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On neutral integrodifferential equations with state-dependent delay in Banach spaces

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Abstract
In this work, we investigate the existence of mild solution for semilinear integro-differential systems and semilinear neutral integro-differential systems with state-dependent delay in Banach spaces. Using Mönch's fixed point theorem, the theory of Grimmer's resolvent operator and the idea of measures of non-compactness, we prove the existence results. At the end, an example is given to further illustrate the conclusions drawn from the theoretical study.

Keywords Integrodifferential equations · State-dependent delay · Mild solution · Resolvent operator · Semigroup theory · Fixed point theorems

Mathematics Subject Classification 34K06 · 34A12 · 37L05 · 93B05

1 Introduction

The term "neutral differential equations" (NDEs) refers to a differential equation in which the derivative of the unknown function is evaluated at both the past time $t - s$ and present time s. Most of the application problems depend on the history of the function. Although delays are inevitable when modelling differential equations, its derivation led us to neutral evolution dynamical systems. There has been growing interest in exploring NDEs due to

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its applications in applied mathematics, engineering, ecology and physics. Besides these, in modelling of network containing lossless transmission lines NDEs appear. These types of networks occur in high-speed computers, when switching circuits are connected by lossless transmission lines, we refer to [9] for additional details. To our knowledge, the majority of the work currently in print focuses on NDEs [5, 31, 35].

On the other hand, integrodifferential equations arise and have gained significant attention in various fields of application such as engineering, electronics, fluid dynamics, physical sciences, and so on. Recently, qualitative and quantitative properties such as existence, uniqueness, stability, and controllability for various types of integrodifferential equations have been extensively studied by many researchers using the resolvent operator theory and fixed point theorems; see [1, 3, 4, 16–19, 21, 22, 29]. The phenomena do not fit into the framework of conventional differential equations. Because of this, integrodifferential equations have been receiving more attention recently from physicists, mathematicians, and engineers. The theory of integrodifferential equations with resolvent operators has therefore received a lot of mathematical attention in recent years, (see [15, 23, 32] and the references therein). In fact, the resolvent operator, which takes the place of the C_0-semigroup in evolution equations, is critical in solving these equations, in both the weak and strict senses.

Furthermore, functional differential equations with state-dependent delay (SDD) exist often in many applications, such as electrodynamics, automatic and remote control, machine cutting, neural networks, population biology, mathematical epidemiology, and economics, as well as the qualitative theory's distinction from discrete and time-dependent delay theories, have made the theory of these equations a subject of intense interest and ongoing research. Chalishajar et al. [11] demonstrated the controllability of impulsive neutral evolution integrodifferential equations with SDD in Banach spaces by introducing a SDD into the equation. According to Kailasavalli et al. [28], the exact controllability of fractional neutral integrodifferential systems with SDD in Banach spaces can be achieved by the use of state-dependent delay. We refer the reader to the book of ca$\tilde{n}$ada et al. [10] for more information on the theory of differential equations with SDD and their applications, as well as the papers [6, 7, 26, 33].

The existence results for integrodifferential equations and neutral integrodifferential equations with SDD and without delay have been extensively studied in the recent years. Suganya et al. [33] analyzed the existence results for an impulsive fractional neutral integrodifferential equation with state-dependent delay (SDD) and non-instantaneous impulses in Banach spaces by using Darbo fixed point theorem combined with the Hausdorff measure of non-compactness. In [21], the authors proved the solutions of neutral functional integrodifferential equations with an initial condition in finite delay. Very recently, authors in [22] considered existence results for a class of impulsive integrodifferential equations with SDD based on fixed point theorems and resolvent operator theory. Ahmed [2] proved the existence of mild solution to Sobolev-type fractional stochastic integrodifferential equation with nonlocal conditions in Hilbert space. Ahmed et al. investigated the existence of mild solution for Sobolev type nonlinear impulsive delay integro-differential system with fractional order $1 < \alpha < 2$.

Inspired by the above discussion, we aim to investigate in this paper the existence of mild solutions for systems (1.1) and (1.2), which are news models. We will prove our results with the help of the theory of Grimmer's resovent operator coupled with Mönch fixed point theorem and the notion of measure of noncompactness.

However, to the best of our knowledge, the study of existence results for systems (1.1) and (1.2) with Mönch fixed point theorem has not yet been done, which is an additional motivation. The aim of this manuscipt is firstly to discuss the existence of mild solutions of semilinear integrodifferential equations with state-dependent delay (SDD) in the form

$$\begin{cases} \zeta^{'}(t) = A\zeta(t) + \displaystyle\int_0^t F(t-s)\zeta(s)ds + g\Big(t, \zeta_{\rho(s,\zeta_s)}, \int_0^t h(t,s,\zeta_{\rho(s,\zeta_s)})ds\Big), \\ \qquad t \in J = [0, +\infty), \\ \\ \zeta(t) = \varphi(t) \in \mathcal{B}, \ t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ on a Banach space $\mathcal{H}$, $g : J \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$, $h : \mathcal{K} \times \mathcal{B} \to \mathcal{H}$, where $\mathcal{K} = \{(t,s) \in J \times J : 0 \leq s \leq t < +\infty\}$, and $\mathcal{B}$ is the phase space to be described later and $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ is a given function which satisfies certain assumptions to be specified later on. $\zeta_t : (-\infty, 0] \to \mathcal{H}, \ \zeta_t(\theta) = \zeta(t+\theta), \ \theta \leq 0$ belongs to an abstract phase space $\mathcal{B}$.

In the second part of this work, we investigate the existence results of the following semilinear neutral integrodifferential systems with SDD of the form

$$\begin{cases} d\big(\zeta(t) - f(t, \zeta_{\rho(t,\zeta_t)})\big) = A\zeta(t)dt \\ \qquad +\bigg(\int_0^t F(t-s)\zeta(s)ds + g\Big(t, \zeta_{\rho(s,\zeta_s)}, \int_0^t h(t,s,\zeta_{\rho(s,\zeta_s)})ds\Big)\bigg)dt, \\ \quad t \in J = [0, +\infty), \\ \\ \zeta(t) = \varphi(t) \in \mathcal{B}, \ t \in (-\infty, 0], \end{cases} \tag{1.2}$$

where $f : J \times \mathcal{B} \to \mathcal{H}$ is a given function and the other functions mentioned in (1.2) are the same as described in (1.1).

Remark 1.1 The advantage of Mönch fixed point theorem is that it helps to find the fixed point of the mild solution of system (2.5) without assuming the compactness of the semigroup. As remarked by Triggiani [34] in an infinite dimensional Banach space, the linear control system is never exactly controllable on a given interval of time, if either a bounded linear operator (from control space to state space) is compact or a semigroup is compact. According to Triggiani [34], this is a typical case for most control systems governed by parabolic partial differential equations, and hence the concept of exact controllability is very limited for many parabolic partial differential equations. We are discussing here neutral integrodifferential equations with state-dependent delay in Banach spaces. So according to Triggiani [34], we have to get rid of the compactness assumption of the semigroup/the resolvant operator. We have used MNC for the same reason. This is one of the novelties of the present work.

The following are the most significant contributions made by this work :

- In order to guarantee the existence of a mild solution for system 1.1 and 1.2 a new set of sufficient conditions has been constructed.
- By applying the theories of resolvent operator in the sense of Grimmer and measure of non-compactness, it is possible to demonstrate the existence of a solution through the use of the Mönch's fixed point theorem.
- The results of this paper are a generalization of the research on differential equations with state-dependent delay that has already been published.
- In addition, an example has been constructed to illustrate our findings.

The structure of this article is as follows. We introduce some basic findings in Sect. 2 that will be used to develop the paper. In Sect. 3, we establish the existence results by means of

Mönch fixed point theorem. At the conclusion, an example is offered in Sect. 4 to illustrate the theoretical results.

2 Preliminaries

This section contains some of the notations used throughout this manuscript and collects some material which is important for our work.

Let $\mathcal{BUC}$ be the space of all bounded uniformly functions from $(-\infty, +\infty)$ into $\mathcal{H}$. Let $\mathcal{BC}$ be the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into $\mathcal{H}$ equipped with the general norm

$$\|\zeta\|_{\mathcal{BC}} = \sup_{t\in(-\infty,+\infty)} |\zeta(t)|.$$

Finally, we denote by $\mathcal{BC}^+$ the Banach space of all bounded and continuous functions from $[0, +\infty)$ into $\mathcal{H}$ equipped with the general norm

$$\|\zeta\|_{\mathcal{BC}^+} = \sup_{t\in[0,+\infty)} |\zeta(t)|.$$

2.1 Phase space $\mathcal{B}$

It must be made clear that once the delay is infinite, we may discuss the theoretical phase space $\mathcal{B}$ in a useful manner. In this manuscript, we discuss phase spaces $\mathcal{B}$, which are the same as those presented in [24, 25]. We therefore skip the details.

We assume that the phase space $\mathcal{B}$ will be a linear space of functions mapping from $(-\infty, 0]$ into $\mathcal{H}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfying the following axioms:

(A_1) If $\zeta : (-\infty, T] \to \mathcal{H}$, $T > 0$ is such that $\zeta_0 \in \mathcal{B}$, for every $t \in J$, then we have:

- (i) $\zeta_t \in \mathcal{B}$.
- (ii) There exists $l > 0$ such that $\|\zeta(t)\| \le l\|\zeta_t\|_{\mathcal{B}}$.
- (iii) $\|\zeta_t\|_{\mathcal{B}} \le \Upsilon_1(t) \sup_{0\le\tau\le t} \|\zeta(\tau)\| + \Upsilon_1(t)\|\zeta_0\|_{\mathcal{B}}$, where $\Upsilon_1 : [0, +\infty) \to [0, +\infty)$ is a continuous map, and $\Upsilon_1 : [0, +\infty) \to [1, +\infty)$ is a locally bounded map, and Υ_1, Υ_2 are independent of $\zeta(\cdot)$.

(A_2) For $\zeta(\cdot)$ in (A_1), $s \mapsto \zeta_s$ is a $\mathcal{B}$-valued continuous map on J.

(A_2) The phase space $\mathcal{B}$ is complete.

We will denote by $\Upsilon_1^* = \sup\{\Upsilon_1(t) : t \in J\}$ and $\Upsilon_2^* = \sup\{\Upsilon_2(t) : t \in J\}$.

Let $\mathcal{R}(\rho^-)$ be the set defined as

$$\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in J \times \mathcal{B},\ \rho(s, \psi) \le 0\}.$$

Assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous and the subsequent hypothesis holds:
(H_ψ) The map $t \mapsto \psi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$, and there exists a bounded and continuous function $\mathcal{N}^\psi : \mathcal{R}(\rho^-) \to [0, +\infty)$ such that

$$\|\psi_t\|_{\mathcal{B}} \le \mathcal{N}^\psi(t)\|\psi\|_{\mathcal{B}}, \text{ for every } t \in \mathcal{R}(\rho^-).$$

The following result will be required in computation.

Lemma 2.1 [27] *Let $\zeta : (-\infty, +\infty) \to \mathcal{H}$ be a continous function such that $\zeta_0 = \psi$. If (H_ψ) holds, then*

$$\|\zeta_\tau\|_{\mathcal{B}} \leq (\Upsilon_2^* + \widetilde{\mathcal{N}}^\psi)\|\psi_t\|_{\mathcal{B}} + \Upsilon_1^* \sup\{\|\zeta(\varsigma)\|_{\mathcal{H}} : \varsigma \in [0, \max\{0, \tau\}]\} \quad \tau \in \mathcal{R}(\rho^-) \cup J,$$

where $\widetilde{\mathcal{N}}^\psi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{N}^\psi(t)$.

Definition 2.1 A function $g : J \times \mathcal{B} \to \mathcal{H}$ is said to be of the Carathéodory if

(i) $\sigma \mapsto g(t, \varphi)$ is measurable for all $\varphi \in \mathcal{B}$;
(ii) $\varphi \mapsto g(t, \varphi)$ is continuous for almost each $t \in J$.

2.2 Integrodifferential equations in Banach spaces

Now, let us review some of the fundamental aspects of the concept of the resolvent operator. Consider the following integrodifferential equation

$$\begin{cases} u^{'}(t) = Au(t) + \displaystyle\int_0^t F(t-s)u(s)ds \quad \text{for } t \geq 0 \\ u(0) = u_0 \in \mathcal{E}, \end{cases} \tag{2.1}$$

where A and $F(t)$ are closed operators on the Banach space $\mathcal{E}$. Let $\mathbb{Y}$ be the Banach space formed from $D(A)$ with the graph norm

$$\|x\|_{\mathbb{Y}} = \|Ax\| + \|x\|, \text{ for } x \in D(A).$$

In what follows, we suppose that A and $\Big(F(t)\Big)_{t\geq 0}$ satisfy the following conditions:

(R_1) A is the infinitesimal generator of C_0-semigroup $\big(S(t)\big)_{t\geq 0}$ in $\mathcal{E}$.
(R_2) For all $t \geq 0$, $F(t) \in \mathcal{L}(\mathbb{Y}, \mathcal{E})$, and for each $u \in \mathbb{Y}$, the function $F(\cdot)u$ is bounded, differentiable and the derivative $F^{'}(\cdot)u$ is bounded and uniformly continuous on $[0, +\infty)$.

Definition 2.2 [23] A family of bounded linear operators $\big(\mathcal{Q}(t)\big)_{t\geq 0} \subset \mathcal{L}(\mathcal{E})$ is said to be a resolvent operator for Eq. (2.1) if $\mathcal{Q}(t)$ satisties the following properties for all $t \geq 0$:

(i) $\mathcal{Q}(0) = I$ (identity operator on $\mathcal{E}$) and $\|\mathcal{Q}(t)\| \leq Me^{\omega t}$ for some constants M and ω.
(ii) For each $u \in \mathcal{E}$, $\mathcal{Q}(t)\xi$ is strongly continuous for $t \geq 0$.
(iii) For $u \in \mathbb{Y}$, $\mathcal{Q}(\cdot)u \in C^1([0, +\infty), \mathcal{E}) \cap C([0, +\infty), \mathbb{Y})$ and

$$\begin{aligned} \mathcal{Q}^{'}(t)u &= A\mathcal{Q}(t)u + \int_0^t F(t-s)\mathcal{Q}(s)uds \\ &= \mathcal{Q}(t)Au + \int_0^t \mathcal{Q}(t-s)F(s)uds \quad \text{for } t \geq 0. \end{aligned}$$

Theorem 2.2 [23] *Assume that (R_1) and (R_2) hold. Then, Eq. (2.1) has a unique resolvent operator $\big(\mathcal{Q}(t)\big)_{t\geq 0}$.*

Theorem 2.3 [15] *Assume that (R_1) and (R_2) hold. Let $\big(\mathcal{Q}(t)\big)_{t\geq 0}$ be the resolvent operator of equation (2.1). Then, $\mathcal{Q}(t)$ for $t > 0$ is operator-norm continuous (or continuous in the uniform operator topology) if only if $S(t)$ is operator-norm continuous for $t > 0$.*

In the following, we give some results for the existence of solutions for the following integrodifferential equation:

$$\begin{cases} u^{'}(t) = Au(t) + \displaystyle\int_0^t F(\sigma - s)u(s)ds + \vartheta(t) \quad \text{for } t \geq 0 \\ u(0) = u_0 \in \mathcal{E}, \end{cases} \tag{2.2}$$

where $\vartheta : [0, +\infty) \to \mathcal{E}$ is a continuous function.

Definition 2.3 [15] A continuous function $u : [0, +\infty) \to \mathcal{E}$ is called a strict solution of equation (2.2) if:

1. $u \in \mathcal{C}^1([0, +\infty), \mathcal{E}) \cap \mathcal{C}([0, +\infty), \mathbb{Y})$,
2. u satisfies equation (2.2) for $t \geq 0$.

Theorem 2.4 [15] *Assume that* (R_1) *and* (R_2) *are satisfied. If* u *is a strict solution of equation* (2.2)*, then the variation of constant formula holds*

$$u(t) = \mathcal{Q}(t)u_0 + \int_0^t \mathcal{Q}(t-s)\vartheta(s)ds, \quad \textit{for } t \geq 0. \tag{2.3}$$

2.3 Measure of noncompactness

Here, we introduce some notions and properties about the Kuratowski measure of noncompactness (KMNC).

Definition 2.4 [8] The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set G of Banach space $\mathcal{H}$ is

$$\alpha(G) = \inf \left\{ \epsilon > 0 \, : G = \cup_{i=1}^{m} G_i \text{ and } \operatorname{diam}(G_i) \leq \epsilon \text{for } i = 1, 2, \ldots, m \right\}.$$

Theorem 2.5 [8, 14] *Let* α *denote Kuratowski measure of noncompactness on the real Banach spaces* $\mathcal{H}$ *and* $G, G_1, G_2 \subseteq \mathcal{H}$ *be bounded. The following properties are satisfied:*

(i) *If* $G_1 \subseteq G_2$*, then* $\alpha(G_1) \leq \alpha(G_2)$ *(Monotonicity).*
(ii) $\alpha(G_1) = \alpha(\bar{G_1}) = \alpha(convG)$*, where* $\bar{G}$ *and* $convG$ *mean the closure and convex hull of* G*, respectively.*
(iii) G *is pre-compact if and only if* $\alpha(G) = 0$ *(Regularity).*
(iv) $\alpha(\lambda G) = |\lambda|\alpha(G)$ *for any* $\lambda \in \mathbb{R}$.
(v) $\alpha(G_1 \cup G_2) \leq \max\{\alpha(G_1), \alpha(G_2)\}$.
(vi) $\alpha(G + \nu) = \alpha(G)$ *for all* $\nu \in \mathcal{H}$.
(vii) $\alpha(G_1 + G_2) \leq \alpha(G_1) + \alpha(G_2)$ *where* $B + C = \{x + y : x \in B, \ y \in C\}$.
(viii) *If the map* $\vartheta : D(\vartheta) \subseteq \mathcal{H} \to \mathcal{U}$ *is Lipschitz continuous with constant* κ*, then*

$$\alpha(\vartheta G) \leq \kappa \, \alpha(G)$$

for any bounded subset $G \subseteq D(\vartheta)$.

Lemma 2.6 [8, 14]Let F_1 and F_2 be two bounded sets of a Banach space $\mathcal{H}$. Then:

(i) If $F_1 \subseteq F_2$ then $\alpha(F_1) \leq \alpha(F_2)$,
(ii) $\alpha(F_1) = 0 \Longleftrightarrow \overline{F_1}$ is compact (F_1 is relatively compact),
(iii) $\alpha(F_1 + F_2) \leq \alpha(F_1) + \alpha(F_2)$.

More details on the Kuratowski's measure of noncompacness can be found in Goebel [8] and Deimling [14].

The next results play an important role in demonstrating our key findings.

Lemma 2.7 [12] *If $G \subset \mathcal{H}$ is bounded for a Banach space $\mathcal{H}$, then a countable subset $G_0 \subset G$ exists, for which $\alpha(G) \leq 2\alpha(G_0)$ exists.*

Lemma 2.8 [13] *Let $\mathcal{H}$ be a Banach space, and $G = \{\zeta_n\} \subset \mathcal{C}([t_1, t_2], \mathcal{H})$ be a bounded and countable set for the constants $-\infty < t_1 < t_2 < +\infty$. Then $\alpha(G(t))$ is a Lebesgue integral on $[t_1, t_2]$ and*

$$\alpha\Big(\Big\{\int_{t_1}^{t_2} \zeta_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq \int_{t_1}^{t_2} \alpha(G(t))dt.$$

Now, we give the following well-known result, referred to as Mönch fixed-point theorem, which is a useful tool for explaining our results.

Theorem 2.9 (Mönch fixed-point) [30] *Let $\mathcal{O}$ be a boundedn closed, and convex subset of a Banach space $\mathcal{G}$ such that $0 \in \mathcal{O}$. Assume that $\Psi : \mathcal{O} \to \mathcal{O}$ is a continuous map which satisfies Mönch's condition, that is,*

$$\mathsf{N} \subseteq \mathcal{O} \textit{ is countable, } \mathsf{N} \subseteq \overline{conv}(\{0\} \cup \Psi(\mathsf{N})) \longrightarrow \mathsf{N} \textit{ is compact.}$$

Then Ψ has a fixed point in $\mathcal{O}$.

To end this current section, we present the mild solutions of the systems (1.1) and (1.2). From Theorem 2.4, we adopt the following concepts of mild solutions of systems (1.1) and (1.2).

Definition 2.5 A function $\zeta : (-\infty, +\infty) \to \mathcal{H}$ is called a mild solution of the system (1.1) when the following conditions are satisfied: $\zeta_0 = \varphi \in \mathcal{B}$ on $(-\infty, 0]$ and the restriction of $\zeta(\cdot)$ to $[0, +\infty)$ is continuous and satisfies the subsequent equation:

$$\zeta(t) = \begin{cases} \psi, \ t \in (-\infty, 0], \\ \mathcal{Q}(t)\varphi(0) + \displaystyle\int_0^t \mathcal{Q}(t-s)g\Big(s, \zeta_{\rho(s,\zeta_s)}, \int_0^s h(s, \tau, \zeta_{\rho(\tau,\zeta_\tau)})d\tau\Big)ds, \ t \in J. \end{cases} \tag{2.4}$$

Definition 2.6 A function $\zeta : (-\infty, +\infty) \to \mathcal{H}$ is called a mild solution of the system (1.2) when the following conditions are satisfied: $\zeta_0 = \varphi \in \mathcal{B}$ on $(-\infty, 0]$ and the restriction of $\zeta(\cdot)$ to $[0, +\infty)$ is continuous and satisfies the subsequent equation:

$$\zeta(t) = \begin{cases} \psi, \ t \in (-\infty, 0], \\ \mathcal{Q}(t)[\varphi(0) - f(0, \varphi)] + f(t, \zeta_{\rho(t,\zeta_t)}) \\ \quad + \displaystyle\int_0^t \mathcal{Q}(t-s)g\Big(s, \zeta_{\rho(s,\zeta_s)}, \int_0^s h(s, \tau, \zeta_{\rho(\tau,\zeta_\tau)})d\tau\Big)ds, \ t \in J. \end{cases} \tag{2.5}$$

3 Existence results

In this section, we discuss the existence results for the systems (1.1) and (1.2) in accordance with the Mönch's fixed-point theorem, which is a useful tool for proving our results. To do this, we first introduce the following hypotheses:

(H_1) The resolvent operator $\big(\mathcal{Q}(t)\big)_{t\geq 0}$ is norm-continuous for $t>0$.
(H_2) The function $h:\mathcal{K}\times\mathcal{B}\to\mathcal{H}$ satisfies the following conditions:

(i) For every $(t,s)\in\mathcal{K}$, the function $h(t,s,\cdot):\mathcal{B}\to\mathcal{H}$ is continuous and for each $x\in\mathcal{B}$, the function $h(\cdot,\cdot,x):\mathcal{K}\to\mathcal{H}$ is strongly measurable.
(ii) There exists an integrable function $v_1:J\to[0,+\infty)$ such that

$$\|h(t,s,x)\|\leq v_1(t)\|x\|_{\mathcal{B}} \text{ for a.e. } t,s\in J,\ \ x\in\mathcal{B}.$$

Assume that the finite bound of $\int_0^t v_1(s)ds$ is a_0.
(iii) There exists an integrable function $l(t,s):J\times J\to(0,+\infty)$ such that

$$\alpha(f(t,s,G_1))\leq l(t,s)\Big[\sup_{-\infty<z\leq 0}\alpha(G_1(z))\Big] \text{ for a.e. } t,s\in J,$$

where $G_1(z)=\{\nu(z):\nu\in G_1\}$; α is the KMNC and denote $l^*=\displaystyle\int_0^t l(s,\nu)d\nu<\infty$.

(H_3) The Carathéodory function $g:J\times\mathcal{B}\times\mathcal{H}\to\mathcal{H}$ satisfies the following conditions: There exists an integrable function $v:(-\infty,+\infty)\to[0,+\infty)$ such that

$$\|g(t,x,\phi)\|\leq v(t)(\|x\|_{\mathcal{B}}+\|\phi\|),\ \ t\in J,\ \ x\in\mathcal{B},\ \ \phi\in\mathcal{H}$$

and

$$v^*:=\sup_{t\in J}\int_0^t v(s)ds<\infty.$$

(H_4) Let $G_2\subset\mathcal{B}$, $\overline{G}\subset\mathcal{H}$ and each $t\in J$, we have

$$\alpha(g(t,G_2,G))\leq v(t)\Big[\sup_{-\infty<z\leq 0}\alpha(G_2(z))+\alpha(\overline{G})\Big],$$

where $G_2(z)=\{x(z):x\in G_2\}$.

Theorem 3.1 Assume that assumptions (R_1)–(R_2) and (H_1)–(H_4) hold, then the system (1.1) has at least one mild solution on $\mathcal{Y}_t$ provided that

$$\mathcal{S}=2(1+l^*)Mv^*\leq 1. \tag{3.1}$$

Proof Consider the space $\mathcal{Y}_t=\{\zeta\in\mathcal{C}(J,\mathcal{H}):\zeta(0)=\psi(0)\}$. We aim to apply Theorem 2.9 to the operator $\mathcal{F}:\mathcal{Y}_t\to\mathcal{Y}_t$ defined as:

$$(\mathcal{F}\zeta)(t)=\begin{cases}\psi,\ t\in(-\infty,0],\\ \mathcal{Q}(t)\psi(0)+\displaystyle\int_0^t\mathcal{Q}(t-s)g\Big(s,\zeta_{\rho(s,\zeta_s)},\int_0^s h(s,\tau,\zeta_{\rho(\tau,\zeta_\tau)})d\tau\Big)ds,\ t\in J.\end{cases}$$

The transformation that we are going to use now is to simplify the calculations and the conditions and not to have a norm as soon as our space is already a Banach space Let $\xi(\cdot):(-\infty,+\infty)\to\mathcal{H}$ be the function defined by

$$\theta(t)=\begin{cases}\psi,\ t\in(-\infty,0],\\ \mathcal{Q}(t)\varphi(0),\ t\in J.\end{cases}$$

Then $\theta_0 = \varphi$. Let $\xi \in \mathcal{C}(J, \mathcal{H})$ with $\xi_0 = 0$, we denote by $\tilde{\xi}$ the function given by

$$\tilde{\xi}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \xi, & t \in J. \end{cases}$$

If $\zeta(\cdot)$ satisfies (2.4), we are able to split it as $\zeta(t) = \theta(t) + \xi(t)$, for ≥ 0 which implies that $\zeta_t = \xi_t + \theta_t$ and also the function $\theta(\cdot)$ satisfies

$$\xi(\sigma) = \int_0^t \mathcal{Q}(t-s) g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} \\ +\theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \xi_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)}) d\tau\Big) ds, \ t \in J.$$

Let $\mathcal{Y}_t^0 = \{\xi \in \mathcal{Y}_t : \xi_0 = 0 \in \mathcal{B}\}$. Let $\xi \in \mathcal{Y}_t^0$, then

$$\|\xi\|_{\mathcal{Y}_t^0} = \|\xi_0\|_{\mathcal{B}} + \sup\{|v(t)| : 0 \leq t \leq +\infty\}.$$

Thus $(\mathcal{Y}_t^0, \|\cdot\|_{\mathcal{Y}_t^0})$ is a Banach space. Next, the operator $\widetilde{\mathcal{F}} : \mathcal{Y}_t^0 \to \mathcal{Y}_t^0$ is defined by

$$(\widetilde{\mathcal{F}}\xi)(t) = \int_0^t \mathcal{Q}(t-s) g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} \\ +\theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \xi_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)}) d\tau\Big) ds, \ t \in J.$$

The claim that system (1.1) admits a mild solution is hence equivalent to the fact that the operator $\mathcal{F}$ has a fixed point. Since $\widetilde{\mathcal{F}}$ also has a fixed point, it follows that the operator $\mathcal{F}$ does as well. Next we prove that $\widetilde{\mathcal{F}}$ has a fixed point by the means of Theorem 2.9. To this end, the proof is splitted into several steps.

Step 1: $\widetilde{\mathcal{F}}$ maps $\mathcal{Y}_t^0$ into $\mathcal{Y}_t^0$.

Using Lemma 2.1 and the phase space axioms, we get for every $t \in J$

$$\begin{aligned} \|\xi_{\rho(t,\xi_t+\theta_t)} + \theta_{\rho(t,\xi_t+\theta_t)}\|_{\mathcal{B}} &\leq \|\xi_{\rho(t,\xi_t+\theta_t)}\|_{\mathcal{B}} + \|\theta_{\rho(t,\xi_t+\theta_t)}\|_{\mathcal{B}} \\ &\leq \Upsilon_1(t)|\xi(t)| + \Upsilon_2(t)\|\xi_0\|_{\mathcal{B}} + \Upsilon_1(t)\|\theta(t)\|_{\mathcal{H}} + \Upsilon_2(t)\|\theta_0\|_{\mathcal{B}} \\ &\leq \Upsilon_1(t)|\xi(t)| + \Upsilon_1(t)\|\mathcal{Q}(t)\psi(0)\|_{\mathcal{H}} + (\Upsilon_2(t) + \mathcal{N}^{\psi})\|\psi\|_{\mathcal{B}} \\ &\leq \Upsilon_1^*|\xi(t)| + \Upsilon_1^* ML\|\psi\|_{\mathcal{B}} + (\Upsilon_2^* + \mathcal{N}^{\psi})\|\psi\|_{\mathcal{B}} \\ &\leq \Upsilon_1^*|\xi(t)| + \Upsilon_1^* ML\|\psi\|_{\mathcal{B}} + (\Upsilon_2^* + \widetilde{\mathcal{N}}^{\psi})\|\psi\|_{\mathcal{B}} \\ &= \Upsilon_1^*|\xi(t)| + (\Upsilon_1^* ML + \Upsilon_2^* + \widetilde{\mathcal{N}}^{\psi})\|\psi\|_{\mathcal{B}}. \end{aligned}$$

Then, we have

$$\|\xi_{\rho(t,\xi_t+\theta_t)} + \theta_{\rho(t,\xi_t+\theta_t)}\|_{\mathcal{B}} \leq c + \Upsilon_1^*|\xi(t)| = c + \Upsilon_1^* r = r', \tag{3.2}$$

where $c = (\Upsilon_1^* ML + \Upsilon_2^* + \widetilde{\mathcal{N}}^{\psi})\|\psi\|_{\mathcal{B}}$ and $|\xi(t)| = \|\xi\|_{\mathcal{Y}_t^0} \leq r$. Then, by (H_2) and (H_3), we have for every $t \in J$

$$\begin{aligned} &\|(\widetilde{\mathcal{F}}\xi)(t)\| \\ &\leq \int_0^t \left\| \mathcal{Q}(t-s) g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s,\tau, \xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)}) d\tau\Big) \right\| ds \\ &\leq M \int_0^t v(s) \Big(\|\xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}\|_{\mathcal{B}} + \left\| \int_0^s h(s, \xi_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)}) d\tau \Big) \right\| \Big) ds \end{aligned}$$

$$
\begin{aligned}
&\leq M \int_0^t v(s)\Big(\|\xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}\|_{\mathcal{B}} + \int_0^s v_1(\tau)\big(\|\xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)}\|_{\mathcal{B}}\big)d\tau\Big)\Big)ds \\
&\leq M \int_0^t v(s)\Big(c + \Upsilon_1^*|\xi(s)| + a_0(c + \Upsilon_1^*|\xi(s)|)\Big)ds \\
&\leq M v^*\Big(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0} + a_0(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0})\Big) \\
&\leq M v^*(1+a_0)(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0}).
\end{aligned}
$$

Therefore $\widetilde{\mathcal{F}}(\xi) \in \mathcal{Y}_t^0$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{M v^*(1+a_0)c}{1 - M v^*(1+a_0)\Upsilon_1^*}$$

and B_r be the closed ball in $\mathcal{Y}_t^0$ centered at the origin and of radius r. Now, take $\xi \in B_r$ and $t \in [0, +\infty)$; then

$$\|(\widetilde{\mathcal{F}}\xi)(t)\| \leq M v^*(1+a_0)c + M v^*(1+a_0)\Upsilon_1^* r.$$

Hence $\|\xi\|_{\mathcal{Y}_t^0} \leq r$, which implies that $\widetilde{\mathcal{F}}(B_r) \subset B_r$.

Step 2: $\widetilde{\mathcal{F}}$ is continuous.

Let $\{\xi^n\}$ be a sequence such that $\xi^n \to \xi$ as $n \to \infty$ in B_r. First, we examine the convergence of the sequence $\{\xi^n_{\rho(s,\xi^n_s)}\}_{n\in\mathbb{N}},\ s \in J$. From the continuity of $\rho(s,\cdot)$, we have $\xi_{\rho(s,\xi^n_s)} \to \xi_{\rho(s,\xi_s)},\ s \in J$, as $n \to \infty$. Let $s \in J$ be such that $\rho(s,\xi_s) > 0$, then we obtain

$$
\begin{aligned}
\big\|\xi^n_{\rho(s,\xi^n_s)} - \xi_{\rho(s,\xi_s)}\big\|_{\mathcal{B}} &\leq \big\|\xi^n_{\rho(s,\xi^n_s)} - \xi_{\rho(s,\xi^n_s)}\big\|_{\mathcal{B}} + \big\|\xi_{\rho(s,\xi^n_s)} - \xi_{\rho(s,\xi_s)}\big\|_{\mathcal{B}} \\
&\leq \Upsilon_1^*\big\|\xi^n - \xi\big\|_{\mathcal{B}} + \big\|\xi_{\rho(s,\xi^n_s)} - \xi_{\rho(s,\xi_s)}\big\|_{\mathcal{B}}
\end{aligned}
$$

which implies that $\xi^n_{\rho(s,\xi^n_s)} \to \xi_{\rho(s,\xi_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,\xi_s) > 0$.

Similarly, if $\rho(s,\xi_s) < 0$, we get

$$\big\|\xi^n_{\rho(s,\xi^n_s)} - \xi_{\rho(s,\xi_s)}\big\|_{\mathcal{B}} = \big\|\psi^n_{\rho(s,\xi^n_s)} - \psi_{\rho(s,\xi_s)}\big\|_{\mathcal{B}} = 0,$$

and we deduce that $\xi^n_{\rho(s,\xi^n_s)} \to \xi_{\rho(s,\xi_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,\xi_s) < 0$.

In the same manner, we can easily show that $\xi^n_{\rho(s,\xi^n_s)} \to \psi$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,\xi_s) = 0$. Therefore, $\xi^n_{\rho(s,\xi^n_s)} \to \xi_{\rho(s,\xi_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$. Also, since g is a function Carathéodoty, we have

$$
\begin{aligned}
&g\Big(s, \xi^n_{\rho(s,\xi^n_s+\theta_s)} + \theta_{\rho(s,\xi^n_s+\theta_s)}, \int_0^s h(s,\tau,\xi^n_{\rho(\tau,\xi^n_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi^n_\tau+\theta_\tau)})d\tau\Big) \\
&\quad \to g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s,\tau,\xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)})d\tau\Big) \text{ as } n \to \infty.
\end{aligned}
$$

It follows from the Lebesgue dominated convergence theorem, that

$$
\begin{aligned}
&\|(\widetilde{\mathcal{F}}\xi^n)(t) - (\widetilde{\mathcal{F}}\xi)(t)\| \\
&\quad \leq \int_0^t \Big\|g\Big(s, \xi^n_{\rho(s,\xi^n_s+\theta_s)} + \theta_{\rho(s,\xi^n_s+\theta_s)}, \int_0^s h(s,\tau,\xi^n_{\rho(\tau,\xi^n_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi^n_\tau+\theta_\tau)})d\tau\Big) \\
&\qquad - g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s,\tau,\xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)})d\tau\Big)\Big\|ds \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Consequently,

$$\|(\widetilde{\mathcal{F}}\xi^n) - (\widetilde{\mathcal{F}}\xi)\|_{\mathcal{B}} \to 0 \text{ as } n \to \infty$$

and the operator $\widetilde{\mathcal{F}}$ is continuous in B_r.
Step 3: $\widetilde{\mathcal{F}}$ maps bounded sets into equi-continuous sets in B_r.
Take $0 \leq t_1 < t_2 \leq \beta$ and for each $\xi \in B_r$, we sustain

$$\begin{aligned}\|(\widetilde{\mathcal{F}}\xi)(_2) - (\widetilde{\mathcal{F}}\xi)(t_1)\|_{\mathcal{H}} \leq & \int_0^{t_1} \|\mathcal{Q}(t_2 - s) - \mathcal{Q}(t_1 - s)\| \Big\| g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} \\ & + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \tau, \xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau^n+\theta_\tau)})d\tau\Big)\Big\| ds \\ & + \int_{t_1}^{t_2} \|\mathcal{Q}(t_2 - s)\| \Big\| g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} \\ & + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \tau, \xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)})d\tau\Big)\Big\| ds.\end{aligned}$$

By (H_3) and (3.2), we have

$$\begin{aligned}&\Big\|g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \tau, \xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)})d\tau\Big)\Big\| \\ &\leq v(s)\Big(\|\xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}\|_{\mathcal{B}} + \int_0^s v_1(\tau)\|\xi_{\rho(\tau,\xi_\tau+\theta_\tau)} + \theta_{\rho(\tau,\xi_\tau+\theta_\tau)}\|_{\mathcal{B}} d\tau\Big) \\ &\leq v(s)(r^{'} + v^* r^{'}) \\ &\leq v(s)(1 + v^*)r^{'}.\end{aligned}$$

Then, we have

$$\begin{aligned}\|(\widetilde{\mathcal{F}}\xi)(_2) - (\widetilde{\mathcal{F}}\xi)(t_1)\|_{\mathcal{H}} \leq & (1 + v^*)r^{'} \int_0^{t_1} \|\mathcal{Q}(t_2 - s) - \mathcal{Q}(t_1 - s)\| v(s) ds \\ & + M(1 + v^*)r^{'} \int_{t_1}^{t_2} v(s) ds.\end{aligned}$$

According to the norm continuity of $\mathcal{Q}(t)$ for $t > 0$, we see that the rigth-hand side of the above inequality tends to 0 as $t_2 \to t_1$. Thus, $\widetilde{\mathcal{F}}(B_r)$ is equicontinuous.
Step 4: $\widetilde{\mathcal{F}}(B_r)$ is equiconvergent.
Let $t \geq 0$ and $\xi \in B_r$, we obtain

$$\|(\widetilde{\mathcal{F}}\xi)(t)\| \leq Mv^*(1 + a_0) \int_0^t v(s) ds.$$

Then, we have

$$\lim_{t \to +\infty} \|(\widetilde{\mathcal{F}}\xi)(t)\| \leq Mv^*(1 + a_0)r^{'}.$$

Hence,

$$\|(\widetilde{\mathcal{F}}\xi)(t) - (\widetilde{\mathcal{F}}\xi)(+\infty)\| \to 0 \text{ as } t \to +\infty$$

and $\widetilde{\mathcal{F}}(B_r)$ is equiconvergent.
Step 5: The Mönch condition holds.
Let U be a subset of B_r such that $U \subset \overline{conv}(\mathcal{F}(U) \cup \{0\})$. We are going to show that $U(t)$ is relatively compact. For this purpose, It is sufficient to prove that $\alpha(U) = 0$. Moreover, in

view of Lemma 2.7, we affirm the existence of a countable set $G_0 = \{\zeta^n\} \subset G$ such that $\alpha(\widetilde{\mathcal{F}}(G)) \leq 2\alpha(\widetilde{\mathcal{F}}(G_0))$ for any bounded set G. Thus for $\{\zeta_n\} \subset G$, the appropriate choice of U. For every $\sigma \in [0, \beta]$, by the virtue of Lemma 2.8 and conditions (H_1)–(H_4) and the properties of the measure α, we obtain

$$
\begin{aligned}
&\alpha(\widetilde{\mathcal{F}})(\xi^n)\\
&= \alpha\Big(\int_0^t \mathcal{Q}(t-s) g\Big(s, \xi^n_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \xi^n_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)}) d\tau\Big) ds\Big)\\
&\leq 2M \int_0^t v(s)\Big(\sup_{-\infty<z\leq 0} \alpha(\xi^n(z+s) + \theta(z+s)) + \int_0^s l(s,\tau) \sup_{-\infty<z\leq 0} \alpha(\xi^n(z+s)\\
&\quad + \theta(z+s)) d\tau\Big) ds\\
&\leq 2M \int_0^t v(s)\Big(\sup_{0<k\leq s} \alpha(\xi^n(k)) + l^* \sup_{0<k\leq s} \alpha(\xi^n(k))\Big) ds\\
&\leq 2(1+l^*) M \int_0^t v(s) \sup_{0<s\leq\beta} \alpha(\xi^n(s)) ds\\
&\leq 2(1+l^*) M \alpha(\{\xi^n\}) \int_0^\sigma v(s) ds\\
&\leq 2(1+l^*) M v^* \alpha(\{\xi^n\}),
\end{aligned}
$$

which gives that

$$\alpha(\widetilde{\mathcal{F}})(U) \leq 2(1+l^*) M v^* \alpha(U).$$

Then, by the hypothesis in Mönch condition, it follows that

$$\alpha(U) \leq \alpha(\widetilde{\mathcal{F}})(U) \leq \mathcal{S}\alpha(U),$$

which implies that

$$\alpha(U)(1-\mathcal{S}) \leq 0.$$

By (3.1), we deduce that $\alpha(U) = 0$ and for $t \in J$; then $U(t)$ is relatively compact in $\mathcal{H}$.

Consequently, by Mönch fixed point Theorem, $\widetilde{\mathcal{F}}$ has a fixed point $\bar{\xi}$. Then $\bar{\zeta} = \bar{\xi} + \theta$ is a fixed point of $\mathcal{F}$, which is a mild solution of the system 1.1. □

Next, we study the existence result for the system (1.2). Now, we make the following additional hypotheses:

(H_5) Let $f : J \times \mathcal{B} \to \mathcal{H}$ be a Carathéodory function and there exists a continuous function $v_f : (-\infty, +\infty) \to [0, +\infty)$ such that:

$$\|f(t,x)\| \leq v_f(t)\|x\|_{\mathcal{B}}, \ t \in J, x \in \mathcal{B}$$

and

$$v_f^* := \sup_{t\in J} \int_0^t v_f(s) ds < \infty.$$

(H_6) Let G be a bounded set, $G \subset \mathcal{B}$ and every $t \in [0, +\infty)$; we have

$$\alpha(f(t,G)) \leq v_f(t)\alpha(G).$$

(H_7) If $G \subset \mathcal{B}$, where G is a bounded set; then, the function $\{t \to f(t, x_\sigma) : x \in G\}$ is equicontinuous on every compact interval $[0, \beta]$ of $[0, +\infty)$, for every $\beta > 0$.

Theorem 3.2 *Assume that the assumptions (R_1)–(R_2) and (H_1)–(H_7) hold, then the system*(1.2) *has at least one mild solution on $\mathcal{Y}_t$ provided that*

$$\mathcal{S}^{'} = 2\Big[(1 + l^*)Mv^* + v_f^*\Big] < 1. \tag{3.3}$$

Proof Define the opertor $\mathcal{P} : \mathcal{Y}_t \to \mathcal{Y}_t$ as follows:

$$(\mathcal{P}\zeta)(t) = \begin{cases} \psi, \ t \in (-\infty, 0], \\ \mathcal{Q}(t)[\psi(0) - f(0, \psi)] + f(t, \zeta_{\rho(t,\zeta_t)}) \\ \quad + \displaystyle\int_0^t \mathcal{Q}(t-s)g\Big(s, \zeta_{\rho(s,\zeta_s)}, \int_0^s h(s, \tau, \zeta_{\rho(\tau,\zeta_\tau)})d\tau\Big)ds, \ t \in J. \end{cases}$$

In perspective of Theorem 3.1, define the operator $\widetilde{\mathcal{P}} : \mathcal{Y}_t^0 \to \mathcal{Y}_t^0$ by

$$\begin{aligned}
&(\widetilde{\mathcal{P}}\xi)(t) \\
&\quad = -\mathcal{Q}(t)f(0, \psi) + f(t, \xi_{\rho(t,\xi_t+\theta_t)} + \theta_{\rho(t,\xi_t+\theta_t)}) \\
&\quad\quad + \int_0^t \mathcal{Q}(t-s)g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \xi_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)})d\tau\Big)ds, \\
&\quad t \in J.
\end{aligned}$$

Clearly $\mathcal{P}$ has a fixed point is equivalent to one $\widetilde{\mathcal{P}}$, then it suffices to prove that $\widetilde{\mathcal{P}}$ has a fixed point. The remaining of this part is splitted as:
Step 1: $\widetilde{\mathcal{P}}$ maps $\mathcal{Y}_t^0$ into $\mathcal{Y}_t^0$.

$$\begin{aligned}
&\|(\widetilde{\mathcal{P}}\xi)(t)\| \\
&\quad \le \|\mathcal{Q}(t)f(0, \psi)\| + \|f(t, \xi_{\rho(t,\xi_t+\theta_t)} + \theta_{\rho(t,\xi_t+\theta_t)})\| \\
&\quad\quad + \int_0^t \|\mathcal{Q}(t-s)g\Big(s, \xi_{\rho(s,\xi_s+\theta_s)} + \theta_{\rho(s,\xi_s+\theta_s)}, \int_0^s h(s, \xi_{\rho(s,\xi_\tau+\theta_\tau)} + \theta_{\rho(s,\xi_\tau+\theta_\tau)})d\tau\Big)\|ds \\
&\quad \le M\|f(0, \psi)\| + v_f(t)\|\xi_{\rho(t,\xi_t+\theta_t)} + \theta_{\rho(t,\xi_t+\theta_t)}\|_{\mathcal{B}} \\
&\quad\quad + M\int_0^t v(s)\Big(c + \Upsilon_1^*|\xi(s)| + a_0(c + \Upsilon_1^*|\xi(s)|)\Big)ds \\
&\quad \le M\|f(0, \psi)\| + v_f(t)(c + \Upsilon_1^*|\xi(t)|) + Mv^*\Big(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0} + a_0(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0})\Big) \\
&\quad \le M\|f(0, \psi)\| + v_f^*(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0}) + Mv^*(1 + a_0)(c + \Upsilon_1^*\|\xi\|_{\mathcal{Y}_t^0}) \\
&\quad \le M\|f(0, \psi)\| + v_f^*c + Mv^*c(1 + a_0) + \Upsilon_1^*[v_f^* + Mv^*(1 + a_0)]\|\xi\|_{\mathcal{Y}_t^0} \\
&\quad := \Theta + \Upsilon_1^*[v_f^* + Mv^*(1 + a_0)]\|\xi\|_{\mathcal{Y}_t^0},
\end{aligned}$$

where $\Theta = M\|f(0, \psi)\| + v_f^*c + Mv^*c(1 + a_0)$.

Thus, $\widetilde{\mathcal{P}}\xi(t) \in \mathcal{Y}_t^0$.

Moreover, choose $r > 0$ such that

$$r \ge \frac{\Theta}{1 - \Upsilon_1^*[v_f^* + Mv^*(1 + a_0)]},$$

and B_r be same as defined in Theorem 3.1. Now take $\xi \in B_r$ and $t \geq 0$; then

$$\|(\widetilde{\mathcal{P}}\xi)(t)\| \leq \Theta + \Upsilon_1^*[v_f^* + Mv^*(1+a_0)]r.$$

Hence $\|\widetilde{\mathcal{P}}\xi\|_{\mathcal{Y}_t^0} \leq r$ and $\widetilde{\mathcal{P}}(B_r) \subset B_r$.

Step 2 and 3: $\widetilde{\mathcal{P}}$ is continuous and equicontinuous in B_r.

By thinking of steps 2 and 3 of Theorem 3.1 and conditions (H_5) and (H_7), we have come to the conclusion that the operator $\widetilde{\mathcal{P}}$ is both continuous and equicontinuous in the space B_r.

Step 4: $\widetilde{\mathcal{P}}$ is equiconvergent.

Let $t \geq 0$ and $\xi \in B_r$; we get

$$\|(\widetilde{\mathcal{P}}\xi)(t)\| \leq \Theta + \Upsilon_1^*[v_f^* + Mv^*(1+a_0)]r.$$

Then, we have

$$\lim_{t\to+\infty} \|(\widetilde{\mathcal{P}}\xi)(t)\| \leq \Theta + \Upsilon_1^*[v_f^* + Mv^*(1+a_0)]r.$$

Hence,

$$\|(\widetilde{\mathcal{P}}\xi)(t) - (\widetilde{\mathcal{P}}\xi)(+\infty)\| \to 0 \text{ as } t \to +\infty,$$

and $\widetilde{\mathcal{P}}$ is equiconvergent.

Step 5: The Mönch conditions hold. As a result of Step 5 of Theorem 3.1 and assumptions (H_5)–(H_7), we have

$$\begin{aligned}\alpha(\widetilde{\mathcal{P}}(\xi^n)) &\leq 2(1+l^*)M\alpha(\{\xi^n\})\int_0^t v(s)ds + 2v_f(\xi)\alpha(\{\xi^n\}) \\ &\leq 2\big[(1+l^*)Mv^* + v_f^*\big]\alpha(\{\xi^n\})\end{aligned}$$

which implies that

$$\alpha(\widetilde{\mathcal{P}}(U)) \leq 2\big[(1+l^*)Mv^* + v_f^*\big]\alpha(U).$$

Therefore, by the hypothesis in the Mönch condition, we have

$$\alpha(U) \leq \alpha(\widetilde{\mathcal{P}}) \leq \mathcal{S}^{'}\alpha(U)$$

and then

$$\alpha(U)(1-\mathcal{S}^{'}) \leq 0.$$

According to (3.3), we see that $\alpha(U) = 0$ and for $\sigma \in J$; then $U(\sigma)$ is relatively compact in $\mathcal{H}$.

Hence, by Mönch fixed point theorem, we conclude that $\widetilde{\mathcal{P}}$ has a fixed point $\bar{\xi}$. Then $\bar{\zeta} = \bar{\xi} + \theta$ is a fixed point of $\mathcal{P}$, which is a mild solution of the system 1.2. □

4 Example

To illustrate our theoretical results, we treat the following partial integrodifferential system

$$\begin{cases} d\Big[z(t,x) - e^{-t}\displaystyle\int_{-\infty}^{t} e^{2(s-t)}\frac{z(s-\rho_1(s)\rho_2(\|z(s)\|),x)}{49}ds\Big] \\ = \Big[\dfrac{\partial^2}{\partial x^2}z(t,x) + \lambda_1\dfrac{\partial}{\partial}z(t,x) + \lambda_2 z(t,x) \\ \qquad + \displaystyle\int_0^t \delta(t-s)\Big(\frac{\partial^2}{\partial x^2}z(s,x) + \lambda_1\frac{\partial}{\partial}z(s,x) + \lambda_2 z(s,x)\Big)ds \\ \qquad + e^{-t}\displaystyle\int_{-\infty}^{t} e^{2(s-t)}\frac{z(s-\rho_1(s)\rho_2(\|z(s)\|),x)}{64}ds \\ \qquad + e^{-t}\displaystyle\int_0^t \sin(s-t)\int_{-\infty}^{s} e^{2(\nu-s)}\frac{z(\nu-\rho_1(\nu)\rho_2(\|z(\nu)\|),x)}{16}ds\Big]dt, \\ z(t,0) = 0 = z(t,\pi),\ t\in[0,+\infty), \\ z(t,x) = 0 = \psi(t,x),\ -\infty < t \le 0, 0 \le x \le \pi, \end{cases} \tag{4.1}$$

where $\rho_j = \mathbb{R} \to \mathbb{R}^+,\ j = 1, 2,\ \psi \in \mathcal{B}$, and $\delta(\cdot)$ is a bounded differentiable function on $\mathbb{R}^+$ with a bounded uniformly continuous derivative $\delta^{'}(\cdot)$ with $\lambda_1, \lambda_2 \in \mathbb{R}$. Let $\mathcal{H} = L^2[0,\pi]$ be the space of square integral functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the norm $\|\cdot\|_{L^2}$ induced by the following inner product

$$\langle\gamma,\varphi\rangle = \int_0^{\pi}\gamma(t)\varphi(t)dt,$$

where γ and φ are square integral functions. $L^2[0,\pi]$ is a separable Hilbert space. Determine the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ by $Ay = y^{''} + \lambda_1 y^{'} + \lambda_2 y$ with the domain

$$D(A) = H^2(0,\pi) \cap H_0^1(0,\pi).$$

Then it is known from [20] that A generates an analytic C_0-semigroup $(S(t))_{t\ge0}$, then (R_1) is fulfilled. In addition, since the semigroup $(S(t))_{t\ge0}$ is analytic, then it is norm-continous for $t > 0$. Hence, in virtue of Theorem 2.3, the corresponding resolvent operator is norm-continous for $t > 0$ and (H_1) is satisfied.

Let $F(t) : D(A) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$F(t)z = \delta(t)Az \text{ for } t \ge 0 \text{ and } z \in D(A).$$

Then, $(F(t))_{t\ge0}$ satisfies (R_2) thanks to the assuption on δ. Therefore, system 1.2 has a unique resolvant operator $(\mathcal{Q}(t))_{t\ge0}$, which is also norm-continous for $t > 0$.

For the phase space, we choose $p(s) = e^{2s},\ s < 0$; then we set $m = \displaystyle\int_{-\infty}^{0} p(s)ds = \frac{1}{2} < \infty$, for $s \le 0$ and determine

$$\|\phi\|_{\mathcal{B}} = \int_{-\infty}^{0} p(s)\sup_{\nu\in[s,0]}\|\phi(\nu)\|_{L^2}ds.$$

Let $(\tau,\phi) \in [0,1] \subset [0,+\infty)\times\mathcal{B}$, where $\phi(\nu)(x) = \phi(\nu,x),\ (\nu,x) \in (-\infty,0]\times[0,\pi]$.To rewrite the system (4.1), we define

$$z(t)(x) = z(t,x), \quad \rho(t,\phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$$

$$f(\sigma,\phi)(x)=e^{-t}\int_{-\infty}^{0}e^{2s}\frac{\phi}{25}ds$$
$$g(t,\phi,\mathcal{T}\phi)(x)=e^{-\tau}\int_{-\infty}^{0}e^{2s}\frac{\phi}{81}ds+(\mathcal{T}\phi)(x),$$

where

$$(\mathcal{T}\phi)(x)(=h(t,s,z)(x)))=e^{-t}\int_0^t\sin(t-s)\int_{-\infty}^{0}e^{2\tau}\frac{\phi}{36}\tau ds.$$

Then, the system (4.1) takes the subsequent abstract form

$$\begin{cases} d\big(\zeta(t)-f(t,\zeta_{\rho(t,\zeta_t)})\big)=A\zeta(t)dt\\ \quad +\Big(\int_0^t F(t-s)\zeta(s)ds+g\Big(\sigma,\zeta_{\rho(s,\zeta_s)},\int_0^t h(t,s,\zeta_{\rho(s,\zeta_s)})ds\Big)\Big)dt,\\ \qquad\qquad t\in J=[0,+\infty),\\ \zeta(t)=\psi(t)\in\mathcal{B}. \end{cases}\tag{4.2}$$

Next, we are going to chek the assumptions $(H_2)--(H_7)$ for the above system (4.1).

Verification of (H_2):

The function $h(t,s,z)(x)$ is Carathéodory and for $t\in[0,1]\subset[0,+\infty)$, $\psi\in\mathcal{B}$, we have

$$\begin{aligned}\|h(t,s,z)\|_{L^2}&\le\left(\int_0^\pi\left(e^{-t}\int_0^t\|\sin(t-s)\|\int_{-\infty}^{0}e^{2\tau}\left\|\frac{\phi}{36}\right\|d\tau ds\right)^2dx\right)^{\frac12}\\ &\le\left(\int_0^\pi\left(\frac{1}{36}e^{-t}\int_{-\infty}^{0}e^{2s}\sup\|\phi\|ds\right)^2dx\right)^{\frac12}\\ &\le\frac{\sqrt{\pi}}{36}e^{-t}\|\phi\|_{\mathcal{B}}\\ &\le v_1(t)\|\phi\|_{\mathcal{B}},\end{aligned}$$

where $v_1(t)=\frac{\sqrt{\pi}}{36}e^{-t}$ and $l^*=\frac{\sqrt{\pi}}{36}\sup_{t\in[0,1]\subset[0,+\infty)}\int_0^\sigma e^{-s}ds=0.031$. Also, we can see that each bounded set $G_1\subset\mathcal{B}$ and

$$\alpha(h(t,s,G_1))\le\frac{\sqrt{\pi}}{36}e^{-t}\sup_{-\infty<n\le0}\alpha(G_1(n))\ \text{ for a.e. } t,s\in[0,1]\subset[0,+\infty).$$

Verification of (H_3) **and** (H_4): The function $g(\sigma,s,z)(x)$ is is Carathéodory and for $t\in[0,1]\subset[0,+\infty)$, $\psi\in\mathcal{B}$, we have

$$\begin{aligned}&\|g(t,s,z)(x)\|_{L^2}\\ &\le\left(\int_0^\pi\left(e^{-t}\int_{-\infty}^{0}e^{2s}\left\|\frac{\phi}{81}\right\|ds+e^{-t}\int_0^t\|\sin(t-s)\|\int_{-\infty}^{0}e^{2\tau}\left\|\frac{\phi}{36}\right\|\tau ds\right)^2dx\right)^{\frac12}\\ &\le\left(\int_0^\pi\left(\frac{1}{81}e^{-t}\int_{-\infty}^{0}e^{2s}\sup\|\phi\|ds+e^{-t}\int_{-\infty}^{0}e^{2s}\sup\|\phi\|\tau ds\right)^2dx\right)^{\frac12}\\ &\le\frac{13\sqrt{\pi}}{324}e^{-t}\|\phi\|_{\mathcal{B}}\end{aligned}$$

$$\leq v(t)\|\phi\|_{\mathcal{B}},$$

where $v(t) = \dfrac{13\sqrt{\pi}}{324}e^{-t}$ et $v^* = \dfrac{13\sqrt{\pi}}{324} \sup\limits_{t\in[0,1]\subset[0,+\infty)} \int_0^t e^{-s}ds = 0.045$. Also, we can see that for each bounded set $G_2 \subset \mathcal{B}$, $\overline{G} \subset \mathcal{H}$, $t \in [0,1] \subset [0,+\infty)$ and

$$\alpha(g(t, G_2, \overline{G})) \leq \frac{13\sqrt{\pi}}{324}e^{-t}\Big(\sup_{-\infty<y\leq 0} \alpha(G_1(y)) + \alpha(\overline{G})\Big),$$

where $G_2) = \{z(y) : z \in G_2\}$.

It follows that the hypotheses (H_3) and (H_4) are satisfied.

Verification of $(H_5)-(H_7)$:

The function $f(t,\phi)(x)$ is Carathéodory and for $t \in [0,1] \subset [0,+\infty)$, $\psi \in \mathcal{B}$, we have

$$\begin{aligned}
\|f(t,\phi)\|_{L^2} &\leq \left(\int_0^{\pi} \left(e^{-t}\int_{-\infty}^{0} e^{2s}\left\|\frac{\phi}{25}\right\|ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{\pi} \left(\frac{1}{25}e^{-t}\int_{-\infty}^{0} e^{2s}\sup\|\phi\|ds\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{\pi}}{25}e^{-t}\|\phi\|_{\mathcal{B}} \\
&\leq v_f(t)\|\phi\|_{\mathcal{B}},
\end{aligned}$$

where $v_f(t) = \dfrac{\sqrt{\pi}}{25}e^{-t}$ and $v_f^* = \dfrac{\sqrt{\pi}}{25} \sup\limits_{t\in[0,1]\subset[0,+\infty)} \int_0^t e^{-s}ds = 0.045$. Also, we can see that for each bounded set $G \subset \mathcal{B}$, we get

$$\alpha(f(t,G)) \leq \frac{\sqrt{\pi}}{25}e^{-t} \sup_{-\infty<y\leq 0} \alpha(G(y)) \text{ for a.e. } t \in [0,1] \subset [0,+\infty).$$

For any $0 \leq \tau_1 < \tau_2 \leq \beta$ and for each $\psi \in B_r$, we obtain

$$\begin{aligned}
\|f(\tau_2,\phi)(x) - f(\tau_1,\phi)(x)\|_{L^2} &\leq \frac{\sqrt{\pi}}{25}(e^{-\tau_2} - e^{-\tau_1})\|\phi\|_{\mathcal{B}} \\
&\to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned}$$

As a consequence, we realize that the hypotheses $(H_5)-(H_7)$ are satisfied.

Remember that $l^* = 0.031$, $v^* = 0.045$, $v_f^* = 0.045$. Moreover, taking $M = 1$, we get for Theorem 3.1

$$\mathcal{S} = 2(1+l^*)Mv^* = 2(1+0.031)\times 0.045 \times 1 = 0.093 < 1$$

and for Theorem 3.2 we get

$$\mathcal{S}' = 2\Big[(1+l^*)Mv^* + v_f^*\Big] = 2[(1+0.031)\times 0.045\times 1 + 0.045] = 0.1828 < 1.$$

Thus 3.1 and 3.3 is verified. Thus, all the conditions of Theorem 3.1 and 3.2 are fulfilled, then the system (1.1) and (1.2) is a mild solution.

Remark 4.1 By adopting the techniques of Theorems 3.1 and 3.2 we can prove the result for the impulsive neutral integrodifferential equations with state-dependent delay of the form:

$$\begin{cases} d\big(\zeta(t) - f(t, \zeta_{\rho(t,\zeta_t)})\big) \\ \quad = A\zeta(t)dt + \left(\int_0^t F(t-s)\zeta(s)ds + g\Big(t, \zeta_{\rho(s,\zeta_s)}, \int_0^t h(t,s,\zeta_{\rho(s,\zeta_s)})ds\Big)\right)dt, \\ \qquad\qquad\qquad\qquad\qquad t \in J = [0, +\infty), \\ \\ \Delta\zeta(t_k) = \zeta(t_k^+) - \zeta(t_k^-) = \mathbb{I}_k(\zeta_{t_k}),\ t = t_k,\ k \in 1, 2, ..., m, \\ \zeta(t) = \varphi(t) \in \mathcal{B},\ t \in (-\infty, 0], \end{cases}$$

5 Conclusion

In this manuscript, we established the existence results for integrodifferential equations (1.1) and (1.2). We proved the existence of mild solutions for considered systems under some conditions and by using Mönch fixed point theorem with the measure of noncompactness. Finally, an example is given to illustrate the effectiveness of the obtained results. However, we can extend the obtained results to some stochastic integrodifferential equations with random impulses.

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