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# Weak solution for time-fractional strongly coupled three species cooperating model

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ABSTRACT

This article investigates the solvability of the time-fractional strongly coupled three-species cooperating system of equations with the Dirichlet boundary conditions. This model expresses the interactions of cooperating species. First, a suitable approximation problem is considered to overcome the strong degeneracy of the original model. Then, we establish the existence of a weak solution for the proposed system using the Faedo–Galerkin method and some compactness arguments.

## 1. Introduction

Modeling and studying biological population dynamics using differential equations is the primary concern of mathematical biology and spatial ecology. Reaction–diffusion equations give rise to these equations, where the reaction term represents inter- and intra-specific equations that include birth and death variables. The word diffusion means the species' random spatial movement. Recent research in this field demonstrates how another may impact the growth of one species. These models fall within the strongly coupled parabolic systems category and are referred to in the literature as cross-diffusion systems, see Refs. 1–7. We breifly review the literature. Wenyan and Ya investigated the existence and nonexistence of the solution of the Lotka–Volterra competition model with cross-diffusions in Ref. 2. Ko and Ahn analyzed a simple food chain model with ratio-dependent functional response cross-diffusion in Ref. 8. Ko and Ryu studied a steady Lotka–Volterra predator–prey system with cross-diffusion in Ref. 9. Xie examined the strongly coupled reaction–diffusion system describing three interacting species in a food chain model in Ref. 10. On the other hand, in recent years, fractional differential equations(FDEs) have been widely used in developing biological models and other fields of science and engineering. In particular, Baleanu et al.[11] proposed a fractional model for the human liver involving the Caputo–Fabrizio derivative with the exponential kernel. Baleanu et al.[12] analyzed the Nipah virus transmission dynamics with the help of the Caputo fractional derivative. Baleanu et al.[13] investigated the real case of a cholera outbreak using the Caputo fractional derivative. The dynamics of the motion of an accelerated mass–spring system within fractional calculus studied in Ref. 14. Jajarmi et al.[15] proposed the new generalization of $\psi-$ Hilfer fractional derivative and examined its basic properties of it.

Shigesda et al.[16] generalized the Lotka–Volterra competing model to describe the spatial segregation of interacting population species in one space dimension. In recent years, much attention has been paid to extending the Shigesda–Kawasaki–Teramoto model in any space dimension. Daoxiang et al.[17] explored a delayed competitor–competitor–mutualist Lotka–Volterra model with infinite integral. Fu et al.[18] studied the existence and uniform boundedness of global solutions for a strongly coupled reaction–diffusion system. Kim and Lin derived the lower and upper bound of blow up estimates for solutions of a parabolic system in Ref. 19. Mimura and Tohma discussed the problem of competitive exclusion or competitor-mediated coexistence using a three-species competition–diffusion system in Ref. 20. Pang and Wang studied a strongly coupled model to understand the dynamics of a two-predator–one-prey ecosystem in Ref. 21. Reisch et al.[22] established the global existence and boundedness of solutions of a reaction–diffusion system modeling liver infections. In Ref. 18, Fu et al. considered the following strongly coupled three-species cooperating model

$$\left.\begin{aligned} u_{1t} &= \Delta(d_1u_1 + \alpha_{11}u_1^2 + \alpha_{12}u_1u_2) + (a_1 - c_1u_1 + e_1u_2)u_1,\\ u_{2t} &= \Delta(d_2u_2 + \alpha_{21}u_1u_2 + \alpha_{22}u_2^2 + \alpha_{23}u_2u_3) + (a_2 + b_2u_1 - c_2u_2 + e_2u_3)u_2,\\ u_{3t} &= \Delta(d_3u_3 + \alpha_{32}u_2u_3 + \alpha_{33}u_3^2) + (a_3 + b_3u_2 - c_3u_3)u_3, \end{aligned}\right\} \tag{1.1}$$

with $x \in \Omega$ which is a bounded domain in $\mathbb{R}^n$ having smooth boundary $\partial\Omega$ and $t > 0$ along with the homogeneous Neumann boundary conditions. Here $\alpha_{ii}, i = 1, 2, 3,$ are self-diffusion pressures and $\alpha_{12}, \alpha_{21}, \alpha_{23}, \alpha_{32}$

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are the cross-diffusion pressures, $a_i, b_i, c_i, d_i, e_i$ are real numbers, $u_i, i = 1, 2, 3$, represent the population densities of three interacting species respectively. They obtained the existence and uniform boundedness of global solutions using energy estimates and Gagliardo–Nirenberg type inequalities and some criteria on the global asymptotic stability of the positive equilibrium points given by the Lyapunov function.

With $\alpha_{ij} = 0$, the above system and the homogeneous Dirichlet boundary conditions have been studied by many authors. One can refer Refs. 19, 23, 24 and references therein. Taking $\alpha_{ij} = 0$ in (1.1), Bhuvaneswari et al.[25] obtained explicit lower bounds for blow-up time for three species cooperating model in a bounded domain in $\mathbb{R}^3$ under different boundary conditions using differential inequality technique under suitable assumptions on the data. Kim and Lin[24] obtained the blow-up estimates for solutions of a semilinear parabolic system that describes the simple food chain model under homogeneous Dirichlet boundary conditions and also obtained the upper and lower bounds of the blow-up rate. They also investigated the three species food chain model in Ref. 19. They proved that solutions exist globally if the intra-specific competitions are strong. In contrast, the blowing up of solutions exists under certain conditions if the intra-specific competitions are weak. Lin et al.[26] considered the two species Lotka–Volterra cooperating model under the homogeneous Dirichlet boundary conditions. They proved the existence of periodic solutions and blow-up of solutions.

By taking all the above papers as motivation, in this work, we consider the model (1.1), with time-fractional derivatives.

$$\left.\begin{aligned}
\partial_t^\alpha u &= \Delta[d_1 u + (\alpha_{11}u + \alpha_{12}v)u] + (a_1 - c_1 u - e_1 v)u && \text{in } Q_T,\\
\partial_t^\alpha v &= \Delta[d_2 v + (\alpha_{21}u + \alpha_{22}v + \alpha_{23}w)v]\\
&\quad + (a_2 - b_2 u - c_2 v - e_2 w)v && \text{in } Q_T,\\
\partial_t^\alpha w &= \Delta[d_3 w + (\alpha_{32}v + \alpha_{33}w)w] + (a_3 - b_3 v - c_3 w)w && \text{in } Q_T,
\end{aligned}\right\} \tag{1.2}$$

along with the initial and boundary conditions

$$\begin{aligned}
&u(x,0) = u_0(x), v(x,0) = v_0(x), w_0(x) = w_0(x) && \text{in } \Omega,\\
&u(x,t) = v(x,t) = w(x,t) = 0, && \text{in } \Sigma_T.
\end{aligned}$$

where $0 < \alpha < 1$, $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, $\Omega$ is a bounded domain in $\mathbb{R}^n, n \leq 3$ with a smooth boundary $\partial\Omega$. Here $u(t), v(t)$, and $w(t)$ represent the population densities of the three cooperating species. The constants $a_i, c_i, i = 1, 2, 3$, are real numbers that represent the intrinsic growth rates and intra-specific competition, respectively. The positive constants $e_1, e_2, b_2, b_3$ are coefficients for inter-specific cooperation. Let $T$ be the maximal existence time. Moreover the Dirichlet boundary condition implies that a hostile environment surrounds the habitat. Here we have assumed that the presence of one species encourages the growth of the preceding one and vice versa.

As per the author's knowledge, there is no article available in the literature to study the time-fractional three-species cooperating model with the Dirichlet boundary conditions. Hence, in this work in Section 2, we established the existence of weak solutions for the time-fractional three species strongly coupled cooperating model using the Faedo–Galerkin method and compactness arguments.

## 2. Weak existence of solutions for time-fractional three species cooperating model

In this section, we prove the existence of a weak solution for the time-fractional three-species cooperating model using the Faedo–Galerkin method, apriori estimates and some compactness arguments. Without loss of generality, the above system (1.2) can be revamped in the following form.

$$\left.\begin{aligned}
\partial_t^\alpha u - \Delta[d_1 u + (\alpha_{11}u + \alpha_{12}v)u] + F_1 &= a_1 u,\\
\partial_t^\alpha v - \Delta[d_2 v + (\alpha_{21}u + \alpha_{22}v + \alpha_{23}w)v] + F_2 &= a_2 v,\\
\partial_t^\alpha w - \Delta[d_3 w + (\alpha_{32}v + \alpha_{33}w)w] + F_3 &= a_3 w,
\end{aligned}\right\} \tag{2.1}$$

where $F_1 := F_1(x,t,u,v,w) = c_1 u^2 + e_1 uv$, $F_2 := F_2(x,t,u,v,w) = b_2 uv + c_2 v^2 + e_2 vw$ and $F_3 := F_3(x,t,u,v,w) = b_3 vw + c_3 w^2$.

**Definition 2.1.** The weak solution $(u,v,w) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ of (2.1) will be of the form,

$$\begin{aligned}
&\int_0^T \langle \partial_t^\alpha u, \phi_1\rangle dt + d_1\int_{Q_T} \nabla u \nabla\phi_1\, dxdt + \int_{Q_T} (\alpha_{11}u + \alpha_{12}v)\nabla u\nabla\phi_1\, dxdt\\
&+ \int_{Q_T} (\alpha_{11}\nabla u + \alpha_{12}\nabla v)u\nabla\phi_1\, dxdt + \int_{Q_T} F_1\phi_1\, dxdt = \int_{Q_T} a_1 u\phi_1\, dxdt,\\
&\int_0^T \langle \partial_t^\alpha v, \phi_2\rangle dt + d_2\int_{Q_T} \nabla v\nabla\phi_2\, dxdt\\
&+ \int_{Q_T} (\alpha_{21}u + \alpha_{22}v + \alpha_{23}w)\nabla v\nabla\phi_2\, dxdt\\
&+ \int_{Q_T} (\alpha_{21}\nabla u + \alpha_{22}\nabla v + \alpha_{23}\nabla w)v\nabla\phi_2\, dxdt\\
&+ \int_{Q_T} F_2\phi_2\, dxdt = \int_{Q_T} a_2 v\phi_2\, dxdt,\\
&\int_0^T \langle \partial_t^\alpha w, \phi_3\rangle dt + d_3\int_{Q_T} \nabla w\nabla\phi_3\, dxdt + \int_{Q_T} (\alpha_{32}v + \alpha_{33}w)\nabla w\nabla\phi_3\, dxdt\\
&+ \int_{Q_T} (\alpha_{32}\nabla v + \alpha_{33}\nabla w)w\nabla\phi_3\, dxdt + \int_{Q_T} F_3\phi_3\, dxdt = \int_{Q_T} a_3 w\phi_3\, dxdt,
\end{aligned}$$

where $\phi_i \in H_0^1(\Omega), i = 1, 2, 3$ and $\langle\cdot,\cdot\rangle$ is the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

**Theorem 2.1.** *The system (2.1) admits a weak solution with an assumption that $u_0, v_0, w_0 \in L^2(\Omega)$.*

Since the problem (2.1) has strong degeneracy in diffusion term, with reference to Ref. 27, we introduce an approximate to (2.1). For $\epsilon > 0$ be a small positive number, let us consider the following regularized system.

$$\left.\begin{aligned}
&\partial_t^\alpha u_\epsilon - d_1\Delta u_\epsilon - div[(2\alpha_{11}f_\epsilon^+(u_\epsilon) + \alpha_{12}f_\epsilon^+(v_\epsilon))\nabla u_\epsilon + \alpha_{12}f_\epsilon^+(u_\epsilon)\nabla v_\epsilon]\\
&\qquad + F_{1,\epsilon} = a_1 u_\epsilon,\\
&\partial_t^\alpha v_\epsilon - d_2\Delta v_\epsilon - div[(\alpha_{21}f_\epsilon^+(u_\epsilon) + 2\alpha_{22}f_\epsilon^+(v_\epsilon) + \alpha_{23}f_\epsilon^+(w_\epsilon))\nabla v_\epsilon\\
&\qquad + \alpha_{21}f_\epsilon^+(u_\epsilon)\nabla u_\epsilon + \alpha_{23}f_\epsilon^+(v_\epsilon)\nabla w_\epsilon] + F_{2,\epsilon} = a_2 v_\epsilon,\\
&\partial_t^\alpha w_\epsilon - d_3\Delta w_\epsilon - div[(\alpha_{32}f_\epsilon^+(v_\epsilon) + 2\alpha_{33}f_\epsilon^+(w_\epsilon))\nabla w_\epsilon + \alpha_{32}f_\epsilon^+(w_\epsilon)\nabla v_\epsilon]\\
&\qquad + F_{3,\epsilon} = a_3 w_\epsilon,
\end{aligned}\right\} \tag{2.2}$$

where $F_{i,\epsilon} := F_{i,\epsilon}(x,t,u_\epsilon,v_\epsilon,w_\epsilon)$, $f_\epsilon(\eta) = \frac{\eta}{1+\epsilon|\eta|}, \eta^+ = \max(0,\eta)$, for any $\eta \in \mathbb{R}$. Let us consider the diffusion matrix $A$ in Box I which is uniformly non negative. Let $\Lambda = [\Lambda_1 \quad \Lambda_2 \quad \Lambda_3]$, where $\Lambda_i \in \mathbb{R}$. Using $\alpha_{11} \geq \frac{\alpha_{12}}{2}$, $\alpha_{22} \geq \frac{\alpha_{21}}{2}, \alpha_{33} \geq \frac{\alpha_{32}}{2}$, $\alpha_{12} = \alpha_{21}, \alpha_{23} = \alpha_{32}$ and also using the inequality $ab \geq a^2 - b^2$, $\forall a, b \in \mathbb{R}$, we get

$$\begin{aligned}
\Lambda A\Lambda^T &= \Lambda_1^2 2\alpha_{11}f_\epsilon^+(u_\epsilon) + \Lambda_1^2\alpha_{12}f_\epsilon^+(v_\epsilon)\\
&\quad + \Lambda_1\Lambda_2\alpha_{21}f_\epsilon^+(v_\epsilon) + \Lambda_1\Lambda_2\alpha_{12}f_\epsilon^+(u_\epsilon)\\
&\quad + \Lambda_2^2\alpha_{21}f_\epsilon^+(u_\epsilon) + 2\Lambda_2^2\alpha_{22}f_\epsilon^+(v_\epsilon)\\
&\quad + \Lambda_2^2\alpha_{23}f_\epsilon^+(w_\epsilon) + \Lambda_2\Lambda_3\alpha_{32}f_\epsilon^+(w_\epsilon)\\
&\quad + \Lambda_2\Lambda_3\alpha_{23}f_\epsilon^+(v_\epsilon) + \Lambda_3^2\alpha_{32}f_\epsilon^+(v_\epsilon) + 2\Lambda^2\alpha_{33}f_\epsilon^+(w_\epsilon)\\
&\geq \Lambda_1^2(2\alpha_{11} - \alpha_{12})f_\epsilon^+(u_\epsilon) + \Lambda_2^2(2\alpha_{22} - \alpha_{21})f_\epsilon^+(v_\epsilon)\\
&\quad + \Lambda_3^2(2\alpha_{33} - \alpha_{32})f_\epsilon^+(w_\epsilon).
\end{aligned} \tag{2.3}$$

We now apply the Faedo Galerkin method to establish the existence result for approximation system (2.2). Consider the spectral problem

$$A = \begin{bmatrix} 2\alpha_{11} f_\epsilon^+(u_\epsilon) + \alpha_{12} f_\epsilon^+(v_\epsilon) & \alpha_{12} f_\epsilon^+(u_\epsilon) & 0 \\ \alpha_{21} f_\epsilon^+(v_\epsilon) & \alpha_{21} f_\epsilon^+(u_\epsilon) + 2\alpha_{22} f_\epsilon^+(v_\epsilon) + \alpha_{23} f_\epsilon^+(w_\epsilon) & \alpha_{23} f_\epsilon^+(v_\epsilon) \\ 0 & \alpha_{32} f_\epsilon^+(w_\epsilon) & \alpha_{32} f_\epsilon^+(v_\epsilon) + 2\alpha_{33} f_\epsilon^+(w_\epsilon) \end{bmatrix},$$

**Box I.**

to find $e \in \Omega$ and a number $k$ such that

$$\left.\begin{array}{rcll} (\nabla e, \nabla\phi) & = & k(e,\phi) & \text{in } \Omega, \\ e & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{2.4}$$

Therefore, the above mentioned spectral problem possess a sequence $\{e_l\}$ of eigenfunctions, which is an orthogonal and orthonormal basis in $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. For a fixed positive integer $n$, let us find functions $u_\epsilon^n$, $v_\epsilon^n$ and $w_\epsilon^n$ for $x \in \bar{\Omega}$, and $t \geq 0$ of the form

$$\begin{aligned} &u_\epsilon^n(x,t) = \sum_{l=1}^n C_{1,n,l}(t) e_l(x), \quad v_\epsilon^n(x,t) = \sum_{l=1}^n C_{2,n,l}(t) e_l(x), \\ &w_\epsilon^n(x,t) = \sum_{l=1}^n C_{3,n,l}(t) e_l(x). \end{aligned} \tag{2.5}$$

We need to find the coefficients $\{C_{j,n,l}\}_{l=1}^n$, $j = 1,2,3$ such that for $m = 1,2,3,\dots,n$, we have

$$\left.\begin{aligned} &\left\langle \partial_t^\alpha u_\epsilon^n, e_m \right\rangle + d_1 \int_\Omega \nabla u_\epsilon^n \nabla e_m dx + \int_\Omega \left(\alpha_{11} f_\epsilon^+(u_\epsilon^n) + \alpha_{12} f_\epsilon^+(v_\epsilon^n)\right) \\ &\quad \times \nabla u_\epsilon^n \nabla e_m dx + \int_\Omega \left(\alpha_{11} \nabla u_\epsilon^n + \alpha_{12} \nabla v_\epsilon^n\right) f_\epsilon^+(u_\epsilon^n) \nabla e_m dx \\ &\quad + \int_\Omega F_{1,\epsilon} e_m dx = \int_\Omega a_1 u_\epsilon^n e_m dx, \\ &\left\langle \partial_t^\alpha v_\epsilon^n, e_m \right\rangle + d_2 \int_\Omega \nabla v_\epsilon^n \nabla e_m dx \\ &\quad + \int_\Omega \left(\alpha_{21} f_\epsilon^+(u_\epsilon^n) + \alpha_{22} f_\epsilon^+(v_\epsilon^n) + \alpha_{23} f_\epsilon^+(w_\epsilon^n)\right) \\ &\quad \times \nabla v_\epsilon^n \nabla e_m dx + \int_\Omega \left(\alpha_{21} \nabla u_\epsilon^n + \alpha_{22} \nabla v_\epsilon^n + \alpha_{23} \nabla w_\epsilon^n\right) f_\epsilon^+(v_\epsilon^n) \nabla e_m dx \\ &\quad + \int_\Omega F_{2,\epsilon} e_m dx = \int_\Omega a_2 v_\epsilon^n e_m dx, \\ &\left\langle \partial_t^\alpha w_\epsilon^n, e_m \right\rangle + d_3 \int_\Omega \nabla w_\epsilon^n \nabla e_m dx \\ &\quad + \int_\Omega \left(\alpha_{32} f_\epsilon^+(v_\epsilon^n) + \alpha_{33} f_\epsilon^+(w_\epsilon^n)\right) \nabla w_\epsilon^n \nabla e_m dx \\ &\quad + \int_\Omega \left(\alpha_{32} \nabla v_\epsilon^n + \alpha_{33} \nabla w_\epsilon^n\right) f_\epsilon^+(w_\epsilon^n) \nabla e_m dx \\ &\quad + \int_\Omega F_{3,\epsilon} e_m dx = \int_\Omega a_3 w_\epsilon^n e_m dx, \end{aligned}\right\} \tag{2.6}$$

and the initial conditions,

$$\begin{aligned} u_\epsilon^n(x,0) &:= u_{0,n}(x) = \sum_{l=1}^n C_{1,n,l}(0) e_l(x), \\ v_\epsilon^n(x,0) &:= v_{0,n}(x) = \sum_{l=1}^n C_{2,n,l}(0) e_l(x), \\ w_\epsilon^n(x,0) &:= w_{0,n}(x) = \sum_{l=1}^n C_{3,n,l}(0) e_l(x). \end{aligned}$$

On the boundary, by using (2.4) we have $u_\epsilon^n(x,t) = v_\epsilon^n(x,t) = w_\epsilon^n(x,t) = 0$. Using normality condition, we write (2.6) as follows

$$\left.\begin{aligned} {}_0^C D_t^\alpha C_{1,n,k} &= -d_1 \int_\Omega \nabla u_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{11} f_\epsilon^+(u_\epsilon^n) + \alpha_{12} f_\epsilon^+(v_\epsilon^n)\right) \nabla u_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{11} \nabla u_\epsilon^n + \alpha_{12} \nabla v_\epsilon^n\right) f_\epsilon^+(u_\epsilon^n) \nabla e_m dx \\ &\quad - \int_\Omega F_{1,\epsilon} e_m dx + \int_\Omega a_1 u_\epsilon^n e_m dx \\ &:= G_1^m(t, \{C_{1,n,l}\}_{l=1}^n, \{C_{2,n,l}\}_{l=1}^n, \{C_{3,n,l}\}_{l=1}^n), \\ {}_0^C D_t^\alpha C_{2,n,k} &= -d_2 \int_\Omega \nabla v_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{21} f_\epsilon^+(u_\epsilon^n) + \alpha_{22} f_\epsilon^+(v_\epsilon^n) + \alpha_{23} f_\epsilon^+(w_\epsilon^n)\right) \\ &\quad \times \nabla u_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{21} \nabla u_\epsilon^n + \alpha_{22} \nabla v_\epsilon^n + \alpha_{23} \nabla w_\epsilon^n\right) f_\epsilon^+(v_\epsilon^n) \nabla e_m dx \\ &\quad - \int_\Omega F_{2,\epsilon}(x,t,u_\epsilon^n, v_\epsilon^n, w_\epsilon^n) e_m dx + \int_\Omega a_2 v_\epsilon^n e_m dx \\ &:= G_2^m(t, \{C_{1,n,l}\}_{l=1}^n, \{C_{2,n,l}\}_{l=1}^n, \{C_{3,n,l}\}_{l=1}^n), \\ {}_0^C D_t^\alpha C_{3,n,k} &= -d_3 \int_\Omega \nabla w_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{32} f_\epsilon^+(v_\epsilon^n) + \alpha_{33} f_\epsilon^+(w_\epsilon^n)\right) \nabla w_\epsilon^n \nabla e_m dx \\ &\quad - \int_\Omega \left(\alpha_{32} \nabla v_\epsilon^n + \alpha_{33} \nabla w_\epsilon^n\right) f_\epsilon^+(w_\epsilon^n) \nabla e_m dx \\ &\quad - \int_\Omega F_{3,\epsilon}(x,t,u_\epsilon^n, v_\epsilon^n, w_\epsilon^n) e_m dx + \int_\Omega a_3 w_\epsilon^n e_m dx \\ &:= G_3^m(t, \{C_{1,n,l}\}_{l=1}^n, \{C_{2,n,l}\}_{l=1}^n, \{C_{3,n,l}\}_{l=1}^n). \end{aligned}\right\} \tag{2.7}$$

The system of Eqs. (2.7) is a system of nonlinear fractional ordinary differential equations. Here all $G_j^m, j = 1,2,3$ are continuous functions of $C_{j,n,l}(t)$. Using standard fractional order differential equations theory, (2.7) has a local solution in the form of (2.6). Given absolutely continuous coefficients $b_{j,n,l}(t)c_l(x), j = 1,2,3$, we set

$$\phi_{j,n}(t,x) = \sum_{l=1}^n b_{j,n,l}(t) c_l(x), j = 1,2,3.$$

It follows from (2.6), that the finite dimensional solution satisfies the following weak formulations for each fixed $t$,

$$\left.\begin{aligned}
&\int_\Omega \partial_t^\alpha u_\epsilon^n \phi_{1,n} dx + d_1 \int_\Omega \nabla u_\epsilon^n \nabla \phi_{1,n} dx \\
&\quad + \int_\Omega \Big(\alpha_{11} f_\epsilon^+(u_\epsilon^n) + \alpha_{12} f_\epsilon^+(v_\epsilon^n)\Big) \nabla u_\epsilon^n \nabla \phi_{1,n} dx \\
&\quad + \int_\Omega \Big(\alpha_{11} \nabla u_\epsilon^n + \alpha_{12} \nabla v_\epsilon^n\Big) f_\epsilon^+(u_\epsilon^n) \nabla \phi_{1,n} dx + \int_\Omega F_{1,\epsilon} \phi_{1,n} dx \\
&= \int_\Omega a_1 u_\epsilon^n \phi_{1,n} dx, \\
&\int_\Omega \partial_t^\alpha v_\epsilon^n \phi_{2,n} dx + d_2 \int_\Omega \nabla v_\epsilon^n \nabla \phi_{2,n} dx \\
&\quad + \int_\Omega \Big(\alpha_{21} f_\epsilon^+(u_\epsilon^n) + \alpha_{22} f_\epsilon^+(v_\epsilon^n) + \alpha_{23} f_\epsilon^+(w_\epsilon^n)\Big) \\
&\quad \times \nabla v_\epsilon^n \nabla \phi_{2,n} dx + \int_\Omega \Big(\alpha_{21} \nabla u_\epsilon^n + \alpha_{22} \nabla v_\epsilon^n + \alpha_{23} \nabla w_\epsilon^n\Big) f_\epsilon^+(v_\epsilon^n) \nabla \phi_{2,n} dx \\
&\quad + \int_\Omega F_{2,\epsilon} \phi_{2,n} dx = \int_\Omega a_2 v_\epsilon^n \phi_{2,n} dx, \\
&\int_\Omega \partial_t^\alpha w_\epsilon^n \phi_{3,n} dx + d_3 \int_\Omega \nabla w_\epsilon^n \nabla \phi_{3,n} dx + \int_\Omega \Big(\alpha_{32} f_\epsilon^+(v_\epsilon^n) + \alpha_{33} f_\epsilon^+(w_\epsilon^n)\Big) \\
&\quad \times \nabla w_\epsilon^n \nabla \phi_{3,n} dx + \int_\Omega \Big(\alpha_{32} \nabla v_\epsilon^n + \alpha_{33} \nabla w_\epsilon^n\Big) f_\epsilon^+(w_\epsilon^n) \nabla \phi_{3,n} dx \\
&\quad + \int_\Omega F_{3,\epsilon} \phi_{3,n} dx = \int_\Omega a_3 w_\epsilon^n \phi_{3,n} dx.
\end{aligned}\right\} \tag{2.8}$$

Upon taking $\phi_{1,n} = u_\epsilon^n; \phi_{2,n} = v_\epsilon^n; \phi_{3,n} = w_\epsilon^n$ and using the Lemma (2.3) in Zhou and Peng,[28] we get

$$\begin{aligned}
&{}_0^C D_t^\alpha \int_\Omega \Big(|u_\epsilon^n|^2 + |v_\epsilon^n|^2 + |w_\epsilon^n|^2\Big) dx \\
&+ \int_\Omega \Big(d_1 |\nabla u_\epsilon^n|^2 + d_2 |\nabla v_\epsilon^n|^2 + d_3 |\nabla w_\epsilon^n|^2\Big) dx \\
&+ \int_\Omega \Big(F_{1,\epsilon} u_\epsilon^n + F_{2,\epsilon} v_\epsilon + F_{3,\epsilon} w_\epsilon\Big) dx \le C \int_\Omega \Big(|u_\epsilon^n|^2 + |v_\epsilon^n|^2 + |w_\epsilon^n|^2\Big) dx,
\end{aligned}$$

where $C$ is a positive constant which is independent of $n$. Therefore, we get

$$\int_\Omega \Big(|u_\epsilon^n|^2 + |v_\epsilon^n|^2 + |w_\epsilon^n|^2\Big) dx \le C, \tag{2.9}$$

for some positive constant $C$ that is independent of $n$. From the above equation we conclude that

$$\|u_\epsilon^n\|_{L^\infty(0,\tilde{T};L^2(\Omega))} + \|v_\epsilon^n\|_{L^\infty(0,\tilde{T};L^2(\Omega))} + \|w_\epsilon^n\|_{L^\infty(0,\tilde{T};L^2(\Omega))} \le C. \tag{2.10}$$

With reference to Eq. (2.9), we get

$$\|\nabla u_\epsilon^n\|_{L^2(Q_{\tilde{T}})} + \|\nabla v_\epsilon^n\|_{L^2(Q_{\tilde{T}})} + \|\nabla w_\epsilon^n\|_{L^2(Q_{\tilde{T}})} \le C. \tag{2.11}$$

Using Eqs. (2.10) and (2.11), we get

$$\|F_{1,\epsilon} u_\epsilon^n\|_{L^1(Q_{\tilde{T}})} + \|F_{2,\epsilon} v_\epsilon^n\|_{L^1(Q_{\tilde{T}})} + \|F_{3,\epsilon} u_\epsilon^n\|_{L^1(Q_{\tilde{T}})} \le C. \tag{2.12}$$

In order to use compactness argument Theorem 2.2 in Temam,[29] we need to establish that zero extension of $u_\epsilon^n$, $v_\epsilon^n$ and $w_\epsilon^n$ are bounded in $\mathcal{W}^\alpha(\mathbb{R}; H_0^1(\Omega); L^2(\Omega))$ which was defined by Zhou et al. in Ref. 30. Further to avoid confusion we use $T$ instead of $\tilde{T}$.

**Lemma 2.1.** *Let $u_0, v_0, w_0 \in L^2(\Omega)$, then sequences $\{\tilde{u_\epsilon}^n\}, \{\tilde{v_\epsilon}^n\}$ and $\{\tilde{w_\epsilon}^n\}$ are bounded set of $W^\alpha(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$ where*

$$\tilde{z}_{\epsilon,n}(x,t) = \begin{cases} z_{\epsilon,n}(x,t), & t \in [0,T], \\ 0, & \mathbb{R}\backslash[0,T]. \end{cases} \tag{2.13}$$

**Proof.** To show that $\{\tilde{u_\epsilon}^n\}, \{\tilde{v_\epsilon}^n\}$ and $\{\tilde{w_\epsilon}^n\}$ are bounded in $W^\alpha(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$, we need to verify,

$$\left.\begin{aligned}
\textstyle\int_{-\infty}^{\infty} |\tau|^{2\beta} |\hat{u_\epsilon}^n(\tau)| d\tau &\le c, \\
\textstyle\int_{-\infty}^{\infty} |\tau|^{2\beta} |\hat{v_\epsilon}^n(\tau)| d\tau &\le c, \\
\textstyle\int_{-\infty}^{\infty} |\tau|^{2\beta} |\hat{w_\epsilon}^n(\tau)| d\tau &\le c,
\end{aligned}\right\} \tag{2.14}$$

for some $\beta > 0$ and $\hat{u_\epsilon}^n$, $\hat{v_\epsilon}^n$ and $\hat{w_\epsilon}^n$ denote the Fourier transform of $\{\tilde{u_\epsilon}^n\}, \{\tilde{v_\epsilon}^n\}$ and $\{\tilde{w_\epsilon}^n\}$. To prove (2.14), let us rewrite (2.6) as follows

$$\left.\begin{aligned}
\big({}_0^C D_t^\alpha \tilde{u_\epsilon}^n, \phi_j\big) &= \Big\langle \tilde{F}_{u_\epsilon^n}, \phi_j \Big\rangle + (u_\epsilon^n(0), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_0 - (u_\epsilon^n(T), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_T, \\
\big({}_0^C D_t^\alpha \tilde{v_\epsilon}^n, \phi_j\big) &= \Big\langle \tilde{F}_{v_\epsilon^n}, \phi_j \Big\rangle + (v_\epsilon^n(0), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_0 - (v_\epsilon^n(T), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_T, \\
\big({}_0^C D_t^\alpha \tilde{w_\epsilon}^n, \phi_j\big) &= \Big\langle \tilde{F}_{w_\epsilon^n}, \phi_j \Big\rangle + (w_\epsilon^n(0), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_0 - (w_\epsilon^n(T), \phi_j)_{-\infty} I_t^{1-\alpha} \delta_T,
\end{aligned}\right\} \tag{2.15}$$

where, $\tilde{F}_{u_\epsilon^n}$, $\tilde{F}_{v_\epsilon^n}$ and $\tilde{F}_{w_\epsilon^n}$ are defined as follows

$$\begin{aligned}
\tilde{F}_{u_\epsilon^n} &= d_1 \Delta u_\epsilon^n + div\Big[(\alpha_{11} f_\epsilon^+(u_\epsilon^n) + \alpha_{12} f_\epsilon^+(v_\epsilon^n)) \nabla u_\epsilon^n + \alpha_{12} f_\epsilon^+(u_\epsilon^n) \nabla v_\epsilon^n\Big] \\
&\quad - F_{1,\epsilon} + a_1 u_\epsilon^n, \\
\tilde{F}_{v_\epsilon^n} &= d_2 \Delta v_\epsilon^n + div\Big[(\alpha_{21} f_\epsilon^+(u_\epsilon^n) + \alpha_{22} f_\epsilon^+(v_\epsilon^n) + \alpha_{23} f_\epsilon^+(w_\epsilon^n)) \nabla v_\epsilon^n \\
&\quad + \alpha_{12} f_\epsilon^+(v_\epsilon^n) \nabla u_\epsilon^n + \alpha_{23} f_\epsilon^+(v_\epsilon^n) \nabla w_\epsilon^n\Big] - F_{2,\epsilon} + a_2 v_\epsilon^n, \\
\tilde{F}_{w_\epsilon^n} &= d_3 \Delta w_\epsilon^n + div\Big[(\alpha_{32} f_\epsilon^+(v_\epsilon^n) + \alpha_{33} f_\epsilon^+(w_\epsilon^n)) \nabla w_\epsilon^n + \alpha_{32} f_\epsilon^+(w_\epsilon^n) \nabla v_\epsilon^n\Big] \\
&\quad - F_{3,\epsilon} + a_3 w_\epsilon^n.
\end{aligned}$$

Using Fourier transform to the first equation of (2.15) we get

$$\begin{aligned}
(2\pi i\tau)^\alpha (\hat{u_\epsilon}^n, \phi_j) &= \Big\langle \tilde{F}_{u_\epsilon^n}, \phi_j \Big\rangle + (u_\epsilon^n(0), \phi_j)(2\pi i\tau)^{\alpha-1} \\
&\quad - (u_\epsilon^n(\tau), \phi_j)(2\pi i\tau)^{\alpha-1} e^{-2\pi i\tau T}.
\end{aligned}$$

Replacing $\phi_j$ as $\hat{u_\epsilon^n}$, we obtain

$$(2\pi i\tau)^\alpha |\hat{u_\epsilon}^n|^2 = \Big\langle \tilde{F}_{u_\epsilon^n}, \hat{u_\epsilon^n} \Big\rangle + (u_\epsilon^n(0), \hat{u_\epsilon^n})(2\pi i\tau)^{\alpha-1} - (u_\epsilon^n(\tau), \hat{u_\epsilon^n})(2\pi i\tau)^{\alpha-1} e^{-2\pi i\tau T}. \tag{2.16}$$

With the help of Eq. (2.10) in (2.16), we get

$$\sup_{\tau\in\mathbb{R}} \|\widehat{F_{u_\epsilon^n}}(\tau)\|_{H^{-1}(\Omega)} \le C.$$

Since $u_\epsilon$ is bounded on $L^\infty(0,T;L^2(\Omega))$ from (2.9) and (2.10) we get $|u_n(0)| \le C$ and $|u_n(T)| \le C$. Therefore (2.16) becomes

$$|\tau|^\alpha |\widehat{u_\epsilon^n}(\tau)|^2 \le C max\{1, |\tau|^{\alpha-1}\} \|\widehat{u_\epsilon^n}\|_{H_0^1(\Omega)}.$$

For fixed $\gamma < \frac{\alpha}{4}$, we see that

$$\begin{aligned}
\int_{-\infty}^{\infty} |\tau|^{2\gamma} |\widehat{u^n}|^2 d\tau &\le C(\gamma) \int_{-\infty}^{\infty} \Big(\frac{1+|\tau|^\alpha}{1+|\tau|^{\alpha-2\gamma}}\Big) |\widehat{u^n}|^2 d\tau \\
&\le C_1(\gamma) \int_{-\infty}^{\infty} \|\hat{u_n}\|^2_{H_0^1(\Omega)} d\tau \\
&\quad + C_2(\gamma) \int_{-\infty}^{\infty} \frac{|\tau|^{\alpha-1} \|\hat{u_n}(\tau)\|_{H_0^1(\Omega)}}{1+|\tau|^{\alpha-2\gamma}} d\tau.
\end{aligned}$$

Applying the Parseval equality, the first integral is bounded as $n \to \infty$. Consider

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|\tau|^{\alpha-1} \|\hat{u_n}(\tau)\|_{H_0^1(\Omega)}}{1+|\tau|^{\alpha-2\gamma}} d\tau \\
&\le \Big(\int_{-\infty}^{\infty} \frac{d\tau}{(1+|\tau|^{\alpha-2\gamma})^2}\Big)^{1/2} \Big(\int_{-\infty}^{\infty} |\tau|^{2\alpha-2} \|\widehat{u_n}(\tau)\|^2_{H_0^1(\Omega)}\Big)^{1/2}.
\end{aligned}$$

Again consider,

$$\begin{aligned}
\int_{-\infty}^{\infty} |\tau|^{2\alpha-2} \|\widehat{u_n}(\tau)\|^2_{H_0^1(\Omega)} d\tau &= \int_0^T \|{}_0 I_t^{1-\alpha} u_\epsilon^n(t)\|^2_{H_0^1(\Omega)} d\tau \\
&\le \Big(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big)^2 \int_0^T \|u_\epsilon^n\|^2_{H_0^1(\Omega)} dt \\
&\le C.
\end{aligned}$$

Therefore, $\tilde{u_\epsilon^n}$ is bounded in $\mathcal{W}^\alpha(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$. Similarly we can prove that $\tilde{v_\epsilon^n}$ and $\tilde{w_\epsilon^n}$ is bounded in $\mathcal{W}^\alpha(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$. □

Using Eqs. (2.10), (2.11) and (2.12) we get
$(u_\epsilon^n, v_\epsilon^n, w_\epsilon^n) \rightharpoonup (u_\epsilon, v_\epsilon, w_\epsilon)$ weakly* in $L^\infty(0,T;L^2(\Omega))$ a.e in $Q_T$,
$(u_\epsilon^n, v_\epsilon^n, w_\epsilon^n) \rightharpoonup (u_\epsilon, v_\epsilon, w_\epsilon)$ weakly in $L^2(0,T;H_0^1(\Omega))$,
$F_{i,\epsilon}(x,t,u_\epsilon^n, v_\epsilon^n, w_\epsilon^n) \rightharpoonup F_{i,\epsilon}(x,t,u_\epsilon, v_\epsilon, w_\epsilon)$ weakly in $L^2(Q_T)$, $i=1,2,3$,
$(\nabla u_\epsilon^n, \nabla v_\epsilon^n, \nabla w_\epsilon^n) \rightharpoonup (\nabla u_\epsilon, \nabla v_\epsilon, \nabla w_\epsilon)$ in $L^2(Q_T)$,
$f_\epsilon^+(u_\epsilon^n) \rightharpoonup f_\epsilon^+(u_\epsilon)$ a.e in $Q_T$,
$f_\epsilon^+(v_\epsilon^n) \rightharpoonup f_\epsilon^+(v_\epsilon)$ a.e in $Q_T$,
$f_\epsilon^+(w_\epsilon^n) \rightharpoonup f_\epsilon^+(w_\epsilon)$ a.e in $Q_T$.

From the estimates (2.10), (2.11) and (2.12), Lemma 2.1 and by using Theorem (2.2) in Temam,[29] we get
$u_\epsilon^n \to u_\epsilon$ strongly in $L^2(0,T;L^2(\Omega))$, $v_\epsilon^n \to v_\epsilon$ strongly in $L^2(0,T;L^2(\Omega))$, $w_\epsilon^n \to w_\epsilon$ strongly in $L^2(0,T;L^2(\Omega))$.

Consider $\phi_j(x,t) = \sum_{l=1}^n a_{j,l}(x,t)e_l(x)$, where $a_{j,l}(t), j = 1,2,3$ are $C^1([0,T])$ functions and $\phi_j(\cdot,T) = 0$. Multiply first equation of (2.2) with $\phi_1(x,t)$ and integrate the resulting equations from 0 to $t_0$ and $t_0$ to $t$ and then subtracting the resultant we get,

$$\begin{aligned}
&(u_{\epsilon,n}(t_0),\phi_1) - (u_{\epsilon,n}(t),\phi_1)\\
&= \int_0^{t_0} \Big\{ (t_0-s)^{\alpha-1} - (t-s)^{\alpha-1} \Big[ (d_1 \nabla u_\epsilon^n, \nabla\phi_1) + \Big( \big(\alpha_{11} f_\epsilon^+(u_\epsilon^n)\\
&\quad + \alpha_{12} f_\epsilon^+(v_\epsilon^n)\big)\nabla u_\epsilon^n, \nabla\phi_1 \Big) + \big( (\alpha_{11}\nabla u_\epsilon^n + \alpha_{12}\nabla v_\epsilon^n) f_\epsilon^+(u_\epsilon^n), \nabla\phi_1 \big)\\
&\quad + (F_{1,\epsilon},\phi_1) - (a_1 u_\epsilon^n, \phi_1) \Big] \Big\} ds\\
&\quad - \int_{t_0}^{t} \Big\{ (t-s)^{\alpha-1} \Big[ (d_1 \nabla u_\epsilon^n, \nabla\phi_1) + \Big( \big(\alpha_{11} f_\epsilon^+(u_\epsilon^n)\\
&\quad + \alpha_{12} f_\epsilon^+(v_\epsilon^n)\big)\nabla u_\epsilon^n, \nabla\phi_1 \Big) + \Big( (\alpha_{11}\nabla u_\epsilon^n + \alpha_{12}\nabla v_\epsilon^n) f_\epsilon^+(u_\epsilon^n), \nabla\phi_1 \Big)\\
&\quad + (F_{1,\epsilon},\phi_1) - (a_1 u_\epsilon^n, \phi) \Big] \Big\} ds.
\end{aligned}$$

Since $u_\epsilon^n \to u_\epsilon$ weakly in $L^2(0,T;H_0^1(\Omega))$, we assume that $u_\epsilon^n(t_0) \rightharpoonup u_\epsilon(t_0)$ for all $t_0 \in [0,T]\backslash K$, for some $K$ satisfying $meas(K) = 0$. Therefore, we get

$$\lim_{n\to\infty} u_{\epsilon,n}(t_0) = u_\epsilon(t_0) \text{ in } L^2(\Omega) \text{ for } t_0 \notin K.$$

Using Lemma 2.1 and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&(u_\epsilon(t_0),\phi_1) - (u_\epsilon(t),\phi_1)\\
&= \int_0^{t_0} (t_0-s)^{\alpha-1} - (t-s)^{\alpha-1} \Big[ (d_1 \nabla u_\epsilon, \nabla\phi_1) + \Big( \big(\alpha_{11} f_\epsilon^+(u_\epsilon)\\
&\quad + \alpha_{12} f_\epsilon^+(v_\epsilon)\big)\nabla u_\epsilon, \nabla\phi_1 \Big) + \Big( (\alpha_{11}\nabla u_\epsilon + \alpha_{12}\nabla v_\epsilon) f_\epsilon^+(u_\epsilon), \nabla\phi_1 \Big)\\
&\quad + (F_{1,\epsilon},\phi_1) - (a_1 u_\epsilon, \phi_1) \Big] ds\\
&\quad - \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[ (d_1 \nabla u_\epsilon, \nabla\phi_1) + \Big( (\alpha_{11} f_\epsilon^+(u_\epsilon) + \alpha_{12} f_\epsilon^+ v_\epsilon)\nabla u_\epsilon, \nabla\phi_1 \Big)\\
&\quad + \Big( (\alpha_{11}\nabla u_\epsilon + \alpha_{12}\nabla v_\epsilon) f_\epsilon^+(u_\epsilon), \nabla\phi_1 \Big) + (F_{1,\epsilon},\phi_1) - (a_1 u_\epsilon, \phi_1) \Big] ds,
\end{aligned}$$

for all $\phi_1 \in L^2(0,T;H_0^1(\Omega))$. As $t \to t_0$, we get

$$(u_\epsilon(t_0),\phi_1) - (u_\epsilon(t),\phi_1) \to 0.$$

Similarly as in Zhou,[31] we can show that

$$(v_\epsilon(t_0),\phi_2) - (v_\epsilon(t),\phi_2) \to 0,$$

$$(w_\epsilon(t_0),\phi_3) - (w_\epsilon(t),\phi_3) \to 0.$$

This concludes the existence of solution to the approximate problem (2.2). Proceeding similarly like the estimates (2.10), (2.11) and (2.12) for $u_\epsilon$, $v_\epsilon$ and $w_\epsilon$, we can infer that

$$\begin{aligned}
\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|w_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} &\le C,\\
\|u_\epsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \|v_\epsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \|w_\epsilon\|_{L^\infty(0,T;H_0^1(\Omega))} &\le C,\\
\|F_{1,\epsilon}(x,t,u_\epsilon,v_\epsilon,w_\epsilon)\|_{L^1(Q_T)} &\le C,\\
\|F_{2,\epsilon}(x,t,u_\epsilon,v_\epsilon,w_\epsilon)\|_{L^1(Q_T)} &\le C,\\
\|F_{3,\epsilon}(x,t,u_\epsilon,v_\epsilon,w_\epsilon)\|_{L^1(Q_T)} &\le C.
\end{aligned} \tag{2.17}$$

**Lemma 2.2.** *Let $u_0$, $v_0$ and $w_0 \in L^2(\Omega)$ and hypothesis (H1)–(H3) holds, then $\tilde{u}_\epsilon, \tilde{v}_\epsilon$ and $\tilde{w}_\epsilon$ are bounded set of $\mathcal{W}^\alpha(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$.*

**Proof.** The proof of lemma follows as in Lemma 2.1. □

**Proof of Theorem 2.1.** From the above estimate (2.17) we can find that sequences $\{u_\epsilon\}, \{v_\epsilon\}$ and $\{w_\epsilon\}$ have convergent subsequences which we label as $u_\epsilon$, $v_\epsilon$ and $w_\epsilon$ such that

$(u_\epsilon, v_\epsilon, w_\epsilon) \rightharpoonup (u,v,w)$ weakly* in $L^\infty(0,T;L^2(\Omega))$,
$(u_\epsilon, v_\epsilon, w_\epsilon) \rightharpoonup (u,v,w)$ weakly in $L^\infty(0,T;H_0^1(\Omega))$.

In view of Lemma 2.2 and Theorem (2.2) in Temam,[29] we obtain,

$(u_\epsilon, v_\epsilon, w_\epsilon) \to (u,v,w)$ strongly in $L^2(0,T;L^2(\Omega))$,
$F_{i,\epsilon}(x,t,u_\epsilon,v_\epsilon,w_\epsilon) \to F_i(x,t,u,v,w)$ strongly in $L^2(0,t;L^2(\Omega)), i=1,2,3.$

Using Holder's and Young's inequalities, we get

$$\begin{aligned}
\|f_\epsilon(u_\epsilon) - u\|_{L^2(Q_T)} &\le \sqrt{2}\|u_\epsilon - u\|_{L^2(Q_T)} + \sqrt{2}\left\| \frac{\epsilon u_\epsilon u}{1+\epsilon u_\epsilon} \right\|_{L^2(Q_T)}\\
&\le \sqrt{2}\|u_\epsilon - u\|_{L^2(Q_T)}\\
&\quad + \sqrt{2}\epsilon^{\frac{2}{3}} \|u_\epsilon\|^{\frac{2}{3}}_{L^\infty(0,T;L^2(\Omega))} \|u\|_{L^2(0,T;L^6(\Omega))}.
\end{aligned}$$

From the Sobolev Embedding $H_0^1(\Omega) \subset L^6(\Omega)$, we get $f(u_\epsilon) \to u$ strongly in $L^2(Q_T)$. Similarly we can prove $f(v_\epsilon) \to v$ and $f(w_\epsilon) \to w$. □

## 3. Conclusion

We established the existence of weak solutions of time-fractional strongly coupled three-species cooperating system of equations with the Dirichlet boundary conditions. We employed the Faedo–Galerkin approximation method and convergence results to prove the existence of solutions of the considered model.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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