|m_4|b\Big)|f_3(u)-f_3(v)|\\
&\le Q_2L_3\|u-v\| \quad \forall x\in\mathcal{J}.\\
\|(T_2u)(x)-(T_2v)(x)\| &\le \sup_{x\in\mathcal{J}}|(T_2u)(x)-(T_2v)(x)|
\end{aligned}
$$

$$\leq \mathcal{Q}_2 L_3 \|u - v\|_Y \leq \|u - v\|.$$

Hence T_2 is contraction.

Step 4: Let $\Omega = \{v \in Y : \lambda T_2(\frac{v}{\lambda}) + \lambda T_1(v) = v, \lambda \in (0, 1)\}$. For every $v \in \Omega$, there exists $\lambda \in (0, 1)$, such that

$$\begin{aligned} v(x) = \lambda\Bigg[& I^{q_1} A(s)v(s)[x] + I^{q_1} F(s)[x] + (m_6 x^2 - m_1 x)[I^{q_2+q_1} A(s)v(s)[\eta] \\ & + I^{q_2+q_1} F(s)[\eta]] + (m_7 x^2 - m_2 x)[I^{q_1} A(s)v(s)[b] + I^{q_1} F(s)[b]] \\ & + (m_8 x^2 - m_3 x)[I^{q_1-1} A(s)v(s)[b] + I^{q_1-1} F(s)[b]] \\ & + (m_9 x^2 - m_4 x) f_3\left(\frac{v}{\lambda}\right) + (m_{10} x^2 - m_5 x - 1)\varphi\Bigg]. \end{aligned}$$

From (H5) and (H6),

$$\begin{aligned} |v(x)| \leq |\lambda|\Bigg[& I^{q_1}|A(s)v(s)|[x] + I^{q_1}|F(s)|[x] + (|m_6|b^2 + |m_1|b)[I^{q_2+q_1}|A(s)v(s)|[\eta] \\ & + I^{q_2+q_1}|F(s)|[\eta]] + (|m_7|b^2 + |m_2|b)[I^{q_1}|A(s)v(s)|[b] + I^{q_1}|F(s)|[b]] \\ & + (|m_8|b^2 + |m_3|b)[I^{q_1-1}|A(s)v(s)|[b] + I^{q_1-1}|F(s)|[b]] \\ & + (|m_9|b^2 + |m_4|b)|f_3\left(\frac{v}{\lambda}\right)| + (|m_{10}|b^2 + |m_5|b + 1)|\varphi|\Bigg] \\ \leq \Bigg[& L_4\|v\| I_1^q[x] + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big) I_1^q[x] \\ & + (|m_6|b^2 + |m_1|b)\Big[L_4\|v\| I^{q_1+q_2}[\eta] \\ & + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big) I^{q_1+q_2}[\eta]\Big] \\ & + (|m_7|b^2 + |m_2|b)\Big[L_4\|v\| I^{q_1}[b] \\ & + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big) I^{q_1}[b]\Big] \\ & + (|m_8|b^2 + |m_3|b)\Big[L_4\|v\| I^{q_1-1}[b] \\ & + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big) I^{q_1-1}[b]\Big]\Bigg] \\ & + (|m_9|b^2 + |m_4|b)|\lambda||f_3(\frac{v}{\lambda})| + (|m_{10}|b^2 + |m_5|b + 1|)|\varphi| \\ \leq\ & L_4\|v\| \mathcal{Q}_1 + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big)\mathcal{Q}_1 \\ & + \mathcal{Q}_2 L_3\|v\| + \mathcal{Q}_3, \qquad \forall x \in \mathcal{J}. \end{aligned}$$

Thus, we have

$$\|v\| \leq L_4\|v\|Q_1 + \Big(\|p\| + \|v\|\,\|p\| + L_2\|v\|\,\|p\| + P_1\|p\|\Big)Q_1 \\ +Q_2L_3\|v\| + Q_3 + \|f_4\|_\infty.$$

Hence, we obtain

$$\|v\| \leq \frac{\|p\|Q_1(1+P_1) + Q_3 + \|f_4\|}{1 - \Big[Q_1(L_4 + \|p\|(1+L_2)) + Q_2L_3\Big]}.$$

It follows that Ω is bounded. Hence the operator T has atleast a fixed point, which is the solution of BVP (1.1). □

4 Numerical Example

In this section, we present an example to demonstrate the application of our results to the class of fractional delay integro-differential equations with and IBCs.

Example 4.1 Consider the fractional delay integro-differential equations associated with IBCs of the type

$$\begin{aligned} {}^C D^{q_1} v(x) &= \frac{1}{80}\left(1 + \sin\left(\frac{\pi x}{2}\right)\right) + \frac{e^{-x}v(x)}{(5+e^x)(1+v(x))} \\ &\quad + \int_0^x \frac{v(s)}{17}ds, \quad 0 < x \leq 3, \\ v(0) &= 1, \\ v'(0) &= I^{0.5}v(0.25), \\ 4v(3) + 6v'(3) &= \frac{1}{5}\int_0^x v(s)ds, \\ v(x) &= \frac{1}{2x^2}, \quad -3 \leq x \leq 0. \end{aligned} \tag{4.1}$$

Solution: In view of problem (4.1), we get $L_1 = -65.568239$, and we observe from equation (3.1) that $m_1 = -1.098093$, $m_2 = 0.001147$, $m_3 = 0.001721$, $m_4 = -2.868206e^{-4}$, $m_5 = 0.620680$, $m_6 = -0.274523$, $m_7 = -0.055269$, $m_8 = -0.082903$, $m_9 = -0.013817$, $m_{10} = 0.113432$.
For every $x \in \mathcal{J} = [0, 3]$, $q_1 = \frac{15}{7}$, we get

$$\|f_1(x,u_1,v_1) - f_1(x,u_2,v_2)\| \leq \left\| \frac{e^{-x}}{(5+e^x)}\left(\frac{u_1(x)}{(1+u_1(x))} - \frac{u_2(x)}{(1+u_2(x))}\right)\right\| \\ + \left\|\Big(F_2(v_1) - F_2(v_2)\Big)\right\|$$

$$\leq \left\| \frac{e^{-x}}{(5+e^{x})} \right\| \Big[\|u_1 - u_2\| + \|F_2(v_1) - F_2(v_2)\| \Big].$$

Similarly for every $x \in \mathcal{J}$, we obtain

$$\|f_2(x,s,u_1) - f_2(x,s,u_2)\| \leq \frac{1}{10} \left\| \int_0^x u_1(s)ds - \int_0^x u_2(s)ds \right\| \leq 0.1 \, \|(u_1 - u_2)\| \, .$$

Hence for $x \in \mathcal{J}$, we obtain

$$\begin{aligned}
\|f_1(x,u_1,v_1) - f_1(x,u_2,v_2)\| &\leq \left\| \frac{e^{-x}}{(5+e^{x})} \right\| \Big[\|u_1 - u_2\| + 0.1 \, \|(u_1 - u_2)\| \Big] \\
&\leq \left\| \frac{1}{(5e^{x} + e^{2x})} + \frac{1}{10} \right\| \|u_1 - u_2\|. \\
\|f_3(v_1) - f_3(v_2)\| &\leq \frac{1}{5} \left\| \int_0^x v_1(s)ds - \int_0^x v_2(s)ds \right\| \leq 0.2 \, \|(v_1 - v_2)\| \, . \\
\|f_1(x,u,v)\| &\leq \left\| \frac{e^{-x}}{(5+e^{x})} \left(\frac{u(x)}{(1+u(x))} \right) \right\| + \left\| \frac{1}{10} \int_0^x v(s)ds \right\| \\
&\leq \left\| \frac{1}{(5e^{x} + e^{2x})} \right\| \Big[1 + \|u\| + \|v\| \Big]. \\
\|A(x)\| &\leq \left\| \frac{1}{80} \left(1 + \sin\left(\frac{\pi x}{2} \right) \right) \right\| \leq \frac{1}{80} \Big(1 + 1 \Big) \leq \frac{1}{40}.
\end{aligned}$$

Hence from the above, we get $\|l\| = 0.0111$, $\|p\| = 0.0023$, $L_2 = 0.0588$, $L_3 = 0.2$, $L_4 = 0.025$, $Q_0 = 0.0451$, $Q_1 = 18.3305$, $Q_2 = 0.1252$, $Q_3 = 3.8829$, and hence from Theorem 3.2,

$$Z_e = Q_1(L_4 + \|p\|(1 + L_2)) + Q_2 L_3 = 0.6807 < 1$$

is satisfied. Similarly from Theorem 3.1,

$$Z_u = Q_1 Q_0 + Q_2 L_3 = 0.8515 < 1$$

is also satisfied. Hence, all the hypotheses are fulfilled. As stated in Theorem 3.1 and Theorem 3.2, the considered problem has a unique continuous solution in $\mathcal{J}$.

It is evident from Figure 1 and Figure 2 that, for the given problem (4.1), the hypotheses of Theorem 3.2 holds true.

Furthermore, it is evident from Figures 3 and 4 that, for the given problem (4.1), the hypotheses of Theorem 3.1 are satisfied. Hence, from Figures 1–4, it follows that there exists a unique continuous solution for the BVP (4.1) in the interval $\mathcal{J}$.

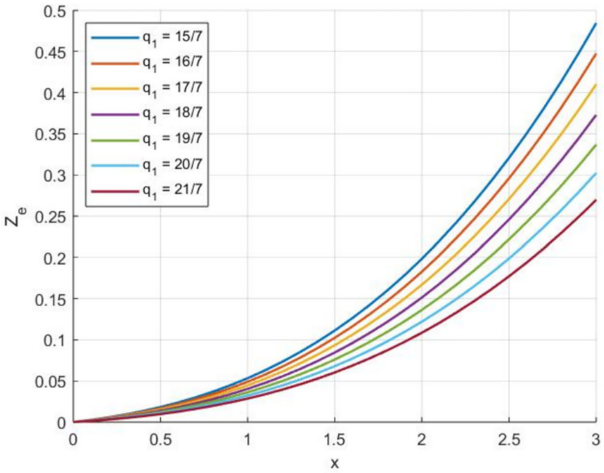

Fig. 1 Z_e values for $x \in [0, 3]$

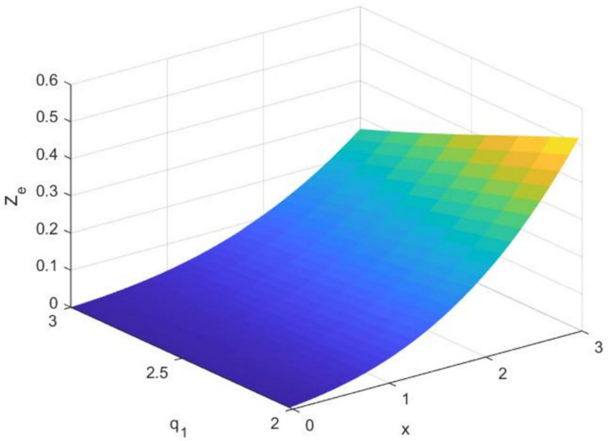

Fig. 2 Z_e values for $x \in [0, 3]$, $q_1 \in (2, 3]$

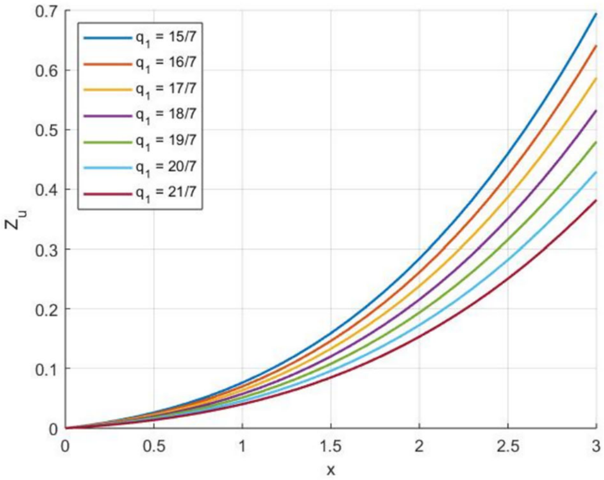

Fig. 3 Z_u values for $x \in [0, 3]$

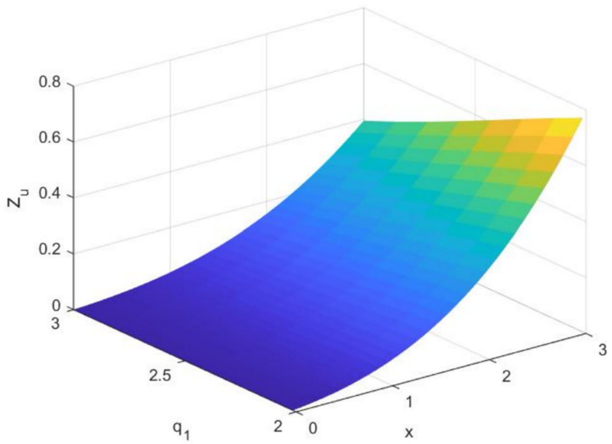

Fig. 4 Z_u values for $x \in [0, 3]$, $q_1 \in (2, 3]$

5 Conclusion

In this study, we have established existence and uniqueness results for fractional delay integro-differential systems of Caputo type subject to nonlocal integral boundary conditions. Uniqueness was demonstrated using the Banach contraction principle, while existence was guaranteed via the Burton-Kirk fixed point theorem. To illustrate the applicability of the theoretical findings, a numerical example was presented, highlighting the effectiveness of the proposed framework in addressing real-world problems. As a direction for future research, this work may be extended to explore controllability and stability properties of such systems, thereby further enhancing their scope and practical relevance.

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Declarations

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