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REAL AND COMPLEX ANALYSIS

Existence and Stability of Integro Differential Equation with Generalized Proportional Fractional Derivative

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Abstract—In this study, integro-differential equations of arbitrary order are studied. The fractional order is expressed in terms of the ψ-Hilfer type proportional fractional operator. This research exposes the dynamical behavior of integro-differential equations with fractional order, such as existence, uniqueness, and stability solutions. The initial value problem and nonlocal conditions are used to prove the results.

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1. INTRODUCTION

Fractional differential equations (FDEs) is considered as a branch of mathematical analysis that deals with the investigation and applications of integrals and derivatives of arbitrary order. Therefore, FDE is an extension of the integer-order calculus that considers integrals and derivatives of any real or complex order, see [9, 11, 13]. The topic of FDEs includes the study of analytic and numerical methods, as well as theoretical features such as existence, uniqueness, periodicity, and asymptotic behaviour. One can refer to [4, 5, 10, 16, 20, 21] for recent works on FDEs.

Nowadays, there are noteworthy potentials that have been spent on getting new classes of fractional operators by introducing more general or new kernels. Vanterler Da C. Sousa recently presented a fractional derivative with kernel of function, and the classical features of current fractional derivatives are explored in [15]. The theoretical analysis and current progress of the ψ-Hilfer fractional derivative can be observed in [16, 17]. In this work, we use the generalized fractional calculus for a special example of the proportional derivatives discussed in [6]. The new fractional derivative operator contains two parameters and has features, including maintaining the semigroup property and convergence to the original function as it tends to zero. Additionally, it is fully behaved and has fundamental features over the classical derivatives with the meaning that it generalizes already existing fractional derivatives in the literature. Some recent contributions on fractional differential equations in terms of the generalized proportional derivatives can be located in the papers, see [1–3].

The existence and uniqueness of the solution play an essential role in the study of FDEs, see [16, 20]. In this paper, we study the existence and uniqueness of solution for a certain type of nonlinear integro-differential equation (IDE) with initial and nonlocal conditions. Further, the stability of solutions is also being discussed.

The paper is constructed as follows: In Section 2, we present the main definitions and interesting results. In Section 3, existence and stability results are established for proposed problems. In Section 4, existence and stability of solutions for nonlocal IDE is discussed.

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2. PRELIMINARIES

Some basic definitions and results introduced in this section. Let C be the Banach space of all continuous functions $\mathfrak{h} : J \to R$ with the norm

$$||\mathfrak{h}||_C = \sup \{|\mathfrak{h}(t)| : t \in J\}.$$

We denote the weighted spaces of all continuous functions defined by

$$C_{\nu,\psi}(J,R) = \{\mathfrak{g} : J \to R : (\psi(t) - \psi(0))^\nu \mathfrak{g}(t) \in C\},\ 0 \leq \nu < 1,$$

with the norm

$$||\mathfrak{g}||_{C_{\nu,\psi}} = \sup_{t\in J} |(\psi(t) - \psi(0))^\nu \mathfrak{g}(t)|.$$

Definition 2.1 [6]. If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$. Then the generalized proportional fractional(GPF) integral

$$\left(\mathcal{I}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \int_0^t \psi'(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t)-\psi(s))^{\alpha-1}}{\vartheta^\alpha \Gamma(\alpha)} \mathfrak{h}(s) ds. \tag{2.1}$$

Definition 2.2 [6]. If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the GPF derivative of order α of the function $\mathfrak{h}$ with respect to another function is defined by with $\psi'(t) \neq 0$ is described as

$$\left(\mathcal{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n$$
$$\times \int_0^t \psi'(s) e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \frac{(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} \mathfrak{h}(s) ds. \tag{2.2}$$

Definition 2.3 [6]. If $\vartheta \in (0,1]$ and $\alpha \in C$ with $\Re(\alpha) > 0$ and $\psi \in C[a,b]$, where $\psi'(s) > 0$, the GPF derivative in Caputo sence of order α of the function $\mathfrak{h}$ with respect to another function is defined by with $\psi'(t) \neq 0$ is described as

$$\left(\mathcal{D}^{\alpha,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathcal{I}^{n-\alpha,\vartheta;\psi}\left(\mathcal{D}^{n,\vartheta;\psi}\mathfrak{h}\right)(t). \tag{2.3}$$

Definition 2.4 [6]. The ψ-Hilfer GPF derivative of order α and type β over $\mathfrak{h}$ with respect to another function is defined by

$$\left(\mathcal{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}\right)(t) = \mathcal{I}^{\beta(1-\alpha),\vartheta;\psi}\left(\mathcal{D}^{1,\vartheta;\psi}\right)\mathcal{I}^{(1-\beta)(1-\alpha),\vartheta;\psi}\mathfrak{h}(t). \tag{2.4}$$

Lemma 2.1. *Let $\alpha, \beta > 0$, Then we have the following semigroup property*

$$(\mathcal{I}^{\alpha,\vartheta;\psi}\mathcal{I}^{\beta,\vartheta;\psi}\mathfrak{g})(t) = (\mathcal{I}^{\alpha+\beta,\vartheta;\psi}\mathfrak{g})(t),$$

and

$$(\mathcal{D}^{\alpha,\vartheta;\psi}\mathcal{I}^{\alpha,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t).$$

Lemma 2.2. *Let $n-1 < \alpha < n$ where $n \in N, \vartheta \in (0,1]$, $0 \leq \beta \leq 1$, with $\nu = \alpha + \beta(n-\alpha)$, such that $n-1 < \nu < n$. If $\mathfrak{g} \in C_\nu$ and $\mathfrak{I}^{n-\nu,\vartheta;\psi}\mathfrak{g} \in C^n_\nu$, then*

$$(\mathcal{I}^{\alpha,\vartheta;\psi}\mathcal{I}^{\alpha,\beta,\vartheta;\psi}\mathfrak{g})(t) = \mathfrak{g}(t) - \sum_{k=1}^{n} \frac{e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\nu-k}}{\vartheta^{\nu-k}\Gamma\nu-k+1}\mathcal{I}^{k-\nu,\vartheta;\psi}(a).$$

Lemma 2.3 ([15], Grönwall's lemma). *Let $\alpha > 0$, $a(t) > 0$ is locally integrable function on J and if $\mathfrak{g}(t)$ be a increasing and nonnegative continuous function on J, such that $|\mathfrak{g}(t)| \leq K$ for some constant K. Moreover if $\mathfrak{h}(t)$ be a nonnegative locally integrable function on J with*

$$\mathfrak{h}(t) \leq a(t) + g(t) \int_0^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} \mathfrak{h}(s) ds, \quad (t) \in J,$$

with some $\alpha > 0$. Then

$$\mathfrak{h}(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\mathfrak{g}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s) \left(\psi(t) - \psi(s)\right)^{n\alpha-1} \right] a(s) ds, \quad (t) \in J.$$

Theorem 2.1 ([8], Schauder fixed point theorem). *Let B be closed, convex and nonempty subset of a Banach space E. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.*

Theorem 2.2 ([8], Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let T_1, T_2 be mapping from Ω into X such that $T_1 x + T_2 y, \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1 x + T_2 x = x$ has a solution on Ω.*

Theorem 2.3 ([8], Banach fixed point theorem). *Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping $\mathfrak{P}$ from Q into itself has a unique fixed point.*

3. SOLUTION OF INITIAL VALUE PROBLEM

In this section, we consider the initial value problem (IVP) for fractional IDE of the form

$$\begin{cases} \mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}\left(t, \mathfrak{h}(t), \int_0^t k(t,s,\mathfrak{h}(s))ds\right), & t \in J := [0,T], \\ \mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t)|_{t=0} = \mathfrak{h}_0, \end{cases} \tag{3.1}$$

where $\mathfrak{D}^{\alpha,\beta;\psi}$ is ψ-Hilfer GPF of orders $\alpha \in (0,1)$, type $\beta \in [0,1]$ and $\vartheta \in [0,1]$, $\mathfrak{h}$ is the given continuous function, $\mathfrak{I}^{1-\nu;\psi}$ is GPF fractional integral of orders $1-\nu (\nu = \alpha + \beta - \alpha\beta)$. Let R be a Banach space, $\mathfrak{g} : J \times R \times R \to R$ is a given continuous function. For brevity let us take

$$H\mathfrak{h}(t) = \int_0^t k(t,s,\mathfrak{h}(s))ds.$$

We make the following hypotheses to prove our main results, for every $t \in J$. We declare

(H1) There exists a constant $\ell_{\mathfrak{g}} > 0$ such that

$$|\mathfrak{g}(s, \mathfrak{h}_1(\cdot), \mathfrak{h}_2(\cdot)) - \mathfrak{g}(s, \mathfrak{y}_1(\cdot), \mathfrak{y}_2(\cdot))| \leq \ell_{\mathfrak{g}} \left(|\mathfrak{h}_1(\cdot) - \mathfrak{y}_1(\cdot)| + |\mathfrak{h}_2(\cdot) - \mathfrak{y}_2(\cdot)|\right),$$

Set $\tilde{\mathfrak{g}} = \mathfrak{g}(s, 0, 0)$.

(H2) For all $\mathfrak{h}, \mathfrak{y} \in R$, there exists a constant $\ell_{\mathfrak{h}} > 0$, such that

$$\int_0^t |k(t,s,\mathfrak{h}) - k(t,s,\mathfrak{y})| \leq \ell_{\mathfrak{h}} \left|\mathfrak{h}(\cdot) - \mathfrak{y}(\cdot)\right|.$$

Set $\tilde{k} = \int_0^s |k_\omega(s,\tau,0)| \, d\tau$.

(H3) There exists $\lambda_\varphi > 0$, we have $\mathfrak{I}^{\alpha;\psi}\varphi(t) \leq \lambda_\varphi \varphi(t)$.

Lemma 3.1. *A function* $\mathfrak{h}$ *is the solution Eq. (3.1), if and only if* $\mathfrak{h}$ *satisfies the random integral equation*

$$\mathfrak{h}(t)=\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1}$$

$$+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \tag{3.2}$$

Theorem 3.1. *Assume that hypotheses* $(H1)$ *and* $(H2)$ *are satisfied. Then, Eq. (3.1) has at least one solution.*

Proof. Consider the operator $\mathfrak{P}: C_{1-\nu,\psi}\to C_{1-\nu,\psi}$, where the equivalent integral Eq. (3.2) can be written in the operator form

$$\mathfrak{P}\mathfrak{h}(t)=\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1}$$

$$+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \tag{3.3}$$

For any $\mathfrak{h}\in J$, we have

$$\left|\mathfrak{P}(\mathfrak{h}(t))\left(\psi(t)-\psi(0)\right)^{1-\nu}\right|\leq\left|\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}\right|$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right|ds\leq\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}$$

$$+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))-\mathfrak{g}(s,0,0)+\mathfrak{g}(s,0,0)\right|ds$$

$$\leq\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left(\ell_{\mathfrak{g}}\left|\mathfrak{h}(s)\right|+\ell_{\mathfrak{g}}\left|H\mathfrak{h}(s)\right|+\left|\tilde{\mathfrak{g}}\right|\right)ds$$

$$\leq\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\Bigg(\ell_{\mathfrak{g}}\left|\mathfrak{h}(s)\right|$$

$$+\ell_{\mathfrak{g}}\int_0^s\left|k(s,\tau,\mathfrak{h}(\tau))-k(s,\tau,0)\right|d\tau+\ell_{\mathfrak{g}}\int_0^s\left|k(s,\tau,0)\right|d\tau+\left|\tilde{\mathfrak{g}}\right|\Bigg)ds$$

$$\leq\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}+\frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})r+\ell_{\mathfrak{g}}\left|\left|\tilde{k}\right|\right|_{C_{1-\nu,\psi}}+||\tilde{\mathfrak{g}}||_{C_{1-\nu,\psi}}\right)=r.$$

This proves that $\mathfrak{P}$ transforms the ball $B_r=\left\{\mathfrak{h}\in C_{1-\nu,\psi}:||\mathfrak{h}||_{C_{1-\nu,\psi}}\leq r\right\}$ into itself. We shall show that the operator $\mathfrak{P}:B_r\to B_r$ satisfies all the conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: $\mathfrak{P}$ is continuous. Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n\to\mathfrak{h}$ in $C_{1-\nu,\psi}$; we derive

$$\left|\left(\mathfrak{P}\mathfrak{h}_n(t)-\mathfrak{P}\mathfrak{h}(t)\right)\left(\psi(t)-\psi(0)\right)^{1-\nu}\right|\leq\frac{\left(\psi(t)-\psi(0)\right)^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}$$

$$\times\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{h}_n(s),H\mathfrak{h}_n(s)))-\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s)))\right|ds$$

$$\leq (\psi(t)-\psi(0))^{1-\nu}\frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(t)-\psi(0))^{\alpha+\nu-1}\left|\left|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),H\mathfrak{h}_n(\cdot)))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),H\mathfrak{h}(\cdot)))\right|\right|_{C_{1-\nu,\psi}}$$

$$\leq \frac{B(\nu,\alpha)}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}\left|\left|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot),H\mathfrak{h}_n(\cdot)))-\mathfrak{g}(\cdot,\mathfrak{h}(\cdot),H\mathfrak{h}(\cdot)))\right|\right|_{C_{1-\nu,\psi}}.$$

Since $\mathfrak{g}$ is continuous, we have $||\mathfrak{P}\mathfrak{h}_n-\mathfrak{P}\mathfrak{h}||_{C_{1-\nu,\psi}}\to 0$ as $n\to\infty$.

Step 2: $\mathfrak{P}(B_r)$ is uniformly bounded. This is clear since $\mathfrak{P}(B_r)\subset B_r$ is bounded.

Step 3: We show that $\mathfrak{P}(B_r)$ is equicontinuous.

Let $t_1,t_2\in J, t_1>t_2$ be a bounded set of $C_{1-\nu,\psi}$ as in Step 2, and $\mathfrak{h}\in B_r$. Then,

$$\left|(\psi(t_1)-\psi(0))^{1-\nu}\mathfrak{P}\mathfrak{h}(t_1)-(\psi(t_2)-\psi(0))^{1-\nu}\mathfrak{P}\mathfrak{h}(t_2)\right|$$

$$\leq \left|\frac{(\psi(t_1)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right.$$

$$\left.-\frac{(\psi(t_2)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right|$$

$$\leq \frac{||\mathfrak{g}||_{C_{1-\nu,\psi}}}{\vartheta^{\alpha}\Gamma(\alpha)}B(\nu,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha}-(\psi(t_2)-\psi(0))^{\alpha}\right|.$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Steps 1–3 together with the Arzela–Ascoli theorem, we can conclude that $\mathfrak{P}$ is continuous and compact. From an application of Theorem 2.1, we deduce that $\mathfrak{P}$ has a fixed point $\mathfrak{h}$ which is a solution of the problem (3.1). □

Lemma 3.2. *Assume that hypotheses (H1) and (H2) are satisfied. If*

$$\frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)}(\psi(T)-\psi(0))^{\alpha}B(\nu,\alpha)<1.$$

Then, (3.1) has a unique solution.

Next, we shall give the definitions of the generalized Ulam–Hyers–Rassias (g-UHR) stability for the problem (3.1). Let $\epsilon>0$ be a positive real number and $\varphi: J\to R^{+}$ be a continuous function. We consider the following inequalities

$$\left|\mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{y}(t)-\mathfrak{g}(t,\mathfrak{y}(t),H\mathfrak{y}(t))\right|\leq\varphi(t). \tag{3.4}$$

Definition 3.1. Equation (3.1) is g-UHR stable with respect to φ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $\mathfrak{y}: R\to C_{1-\nu,\psi}$ of inequality (3.4) there exists a solution $\mathfrak{h}: R\to C_{1-\nu,\psi}$ of Eq. (3.1) with

$$|\mathfrak{y}(t)-\mathfrak{h}(t)|\leq C_{f,\varphi}\varphi(t),\quad t\in J.$$

Theorem 3.2. *The hypotheses (H1), (H2) and (H3) hold. Then Eq. (3.1) is g-UHR stable.*

Proof. Let $\mathfrak{y}$ be solution of inequality (3.4) and by Lemma 3.2 there exists a unique solution $\mathfrak{h}$ for the Eq. (3.1). Thus, we have

$$\mathfrak{h}(t)=\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-(\psi(0))^{\nu-1}$$

$$+\frac{1}{\vartheta^{\alpha}\Gamma(\alpha)}\int_0^{t}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds.$$

By differentiating inequality (3.4) for each $t\in J$, we have

$$\left|\mathfrak{y}(t)-\frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}(\psi(t)-\psi(0))^{\nu-1}\right.$$

$$- \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds \Bigg| \le \lambda_{\varphi}\varphi(t).$$

Hence, it follows

$$\begin{aligned}
|\mathfrak{y}(t)-\mathfrak{h}(t)| \le \Bigg| \mathfrak{y}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} (\psi(t)-(\psi(0))^{\nu-1} \\
+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \Bigg| \\
\le \Bigg| \mathfrak{y}(t) - \frac{\mathfrak{h}_0}{\vartheta^{\nu-1}\Gamma(\nu)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} (\psi(t)-(\psi(0))^{\nu-1} \\
- \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds \Bigg| \\
+ \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \left|\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s)) - \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right| ds \\
\le \lambda_{\varphi}\varphi(t) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |\mathfrak{y}(s)-\mathfrak{h}(s)|\, ds \\
\le \lambda_{\varphi}\varphi(t) + \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})}{\vartheta^{\alpha}\Gamma(\alpha)} \int\limits_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \lambda_{\varphi}\varphi(s)ds := C_{f,\varphi}\varphi(t).
\end{aligned}$$

Thus, Eq. (3.1) is g-UHR stable. □

4. SOLUTION OF FRACTIONAL NONLOCAL IVP

In this section, we study the existence, uniqueness, and stability of IDE involving ψ-Hilfer derivative given by

$$\begin{cases} \mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t),H\mathfrak{h}(t)), \\ \mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t) = \sum_{i=1}^m c_i \mathfrak{h}(\tau_i), \quad \tau_i \in J, \end{cases} \tag{4.1}$$

where $\tau_i, i=0,1,...,m$ are prefixed points satisfying $0<\tau_1\le ... \le \tau_m < b$ and c_i is real numbers. Here, nonlocal condition $\mathfrak{h}(0) = \sum_{i=1}^m c_i\mathfrak{h}(\tau_i)$ can be applied in physical problems yielding better effect than the initial conditions $\mathfrak{h}(0)=\mathfrak{h}_0$. Further, Eq. (3.1) is equivalent to mixed integral type of the form

$$\mathfrak{h}(t) = \begin{cases} \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^m c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) \\ \times (\psi(\tau_i)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds \\ + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds, \end{cases} \tag{4.2}$$

where

$$T = \frac{1}{\vartheta^{\nu-1}\Gamma(\nu) - \sum_{i=1}^m c_i e^{\frac{\vartheta-1}{\vartheta}(\psi(\tau_i)-\psi(0))} (\psi(\tau_i)-\psi(0))^{\nu-1}}.$$

Theorem 4.1. *Assume that (H1) and (H2) are satisfied. Then, Eq. (4.1) has at least one solution.*

Consider the operator $\mathcal{N}: C_{1-\nu,\psi} \to C_{1-\nu,\psi}$, it is well defined and given by

$$\mathcal{N}\mathfrak{h}(t) = \begin{cases} \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) \\ \times (\psi(\tau_i)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds \\ + \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds. \end{cases} \tag{4.3}$$

Set $\widetilde{\mathfrak{g}}(s) = \mathfrak{g}(s,0,0)$. Consider the ball $B_r = \left\{ \mathfrak{h} \in C_{1-\nu,\psi} : ||\mathfrak{h}||_{C_{1-\nu,\psi}} \leq r \right\}$.

Now, we subdivide the operator $\mathcal{N}$ into two operator $\mathcal{N}_1$ and $\mathcal{N}_2$ on B_r as follows:

$$\mathcal{N}_1\mathfrak{h}(t) = \frac{T(\psi(t)-\psi(0))^{\nu-1}}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s)$$
$$\times (\psi(\tau_i)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds,$$

and

$$\mathcal{N}_2\mathfrak{h}(t) = \frac{1}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))} \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} \mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds.$$

The proof is divided into several steps.

Step 1. $\mathcal{N}_1\mathfrak{h} + \mathcal{N}_2\mathfrak{y} \in B_r$ for every $\mathfrak{h}, \mathfrak{y} \in B_r$.

$$\left| \mathcal{N}_1\mathfrak{h}(t) (\psi(t)-\psi(0))^{1-\nu} \right|$$
$$\leq \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi'(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} |\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))| ds$$
$$\leq \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_0^{\tau_i} \psi'(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} (|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s)) - \mathfrak{g}(s,0,0)| + |\mathfrak{g}(s,0,0)|) ds$$
$$\leq \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_a^{\tau_i} \psi'(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} (\ell_{\mathfrak{g}} (|\mathfrak{h}(s)| + |H\mathfrak{h}(s))|) + |\widetilde{\mathfrak{g}}(s)|) ds$$
$$\leq \frac{T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_a^{\tau_i} \psi'(s) (\psi(\tau_i)-\psi(s))^{\alpha-1} \left(\ell_{\mathfrak{g}} \left(|\mathfrak{h}(s)| + \ell_{\mathfrak{h}} |\mathfrak{h}(s))| + \left| \tilde{\mathfrak{k}}(s) \right| \right) + |\tilde{\mathfrak{g}}(s)| \right) ds.$$

This gives

$$||\mathcal{N}_1\mathfrak{h}||_{C_{1-\nu,\psi}} \leq \frac{B(\nu,\alpha)T}{\vartheta^{\alpha}\Gamma(\alpha)} \sum_{i=1}^{m} c_i (\psi(\tau_i)-\psi(a))^{\alpha+\nu-1}$$
$$\times \left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}}) ||\mathfrak{h}||_{C_{1-\nu,\psi}} + \ell_{\mathfrak{g}} \left|\left| \tilde{\mathfrak{k}} \right|\right|_{C_{1-\nu,\psi}} + ||\widetilde{\mathfrak{g}}||_{C_{1-\nu,\psi}} \right). \tag{4.4}$$

For operator $\mathcal{N}_2$

$$\left| \mathcal{N}_2\mathfrak{h}(t) (\psi(t)-\psi(0))^{1-\nu} \right| \leq \frac{(\psi(t)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t)-\psi(s))^{\alpha-1} |\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))| ds$$
$$\leq \frac{(\psi(t)-\psi(0))^{1-\nu}}{\vartheta^{\alpha}\Gamma(\alpha)} (\psi(t)-\psi(0))^{\alpha+\nu-1} B(\nu,\alpha)$$

$$\times \left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\,||\mathfrak{h}||_{C_{1-\nu,\psi}} + \ell_{\mathfrak{g}} \left|\left|\tilde{\mathfrak{k}}\right|\right|_{C_{1-\nu,\psi}} + ||\widetilde{\mathfrak{g}}||_{C_{1-\nu,\psi}}\right).$$

Thus, we obtain

$$||\mathcal{N}_2\mathfrak{h}||_{1-\nu} \leq \frac{B(\nu,\alpha)}{\vartheta^\alpha\Gamma(\alpha)} (\psi(t)-\psi(0))^\alpha \left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\,||\mathfrak{h}||_{C_{1-\nu,\psi}} + \ell_{\mathfrak{g}} \left|\left|\tilde{\mathfrak{k}}\right|\right|_{C_{1-\nu,\psi}} + ||\widetilde{\mathfrak{g}}||_{C_{1-\nu,\psi}}\right). \quad (4.5)$$

Linking (4.4) and (4.5), for every $\mathfrak{h}, \mathfrak{y} \in B_r$,

$$||\mathcal{N}_1\mathfrak{h} + \mathcal{N}_2\mathfrak{y}||_{C_{1-\nu,\psi}} \leq ||\mathcal{N}_1\mathfrak{h}||_{C_{1-\nu,\psi}} + ||\mathcal{N}_2\mathfrak{y}||_{C_{1-\nu,\psi}} \leq r.$$

Step 2. $\mathcal{N}_1$ is a contraction mapping. For any $\mathfrak{h}, \mathfrak{y} \in B_r$

$$\left|(\mathcal{N}_1\mathfrak{h}(t) - \mathcal{N}_1\mathfrak{y}(t))\,(\psi(t)-\psi(0))^{1-\nu}\right|$$

$$\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i} \psi'(s)\,(\psi(\tau_i)-\psi(s))^{\alpha-1}\,|\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s)) - \mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))|\,ds$$

$$\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_a^{\tau_i} \psi'(s)\,(\psi(\tau_i)-\psi(s))^{\alpha-1}\,|\mathfrak{h}(s)-\mathfrak{y}(s)|\,ds$$

$$\leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^m c_i\,(\psi(\tau_i)-\psi(0))^{\alpha+\nu-1}\,B(\nu,\alpha)\,||\mathfrak{h}-\mathfrak{y})||_{C_{1-\nu,\psi}}.$$

This gives

$$||(\mathcal{N}_1\mathfrak{h} - \mathcal{N}_1\mathfrak{y})||_{C_{1-\nu,\psi}} \leq \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})T}{\Gamma(\alpha)} \sum_{i=1}^m c_i\,(\psi(\tau_i)-\psi(0))^{\alpha+\nu-1}\,B(\nu,\alpha)\,||\mathfrak{h}-\mathfrak{y})||_{C_{1-\nu,\psi}}.$$

The operator $\mathcal{N}_1$ is contraction.

Step 3. The operator $\mathcal{N}_2$ is compact and continuous.

According to Step 1, we know that

$$||\mathcal{N}_2\mathfrak{h}||_{1-\nu,\psi} \leq \frac{B(\nu,\alpha)}{\Gamma(\alpha)} (\psi(t)-\psi(0))^\alpha \left(\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})\,||\mathfrak{h}||_{C_{1-\nu,\psi}} + \ell_{\mathfrak{g}} \left|\left|\tilde{\mathfrak{k}}\right|\right|_{C_{1-\nu,\psi}} + ||\widetilde{\mathfrak{g}}||_{C_{1-\nu,\psi}}\right).$$

So, operator $\mathcal{N}_2$ is uniformly bounded.

Now, we prove the compactness of operator $\mathcal{N}_2$.

For $0 < t_1 < t_2 < T$, we have

$$|\mathcal{N}_2\mathfrak{h}(t_1) - \mathcal{N}_2\mathfrak{h}(t_2)| \leq \left|\frac{1}{\vartheta^\alpha\Gamma(\alpha)} \int_0^{t_1} \psi'(s)\,(\psi(t_1)-\psi(s))^{\alpha-1}\,\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right.$$

$$\left. - \frac{1}{\vartheta^\alpha \frac{1}{\vartheta^\alpha\Gamma(\alpha)}\Gamma(\alpha)} \int_0^{t_2} \psi'(s)\,(\psi(t_2)-\psi(a))^{\alpha-1}\,\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right|$$

$$\leq ||\mathfrak{g}||_{C_{1-\nu,\psi}}\,B(\nu,\alpha)\left|(\psi(t_1)-\psi(0))^{\alpha+\nu-1} - (\psi(t_2)-\psi(0))^{\alpha+\nu-1}\right|$$

tending to zero as $t_1 \to t_2$. Thus, $\mathcal{N}_2$ is equicontinuous. Hence, the operator $\mathcal{N}_2$ is compact on B_r by the Arzela–Ascoli Theorem. It follows from Theorem 2.2 that problem (4.1) has at least one solution.

Theorem 4.2. *If hypothesis (H1) and the constant*

$$\delta = \frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{h}})B(\nu,\alpha)}{\vartheta^\alpha\Gamma(\alpha)} \left(T\sum_{i=1}^m c_i\,(\psi(\tau_i)-\psi(a))^{\alpha+\nu-1} + (\psi(b)-\psi(a))^\alpha\right) < 1$$

holds. Then, Eq. (4.1) has a unique solution.

Next, we shall give the definitions of g-UHR stability for Eq. (4.1)

$$\left|\mathfrak{D}^{\alpha,\beta;\psi}\mathfrak{y}(t)-\mathfrak{g}(t,\mathfrak{h}(t),H\mathfrak{h}(t))\right|\leq\varphi(t). \tag{4.6}$$

Definition 4.1. *Eq. (4.1) is g-UHR stable with respect to $\varphi\in C_{1-\nu,\psi}$ if there exists a real number $C_{f,\varphi}>0$ such that for each solution $\mathfrak{y}\in C_{1-\nu,\psi}$ of the inequality (4.6) there exists a solution $\mathfrak{h}\in C_{1-\nu,\psi}$ of Eq. (4.1) with*

$$|\mathfrak{y}(t)-\mathfrak{h}(t)|\leq C_{\mathfrak{g},\varphi}\varphi(t).$$

Theorem 4.3. *Let hypotheses (H1)−(H3) be fullfilled. Then Eq. (4.1) is g-UHR stable.*

Let $\mathfrak{y}$ be a solution of inequality (4.6) and by Theorem 4.2 $\mathfrak{h}$ is the unique solution of equation

$$\mathfrak{D}^{\alpha,\beta,\vartheta;\psi}\mathfrak{h}(t)=\mathfrak{g}(t,\mathfrak{h}(t),H\mathfrak{h}(t)),$$
$$\mathfrak{I}^{1-\nu,\vartheta;\psi}\mathfrak{h}(t)=\sum_{i=1}^{m}c_i\mathfrak{h}(\tau_i),\quad \tau_i\in J,$$

given by

$$\mathfrak{h}(t)=A_{\mathfrak{h}}+\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds,$$

where

$$A_{\mathfrak{h}}=\frac{T\left(\psi(t)-\psi(0)\right)^{\nu-1}}{\vartheta^\alpha\Gamma(\alpha)}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(0))}\sum_{i=1}^{m}c_i\int_0^{\tau_i}e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}$$
$$\times\psi'(s)\left(\psi(\tau_i)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds.$$

Thus, $A_{\mathfrak{h}}=A_{\mathfrak{y}}$.

By differentiating inequality (4.6), we have

$$\left|\mathfrak{y}(t)-A_{\mathfrak{y}}-\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds\right|\leq\lambda_\varphi\varphi(t).$$

Hence, it follows,

$$\begin{aligned}|\mathfrak{y}(t)-\mathfrak{h}(t)|&\leq\left|\mathfrak{y}(t)-A_{\mathfrak{h}}-\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))ds\right|\\&\leq\left|\mathfrak{y}(t)-A_{\mathfrak{h}}-\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\vartheta-1}{\vartheta}(\psi(t)-\psi(s))}\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))ds\right|\\&\quad+\frac{1}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{g}(s,\mathfrak{y}(s),H\mathfrak{y}(s))-\mathfrak{g}(s,\mathfrak{h}(s),H\mathfrak{h}(s))\right|ds\\&\leq\lambda_\varphi\varphi(t)+\frac{\ell_{\mathfrak{g}}(1+\ell_{\mathfrak{g}})}{\vartheta^\alpha\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\left|\mathfrak{y}(t)-\mathfrak{h}(t)\right|ds.\end{aligned}$$

By Lemma 2.3, there exists a constant $M^*>0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$|\mathfrak{y}(t)-\mathfrak{h}(t)|\leq M^*\lambda_\varphi\varphi(t):=C_{f,\varphi}\varphi(t).$$

Thus, Eq. (4.1) is g-UHR stable.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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