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ORIGINAL RESEARCH

Existence and Trajectory Controllability of Conformable Fractional Neutral Stochastic Integrodifferential Systems with Infinite Delay

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Abstract
The present work deals with the existence and the trajectory (T-)controllability of conformable fractional neutral stochastic integrodifferential systems with infinite delay. The conformable fractional stochastic integrodifferential system with infinite delay is modeled and the existing result of the mild solution is established using stochastic analysis techniques and Banach fixed point theorem. Then the neutral condition is incorporated into the previously modeled system and the existence results of the corresponding mild solutions are studied. Moreover, the trajectory controllability results of the modeled system are investigated followed by the illustration of the proven results. Our work generalizes the previous work of Dhayal et al. (Asian J Control 23(6): 2669–2680, 2020), Durga et al. (Optimization 1–27, 2022), and Chalishajar et al. (Differ Equ Dyn Syst, 2023). Numerical simulation has not been studied by other authors making this manuscript more interesting.

Introduction

T-controllability is a new concept that enables us to address a number of natural questions concerning control theory. In T-controllability problems, we seek a control that steers the system along a predefined trajectory rather than the control that steers the system from a given initial state to a desired final state. The advantages of T-controllability include;

1. Minimizing the cost involved in steering the system from initial state to desired final state.

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2. Safeguarding the system. For example, For cost-effectiveness and collision avoidance, it may be beneficial to know an exact path along with the intended destination before launching a rocket into orbit.

The theory of stochastic differential equations and its applications has grown in popularity in recent years, as stochastic modeling plays an important role in the formulation and analysis of economic, biological, and physical dynamical systems with significant uncertainty. We refer to [1–5] and the references cited therein for more information on stochastic differential equations. The existence, uniqueness, and stability of mild solutions to second-order stochastic evolution equations with infinite delay and Poisson jumps were studied by Ren and Sakthivel [6]. In their 2007 paper, Yang and Zhu [7] demonstrated that the mild solution to a class of Sobolev-type stochastic neutral functional differential equations with Poisson jump exists, is unique, and is stable.

Many researchers used an integral form of the fractional derivative(FD). There are several definitions of FD; it is not unique. Caputo, Riemann-Liouville, Weyl, Riesz, Grunwald-Letnikov, Riesz-Caputo FD, etc. are a few of these derivatives. Due to their usage of fractional integrals in their denotation, all of these FDs exhibit nonlocal features. The majority of physical systems are classified as FOSs due to the memory and heredity characteristics of FDs. However, because these derivatives do not adhere to the quotient rule, product rule, chain rule, etc., there are several challenges that arise when modeling physical systems with these FDs. Khalil et al. [8] presented a limit-based FD called a conformable FD to address the issues with nonlocal FDs. The majority of the features, such as the integer order derivative, are observed in this local FD (Abdeljawad [9]). Hence, conformable FD is better equipped to handle the modeling of physical systems.

On the other hand, a control system is an arrangement of parts that together constitute a system configuration and deliver the required system response. The capacity to lead a dynamic system from an arbitrary initial state to an arbitrary final state using the set of authorized controls is known as controllability, and it is a structural characteristic of dynamical systems. Controllability (introduced by Kalman 1960) leads to some very important conclusions about the behavior of linear and nonlinear dynamical systems. There are various concepts of controllability, such as complete [12], approximate [13, 14], exact [15, 16], null controllability [17, 18], etc.

Using measure of noncompactness, Vijayakumar et al. [19] discussed on the existence of Sobolev-type Hilfer fractional neutral integrodifferential systems. In the same direction, Williams et al. [20] focused on the controllability of nonlocal mixed Volterra-Fredholm type fractional delay integrodifferential equations of order $1 < r < 2$. By using the results on fractional calculus, cosine and sine operators, and Schauder's fixed point theorem, Mohan Raja et al. [21] formulated a new set of sufficient conditions which guarantees the approximate controllability of fractional differential evolution systems of order $1 < r < 2$ in Hilbert spaces. Nisar and Vijayakumar et al. [22] studied an approximate controllability for non-densely defined Sobolev-type Hilfer fractional neutral differential system with infinite delay using fractional theory and the fixed-point method. Using Martelli's fixed point theorem, Dineshkumar et al. [23] formulated a new set of sufficient conditions for the approximate controllability of fractional evolution stochastic integrodifferential delay inclusions of order $r \in (0, 1)$ with nonlocal conditions in Hilbert space. The new notion of T-controllability was first introduced by Chalishajar et al. [24]. An exact controllability, in general, means that a dynamical control system can be steered from an arbitrary initial state to an arbitrary final desired state using the set of admissible controls but T-controllability means that a dynamical control system can be steered from an arbitrary initial state to an arbitrary final desired state through the prescribed trajectories using the set of admissible controls. So naturally, T-control is the strongest notion than all other existing control definitions. The first and second order T-controllability in infinite dimension with numerical simulation was initiated by Chalishajar et at. [25, 26]. A few years back, Malik and George [27] investigated the T-controllability of a fractional order system. Dhayal et al. [28] then studied the approximate and T-controllability for a class of fractional stochastic differential equations driven by fBm with non-instantaneous impulses and Poisson jumps by employing the η-resolvent family and Krasnoselskii's fixed point theorem. Recently, Durga et al. [29] used the T-control for Hilfer fractional order stochastic system. Very recently, Chalishajar et al. [30] discussed the existence of the Hilfer fractional system with deviated argument and its T-controllability. In this manuscript, we consider the T-controllability of conformable fractional stochastic systems with infinite delay in time which generalizes all previous work of ([24–30]).

Because FDEs are important and beneficial, they are widely used, as are the methods for solving them. Since analytical solutions, such as those utilizing matrix Mittag-Leffler functions or Laplace transforms, are occasionally difficult or impracticable for more complex FDEs, numerical approaches are also crucial for solving FDEs in practice. Many such methods that have been established in the literature, including the q-homotopy analysis transform method, the B-spline collocation method, the predictor-corrector method, the space-spectral time-fractional Adams-Bashforth-Moulton method, the fractional Taylor operational matrix method, the Bernoulli polynomials method, the differential transform method, and others. Here, we use a numerical plan to support the theory. The highly successful results and applicability of the suggested method can significantly advance the field of numerical methods.

There are publications that examine the T-controllability of differential equations and stochastic integro-differential equations (see [24–29] and the references therein). Nevertheless, the T-controllability results on conformable fractional stochastic differential equations with infinite delay remain untreated in the literature. Consequently, the following is a list of the manuscript's main contributions and innovations:

- Initially, a conformable fractional stochastic integrodifferential equation(SIDE) with infinite delay is modeled and the existence results are investigated.
- Then, the conformable fractional SIDE with neutral conditions is built up and the existence results are examined using the stochastic techniques and Banach contraction principle.
- To the best of the authors' knowledge, there is no work that depicts the T-controllability results of conformable fractional SIDEs with infinite delay, which were inspired by the studies previously stated.

Motivated by the above facts, let us take into account the conformable fractional stochastic integrodifferential system with infinite delay of the form:

$$\begin{aligned} \mathcal{D}^{\alpha}\mathfrak{x}(\mathfrak{t}) =& \mathfrak{A}\mathfrak{x}(\mathfrak{t}) + \mathfrak{z}(\mathfrak{t}) + \mathfrak{f}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}, \int_0^{\mathfrak{t}} \mathfrak{g}\big(\mathfrak{t}, s, \mathfrak{x}_s\big)ds\right) + \mathfrak{h}\left(\mathfrak{t}, \mathfrak{x}_{\mathfrak{t}}, \int_0^{\mathfrak{t}} \tilde{\mathfrak{g}}\big(\mathfrak{t}, s, \mathfrak{x}_s\big)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}, \\ \mathfrak{x}(\mathfrak{t}) =& \zeta(\mathfrak{t}), \quad \in \mathcal{L}^2(\Omega, \mathfrak{G}_{\mathcal{J}}), \mathfrak{t} \in (-\infty, 0]. \end{aligned} \tag{1.1}$$

- $\mathcal{D}^{\alpha}$ is the conformable FD for $\mathfrak{t} \in \mathcal{E}' = (0, \mathfrak{v}]$ and $0 < \alpha < 1$.
- The infinitesimal generator of $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathcal{Y} \to \mathcal{Y}$ generates a strongly continuous semigroup $\{\mathfrak{G}(\mathfrak{t})\}\mathfrak{t} \geq 0$ on a Hilbert space $\mathcal{Y}$ with $\langle .,. \rangle \mathcal{Y}$ and norm $\|.\|_{\mathcal{Y}}$.
- Allowing $\mathcal{E} = [0, \mathfrak{v}]$. Then, values are received by the control function, $\mathfrak{z}$, from the reflexive Hilbert space, $\mathcal{K}$.
- The appropriate functions are $\mathfrak{f} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{Y}$, $\mathfrak{h} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{L}_2^0$ and $\mathfrak{g}, \tilde{\mathfrak{g}} : \mathcal{E} \times \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \to \mathcal{Y}$.
- Let $\mathcal{Z}$ be a different real separable Hilbert space with $\langle .,. \rangle_{\mathcal{Z}}$ and $\|.\|_{\mathcal{Z}}$ as its norm.
- We may assume that $\{\omega(\mathfrak{t}), \mathfrak{t} \geq 0\}$ is a $\mathcal{Z}$–valued Brownian motion with finite-trace nuclear covariance operator $\mathcal{Q} \geq 0$.
- The abstract phase space $\mathfrak{G}_{\mathcal{J}}$ has the element $\mathfrak{x}_{\mathfrak{t}} : (-\infty, 0] \to \mathcal{Y}$, which is described by $\mathfrak{x}_{\mathfrak{t}}(s) = \mathfrak{x}(\mathfrak{t} + s)$.
- The initial condition $\zeta = \{\zeta(\mathfrak{t}) : \mathfrak{t} \in (-\infty, 0]\}$ is independent of the Wiener process $\{\omega(\mathfrak{t})\}$ with the finite second moment which is $\mathfrak{G}_{\mathcal{J}}-$ valued random variable and $\mathfrak{I}_0-$ measurable.

The following is the manuscript's outline: Sect. 2 is devoted to the preliminaries required to study the aforementioned system (1.1). In Sect. 3, for the modeled system, the existence results of mild solutions are investigated. In Sect. 4, a conformable fractional stochastic neutral integrodifferential system is framed, and the existence results of mild solutions are presented. In Sect. 5, the optimal result of the system (1.1) is investigated. To validate the obtained results, an example is provided in Sect. 6 depicts the cost functional considered and attains its minimum. At the last, in Sect. 7, numerical simulation is presented to show the application of the studied system followed y the detailed conclusion with future aspects of this research work.

Preliminaries

Assume that $(\Omega, \mathfrak{I}, \mathcal{P})$ is a complete probability space, that $\mathfrak{I}_{\mathfrak{t}\mathfrak{t}\in\mathcal{E}}$ represents the normal filtration as being right continuous, and that $\{\mathfrak{I}_0\}$ contains $\mathcal{P}$–null sets. With the covariance operator $\mathcal{Q}$, ω is a $\mathcal{Q}$-Weiner process on $(\Omega, \mathfrak{I}_{\mathfrak{v}}, \mathcal{P}) \ni Tr(\mathcal{Q}) < \infty$. Consider the sequence of bounded non-negative real number $\{h_k\}_{k\geq 0}$ and the complete orthonormal basis $\{\xi_k\}_{k\geq 1}$ in $\mathcal{Z} \ni \mathcal{Q}\xi_k = h_k\xi_k$. The sequence of independent Brownian motion $\{\mathfrak{w}_k\}_{k\geq 1}$ follows $\langle \omega(\mathfrak{t}), \xi \rangle_{\mathcal{Z}} = \sum_{k=1}^{\infty} \sqrt{h_k} \langle \xi_k, \xi \rangle \mathfrak{w}_k(\mathfrak{t})$, $\xi \in \mathcal{Z}$, $\mathfrak{t} \in \mathcal{E}$. Take into account $\mathcal{L}_2^0 = \mathcal{L}_2\big(\mathcal{Q}^{1/2}\mathcal{Z};\mathcal{Y}\big)$ as the space containing all Hilbert-Schmidt operators from $\mathcal{Q}^{1/2}\mathcal{Z}$ to $\mathcal{Y}$ with $\|\phi\|_{\mathcal{Q}}^2 = Tr(\phi\mathcal{Q}\phi^*)$, where ϕ^* is the adjoint operator of ϕ. The expression $\mathcal{L}_2(\Omega, \mathfrak{I}, \mathcal{P};\mathcal{Y}) \equiv \mathcal{L}_2(\Omega;\mathcal{Y})$ is used to describe the set of all strongly measurable square integrable $\mathcal{Y}$–valued random operators, is a Banach space equipped with the norm $\|\mathfrak{x}(.)\|_{\mathcal{L}_2} = \left(\mathbb{E}\big\|\mathfrak{x}(., v_0)\big\|_{\mathcal{Y}}^2\right)^{1/2}$, where $\mathbb{E}\|h_0\| = \int_{\Omega} h_0(v_0)d\mathcal{P}$ defines the expectation $\mathbb{E}$. Let $\mathcal{C}\big(\mathcal{E}, \mathcal{L}_2(\Omega;\mathcal{Y})\big)$ be the Banach space consisting of all continuous functions $\mathcal{E}$ into $\mathcal{L}_2(\Omega;\mathcal{Y})$ satisfying $\sup_{\mathfrak{t}\in\mathcal{E}} \mathbb{E}\|\mathfrak{x}(\mathfrak{t})\|^2 < \infty$.

Definition 2.1 [8] For a function $\mathfrak{p}(.)$ with $\mathfrak{t} > 0$, the conformable FD of order v is defined as follows:

$$\frac{d^v \mathfrak{p}(\mathfrak{t})}{d\mathfrak{t}^v} = \lim_{v\to 0} \frac{\mathfrak{p}(\mathfrak{t} + v\mathfrak{t}^{1-v}) - \mathfrak{p}(\mathfrak{t})}{v}, \quad 0 < v < 1.$$

For the specific condition $\mathfrak{t} = 0$, the following definition is derived:

$$\frac{d^\nu \mathfrak{p}(0)}{d\mathfrak{t}^\nu} = \lim_{\mathfrak{t}\to 0^+} \frac{d^\nu \mathfrak{p}(\mathfrak{t})}{d\mathfrak{t}^\nu}.$$

The conformable FD of order ν of a function $\mathfrak{s}(.)$ is related with a fractional integral $\mathcal{I}^\nu(.)$ defined by

$$\mathcal{I}^\nu(\mathfrak{p})(\mathfrak{t}) = \int_0^{\mathfrak{t}} \mathfrak{s}^{\nu-1}\mathfrak{p}(\mathfrak{s})d\mathfrak{s}.$$

Following is the abstract phase space $\mathfrak{G}_{\mathcal{J}}$ [31]:

$$\mathfrak{G}_{\mathcal{J}} = \begin{cases} \zeta : (-\infty, 0] \to \mathcal{Y}, \forall\, c > 0,\ (\mathbb{E}\|\zeta(\eta)\|^2)^{1/2} \text{ is a bounded and measurable function on } [-c, 0] \\ \text{with } \int_{-\infty}^0 \mathcal{J}(s) \sup\limits_{s\le\eta\le 0} \left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2} ds < +\infty, \end{cases}$$

where $\mathcal{J} : (-\infty, 0] \to (0, +\infty)$ be continuous with $\mathfrak{l} = \int_{-\infty}^0 \mathcal{J}(\mathfrak{t})d\mathfrak{t} < +\infty$ and

$$\|\zeta\|_{\mathfrak{G}_{\mathcal{J}}} = \int_{-\infty}^0 \mathcal{J}(s) \sup_{s\le\eta\le 0} \left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2} ds,\ \ \forall\ \zeta \in \mathfrak{G}_{\mathcal{J}}.$$

Clearly $\left(\mathfrak{G}_{\mathcal{J}}, \|.\|_{\mathfrak{G}_{\mathcal{J}}}\right)$ is a Banach space.

The $\mathcal{C}((-\infty, \nu], \mathcal{Y})$ be the space of all continuous $\mathcal{Y}$–valued stochastic processes $\{\xi(\mathfrak{t}) : \mathfrak{t} \in (-\infty, \nu]\}$. Also, $\mathfrak{G}'_{\mathcal{J}} = \{\mathfrak{x} : \mathfrak{x} \in \mathcal{C}(-\infty, \nu], \mathcal{Y})\}$ endowed with the seminorm $\mathfrak{G}'_{\mathcal{J}}$ defined as

$$\|\mathfrak{x}\|_{\mathfrak{v}} = \|\zeta\|_{\mathfrak{G}_{\mathcal{J}}} + \sup_{s\in[0,\mathfrak{v}]} \left(\mathbb{E}\|\mathfrak{x}(s)\|^2\right)^{1/2},\ \mathfrak{x} \in \mathfrak{G}'_{\mathcal{J}}.$$

Lemma 2.1 [31] *If* $\mathfrak{x}_0 = \zeta \in \mathfrak{G}_{\mathcal{J}}, \mathfrak{x} \in \mathfrak{G}'_{\mathcal{J}}$, *then for* $\mathfrak{t} \in \mathcal{E}, \mathfrak{x}_{\mathfrak{t}} \in \mathfrak{G}_{\mathcal{J}}$. *Moreover,*

$$\mathfrak{l}\left(\mathbb{E}\|\mathfrak{x}(\mathfrak{t})\|^2\right)^{1/2} \le \|\mathfrak{x}_{\mathfrak{t}}\|_{\mathfrak{G}_{\mathcal{J}}} \le \|\mathfrak{x}_0\|_{\mathfrak{G}_{\mathcal{J}}} + \mathfrak{l} \sup_{s\in[0,\mathfrak{t}]} \left(\mathbb{E}\|\mathfrak{x}(s)\|^2\right)^{1/2},$$

where $\mathfrak{l} = \int_{-\infty}^0 \mathcal{J}(s)ds < +\infty$.

Definition 2.2 If the mild solution $\mathfrak{x}(.)$ satisfies $\delta(\mathfrak{t}) = \mathfrak{x}(\mathfrak{t})$ a.e., then the system (1.1) is trajectory controllable on $[\mathfrak{t}_0, \mathrm{T}]$.

Lemma 2.2 (Generalized Gronwall's inequality [32]): *If* $\beta > 0$, $\tilde{\mathfrak{a}}(\mathfrak{t})$ *is a non-negative function locally integrable on* $0 \le \mathfrak{t} < \mathrm{T}$, *for some* $(\mathrm{T} < +\infty)$, *and* $\mathfrak{g}(\mathfrak{t})$ *is a non-negative, non-decreasing continuous function on* $0 \le \mathfrak{t} \le \mathrm{T}$, $\mathfrak{g}(\mathfrak{t}) \le c$ *being a constant and suppose* $\tilde{\mathfrak{u}}(\mathfrak{t})$ *is non-negative and locally integrable on* $0 \le \mathfrak{t} < \mathrm{T}$ *with* $\tilde{\mathfrak{u}}(\mathfrak{t}) \le \tilde{\mathfrak{a}}(\mathfrak{t}) + \mathfrak{g}(\mathfrak{t}) \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{\beta-1}\tilde{\mathfrak{u}}(s)ds$, *on the interval. Then*

$$\tilde{\mathfrak{u}}(\mathfrak{t}) \le \tilde{\mathfrak{a}}(\mathfrak{t}) + \int_0^{\mathfrak{t}} \sum_{n=1}^{\infty} \frac{(\mathfrak{g}(\mathfrak{t})\Gamma(\beta))^n}{\Gamma(n\beta)} (\mathfrak{t} - s)^{\beta-1}\tilde{\mathfrak{a}}(s)ds,\ \ 0 \le \mathfrak{t} < \mathrm{T}.$$

In particular, when $\tilde{\mathfrak{a}}(\mathfrak{t}) = 0$, *then* $\tilde{\mathfrak{u}}(\mathfrak{t}) = 0$ *for all* $0 \le \mathfrak{t} < \mathrm{T}$.

Definition 2.3 [33] A mild solution of (1.1) is $\mathfrak{F}_{\mathfrak{t}}$−adapted stochastic process $\mathfrak{x} : (-\infty, \mathfrak{b}] \to \mathcal{Y}$ with $\zeta \in \mathcal{L}^2(\Omega, \mathfrak{G}_{\mathcal{J}})$ on $(-\infty, 0]$, $\mathfrak{x}_0 \in \mathcal{L}_2^0(\Omega, \mathcal{Y})$ and the integral equation below is satisfied:

$$\begin{aligned}\mathfrak{x}(\mathfrak{t}) = &\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\Big(s, \mathfrak{x}_s, \int_0^s \mathfrak{g}\big(s, \rho, \mathfrak{x}_\rho\big)d\rho\Big)ds \\ &+ \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s, \mathfrak{x}_s, \int_0^s \tilde{\mathfrak{g}}\big(s, \rho, \mathfrak{x}_\rho\big)d\rho\Big)d\omega(s) + \mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}\Big)\zeta(0).\end{aligned} \tag{2.1}$$

Main Results

The following are the assumptions to discuss the existence and uniqueness of mild solution as well as the optimal control of the evolution equation:

(A1) The linear operator $\mathfrak{A} : \mathcal{Y} \to \mathcal{Y}$ in (1.1) generates C_0−semigroup $\mathcal{T}(.)$. Thus there exists $\mathcal{M} > 0$ being constant such that $\|\mathcal{T}(\mathfrak{t})\| \le \mathcal{M}\, \forall\, \mathfrak{t} \in \mathcal{E}'$.

(A2) For $\mathfrak{t} \in \mathcal{E}$, the function $\mathfrak{f} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{Y}$ is continuous. $\mathfrak{u}_1, \tilde{\mathfrak{u}}_1 \in \mathfrak{G}_{\mathcal{J}}, \mathfrak{u}_2, \tilde{\mathfrak{u}}_2 \in \mathcal{Y}$ and there exist positive constants $N_{\mathfrak{f}}, \hat{N}_{\mathfrak{f}}$

$$\begin{aligned}\mathbb{E}\Big\|\mathfrak{f}\big(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\big) - \mathfrak{f}\big(\mathfrak{t}, \tilde{\mathfrak{u}}_1, \tilde{\mathfrak{u}}_2\big)\Big\|^2 \le& N_{\mathfrak{f}}\Big(\|\mathfrak{u}_1 - \tilde{\mathfrak{u}}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2 - \tilde{\mathfrak{u}}_2\|^2\Big), \\ \mathbb{E}\Big\|\mathfrak{f}\big(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\big)\Big\|^2 \le& \hat{N}_{\mathfrak{f}}\Big(1 + \|\mathfrak{u}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2\|^2\Big).\end{aligned}$$

(A3) For $\mathfrak{t} \in \mathcal{E}$, the function $\mathfrak{h} : \mathcal{E} \times \mathfrak{G}_{\mathcal{J}} \times \mathcal{Y} \to \mathcal{L}_2^0$ is continuous. $\mathfrak{u}_1, \tilde{\mathfrak{u}}_1 \in \mathfrak{G}_{\mathcal{J}}, \mathfrak{u}_2, \tilde{\mathfrak{u}}_2 \in \mathcal{Y}$ and there exist positive constants $N_{\mathfrak{h}}, \hat{N}_{\mathfrak{h}}$

$$\begin{aligned}\mathbb{E}\Big\|\mathfrak{h}\big(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\big) - \mathfrak{h}\big(\mathfrak{t}, \tilde{\mathfrak{u}}_1, \tilde{\mathfrak{u}}_2\big)\Big\|^2 \le& N_{\mathfrak{h}}\Big(\|\mathfrak{u}_1 - \tilde{\mathfrak{u}}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2 - \tilde{\mathfrak{u}}_2\|^2\Big), \\ \mathbb{E}\Big\|\mathfrak{h}\big(\mathfrak{t}, \mathfrak{u}_1, \mathfrak{u}_2\big)\Big\|^2 \le& \hat{N}_{\mathfrak{h}}\Big(1 + \|\mathfrak{u}_1\|^2_{\mathfrak{G}_{\mathcal{J}}} + \mathbb{E}\|\mathfrak{u}_2\|^2\Big).\end{aligned}$$

(A4) For each $(\mathfrak{t}, s) \in \mathcal{E}^2$, the functions $\mathfrak{g}, \tilde{\mathfrak{g}} : \mathcal{E}^2 \times \mathfrak{G}_{\mathcal{J}} \to \mathcal{Y}$ are continuous. For all $\mathfrak{u}, \tilde{\mathfrak{u}} \in \mathfrak{G}_{\mathcal{J}}$, there exist positive constants $\mathfrak{m}_1, \mathfrak{m}_2, \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$

$$\begin{aligned}\mathbb{E}\|\mathfrak{g}(\mathfrak{t}, s, \mathfrak{u}) - \mathfrak{g}(\mathfrak{t}, s, \tilde{\mathfrak{u}})\|^2 \le& \mathfrak{m}_1\|\mathfrak{u} - \tilde{\mathfrak{u}}\|^2_{\mathfrak{G}_{\mathcal{J}}}, \\ \mathbb{E}\|\tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u}) - \tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le& \mathfrak{m}_2\|\mathfrak{u} - \tilde{\mathfrak{u}}\|^2_{\mathfrak{G}_{\mathcal{J}}}, \\ \mathbb{E}\|\mathfrak{g}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le& \tilde{\mathfrak{m}}_1\Big(1 + \|\mathfrak{u}\|^2_{\mathfrak{G}_{\mathcal{J}}}\Big), \\ \mathbb{E}\|\tilde{\mathfrak{g}}(\mathfrak{t}, s, \mathfrak{u})\|^2 \le& \tilde{\mathfrak{m}}_2\Big(1 + \|\mathfrak{u}\|^2_{\mathfrak{G}_{\mathcal{J}}}\Big).\end{aligned}$$

Theorem 3.1 *With the assumptions* (A1)-(A5), *a unique mild solution* (1.1) *exists if*

$$3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\big(1+\tilde{\mathfrak{m}}_1\mathfrak{v}^2\big)+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\big(1+\tilde{\mathfrak{m}}_2\mathfrak{v}^2\big)<1.$$

Proof Let the operator $\psi : \mathfrak{G}'_{\mathcal{J}} \to \mathfrak{G}'_{\mathcal{J}}$ be

$$\psi\mathfrak{x}(\mathfrak{t})=\begin{cases}\zeta(\mathfrak{t}), & \mathfrak{t}\in(-\infty,0]\\ \int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\Big(s,\mathfrak{x}_s,\int_0^s\mathfrak{g}\big(s,\rho,\mathfrak{x}_\rho\big)d\rho\Big)ds & \\ +\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s,\mathfrak{x}_s,\int_0^s\tilde{\mathfrak{g}}\big(s,\rho,\mathfrak{x}_\rho\big)d\rho\Big)d\omega(s), & \mathfrak{t}\in\mathcal{E}.\end{cases}$$

For $\zeta \in \mathfrak{G}_{\mathcal{J}}$, we may define $\overline{\zeta}$ as

$$\overline{\zeta}(\mathfrak{t})=\begin{cases}\zeta(\mathfrak{t}), & \mathfrak{t}\in(-\infty,0]\\ \zeta(0), & \mathfrak{t}\in\mathcal{E};\end{cases}$$

then $\overline{\zeta} \in \mathfrak{G}'_{\mathcal{J}}$. Let $\mathfrak{x}(\mathfrak{t}) = \mathfrak{w}(\mathfrak{t}) + \overline{\zeta}(\mathfrak{t})$, $-\infty < \mathfrak{t} \le \mathfrak{v}$. Clearly $\mathfrak{x}$ is satisfied if and only if $\mathfrak{w}$ fulfills $\mathfrak{w}_0 = 0$.

$$\begin{aligned}\mathfrak{w}(\mathfrak{t})=&\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds\\&+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)ds\\&+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)d\omega(s).\end{aligned}$$

Let $\mathfrak{G}''_{\mathcal{J}} = \{\mathfrak{w} \in \mathfrak{G}'_{\mathcal{J}}; \mathfrak{w}_0 = 0 \in \mathfrak{G}_{\mathcal{J}}\}$. For any $\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}}$.

$$\|\mathfrak{w}\|_{\mathfrak{v}}=\|\mathfrak{w}_0\|_{\mathfrak{G}_{\mathcal{J}}}+\sup_{0\le s\le\mathfrak{v}}\big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big)^{1/2}=\sup_{0\le s\le\mathfrak{v}}\big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big)^{1/2}.$$

This demonstrates that $\Big(\mathfrak{G}''_{\mathcal{J}}, \|.\|_{\mathfrak{v}}\Big)$ is a Banach space. We assume $\mathcal{B}_{\mathfrak{p}} = \{\mathfrak{w} \in \mathfrak{G}''_{\mathcal{J}} : \|\mathfrak{w}\|^2_{\mathfrak{v}} \le \mathfrak{p}\}$; for some $\mathfrak{p} > 0$. then, $\mathcal{B}_{\mathfrak{p}} \subseteq \mathfrak{G}''_{\mathcal{J}}$ is uniformly bounded, $\mathfrak{w} \in \mathcal{B}_{\mathfrak{p}}$, $\forall\, \mathfrak{p}$. By virtue of Lemma 2.1,

$$\begin{aligned}\|\mathfrak{w}_{\mathfrak{t}}+\overline{\zeta}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}\le&2\Big(\|\mathfrak{w}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}+\|\overline{\zeta}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}\Big)\\\le&4\Big(\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{l}^2\sup_{s\in[0,\mathfrak{t}]}\big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big)+\|\overline{\zeta}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{l}^2\sup_{s\in[0,\mathfrak{t}]}\Big(\mathbb{E}\|\overline{\zeta}(s)\|^2\Big)\Big)\\\le&4\Big(0+\mathfrak{l}^2\|\mathfrak{w}\|^2_{\mathfrak{v}}+\|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\Big)\\\le&4\Big(\mathfrak{l}^2\mathfrak{p}+\|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\Big)=\mathfrak{p}'.\end{aligned}\tag{3.1}$$

Now we may define $\tilde{\Psi} : \mathfrak{G}''_{\mathcal{J}} \to \mathfrak{G}''_{\mathcal{J}}$ as

$$\tilde{\Psi}\mathfrak{w}(\mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \in (-\infty, 0] \\ \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds \\ + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\big)d\rho\big)ds \\ + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)d\omega(s), & \mathfrak{t} \in \mathcal{E}, \end{cases} \tag{3.2}$$

implies that $\tilde{\Psi}$ has a unique fixed point. Let us divide the proof into several steps.

Step 1:

To claim there exists $\mathfrak{p} > 0 \ni \tilde{\Psi}(\mathcal{B}_{\mathfrak{p}}) \subset \mathcal{B}_{\mathfrak{p}}$. On the contrary let us assume that for $\mathfrak{p} > 0$ there exists $\mathfrak{w}^p(.) \in \mathcal{B}_{\mathfrak{p}}$ and $\tilde{\Psi}(\mathfrak{w}^p) \notin \mathcal{B}_{\mathfrak{p}}$, (i.e), $\mathbb{E}\|(\tilde{\Psi}\mathfrak{w}^{\mathfrak{p}})(\mathfrak{t})\|^2 > \mathfrak{p}$ for some $\mathfrak{t} \in \mathcal{E}$. Lemma 2.2 and the assumptions (A1)-(A6) yield

$$\begin{aligned} \mathfrak{p} \le& \mathbb{E}\|\big(\tilde{\Psi}\mathfrak{w}^{\mathfrak{p}}\big)(\mathfrak{t})\|^2 \\ \le& 3\Bigg[\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds\Bigg\|^2 \\ &+ \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\big(s, \mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s, \int_0^s \mathfrak{g}\big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\big)d\rho\big)ds\Bigg\|^2 \\ &+ \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s, \mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\Big)d\rho\Big)d\omega(s)\Bigg]\Bigg\|^2 \\ \le& 3[\mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3], \end{aligned}$$

where

$$\begin{aligned} \mathfrak{S}_1 =& \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds\Bigg\|^2 \le \mathcal{M}^2 \int_0^{\mathfrak{t}} \mathfrak{s}^{2(\alpha-1)}\mathbb{E}\|\mathfrak{z}(s)\|^2 ds \\ \le& \mathcal{M}^2 \frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}, \\ \mathfrak{S}_2 \le& \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\big(s, \mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s, \int_0^s \mathfrak{g}\big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\big)d\rho\big)ds\Bigg\|^2 \\ \le& \mathcal{M}^2 \int_0^{\mathfrak{t}} \mathfrak{s}^{2(\alpha)-1}\hat{N}_{\mathfrak{f}}\Bigg(1 + \|\mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s\|^2 + \mathbb{E}\Bigg\|\int_0^s \mathfrak{g}\Big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\Big)d\rho\Bigg\|^2\Bigg)ds \\ \le& \mathcal{M}^2 \frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\big(1 + \mathfrak{p}' + \tilde{\mathfrak{m}}_1\mathfrak{v}^2(1 + \mathfrak{p}')\big), \\ \mathfrak{S}_3 \le& \mathbb{E}\Bigg\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\big(s, \mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\big)d\rho\big)d\omega(s)\Bigg\|^2 \\ \le& \mathcal{M}^2 Tr(\mathcal{Q}) \int_0^{\mathfrak{t}} \mathfrak{s}^{2(\alpha)-1}\hat{N}_{\mathfrak{h}}\Bigg(1 + \|\mathfrak{w}_s^{\mathfrak{p}} + \overline{\zeta}_s\|^2 + \mathbb{E}\Bigg\|\int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho^{\mathfrak{p}} + \overline{\zeta}_\rho\Big)d\rho\Bigg\|^2\Bigg)ds \\ \le& \mathcal{M}^2 Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\big(1 + \mathfrak{p}' + \tilde{\mathfrak{m}}_2\mathfrak{v}^2(1 + \mathfrak{p}')\big). \end{aligned}$$

Thus combining the above form we obtain

$$\begin{aligned}\mathfrak{p} \leq & 3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}+3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\big(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}')\big)\\ &+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\big(1+\mathfrak{p}'+\tilde{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}')\big).\end{aligned}$$

Dividing throughout by $\mathfrak{p}$ and by taking $\mathfrak{p}\to\infty$,

$$1\leq 3\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\big(1+\tilde{\mathfrak{m}}_1\mathfrak{v}^2\big)+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\big(1+\tilde{\mathfrak{m}}_2\mathfrak{v}^2\big);$$

which goes against what we assumed. Therefore, for some $\mathfrak{p}>0$, $\tilde{\Psi}(\mathcal{B}_{\mathfrak{p}})\subset\mathcal{B}_{\mathfrak{p}}$.

Step 2:

To claim $\tilde{\Psi}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$. Let us consider $\mathfrak{w},\hat{\mathfrak{w}}\in\mathcal{B}_{\mathfrak{p}}$, then $\mathbb{E}\big\|\tilde{\Psi}\mathfrak{w}(\mathfrak{t})-\tilde{\Psi}\hat{\mathfrak{w}}(\mathfrak{t})\big\|^2$

$$\begin{aligned}&\leq 2\Bigg[\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\Big[\mathfrak{f}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\\
&\quad-\mathfrak{f}\Big(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\Big(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\Big]ds\Bigg\|^2+\mathbb{E}\Bigg\|\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\\
&\quad\times\Bigg[\mathfrak{h}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\\
&\quad-\mathfrak{h}\Big(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\Big(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\Bigg]d\omega(s)\Bigg\|^2\Bigg]\\
&\leq 2\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{f}}\Big(\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}+\mathfrak{m}_1\mathfrak{v}^2\|\mathfrak{w}_\rho-\hat{\mathfrak{w}}_\rho\|^2\Big)\\
&\quad+2\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\big(\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\
&\qquad+\mathfrak{m}_2\mathfrak{v}^2\|\mathfrak{w}_\rho-\hat{\mathfrak{w}}_\rho\|^2\big)\\
&\leq\Bigg[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\Bigg]\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\
&\leq\Bigg[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\Bigg]\\
&\quad\times\Big(\mathfrak{l}^2\sup_{s\in\mathcal{E}}\mathbb{E}\big\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\big\|^2+\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\|\hat{\mathfrak{w}}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}\Big)\\
&\leq\mathfrak{l}^2\Bigg[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\Bigg]\sup_{s\in\mathcal{E}}\mathbb{E}\big\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\big\|^2\\
&\leq\mathsf{P}^*\sup_{s\in\mathcal{E}}\mathbb{E}\big\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\big\|^2,\end{aligned}$$

here

$$\mathsf{P}^*=\mathfrak{l}^2\Bigg[2\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\mathfrak{m}_1\mathfrak{v}^2)+2\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}(1+\mathfrak{m}_2\mathfrak{v}^2)\Bigg].$$

Thus by taking supreme over $\mathfrak{t}$, we obtain $\|\tilde{\Psi}\mathfrak{w}-\tilde{\Psi}\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}\leq\mathsf{P}^*\|\mathfrak{w}-\hat{\mathfrak{w}}\|^2_{\mathfrak{v}}$.

As a result, $\tilde{\Psi}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$ and has a unique fixed point $\mathfrak{w}(.)\in\mathcal{B}\mathfrak{p}$, which is the mild solution of (1.1). Hence the facts. □

Systems of Neutral Stochastic Differential Equations with Infinite Delay

Neutral differential systems have gained popularity in applied mathematics recently. A number of partial differential systems, such as heat flow in materials, wave propagation, and various natural phenomena, get assistance from neutral systems with or without delay. Let's now consider the following neutral stochastic infinite-delay integrodifferential system of the following form:

$$\begin{aligned} \mathcal{D}^{\alpha}\left[\mathfrak{x}(\mathfrak{t})-\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right] =&\mathfrak{A}\left[\mathfrak{x}(\mathfrak{t})-\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right]+\mathfrak{z}(\mathfrak{t})+\mathfrak{f}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right)\\ &+\mathfrak{h}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}},\\ \mathfrak{x}(\mathfrak{t}) =&\zeta(\mathfrak{t}),\quad \in\mathcal{L}^2(\Omega,\mathfrak{G}_{\mathcal{J}}),\mathfrak{t}\in(-\infty,0]. \end{aligned} \tag{4.1}$$

A strongly continuous semigroup $\{\mathfrak{G}(\mathfrak{t})\}_{\mathfrak{t}\geq 0}$ is generated by $\mathfrak{A}$ on $\mathcal{Y}$.

Consider the following hypotheses:

(A5) $\mathsf{k}:[0,\mathfrak{b}]\times\mathfrak{G}_{\mathcal{J}}\to\mathcal{Y}$ is a continuous function such that it satisfies the following requirement

$$\begin{aligned} \mathbb{E}\|\mathsf{k}(\mathfrak{t},\mathfrak{x})-\mathsf{k}(\mathfrak{t},\hat{\mathfrak{x}})\|^2 \leq& N_{\mathsf{k}}\|\mathfrak{x}-\hat{\mathfrak{x}}\|^2_{\mathfrak{G}_{\mathcal{J}}},\ \ \mathfrak{x},\hat{\mathfrak{x}}\in\mathfrak{G}_{\mathcal{J}},\ \mathfrak{t}\in\mathfrak{G}_{\mathcal{J}},\\ \mathbb{E}\|\mathsf{k}(\mathfrak{t},\mathfrak{x})\|^2 \leq& \hat{N}_{\mathsf{k}}\left(1+\|\mathfrak{x}\|^2_{\mathfrak{G}_{\mathcal{J}}}\right),\ \ \mathfrak{x}\in\mathfrak{G}_{\mathcal{J}},\ \mathfrak{t}\in\mathfrak{G}_{\mathcal{J}}. \end{aligned}$$

Theorem 4.1 *Assume that* (*A*1)-(*A*6) *get satisfied. Then* (4.1) *has a unique mild solution provided*,

$$5\hat{N}_{\mathsf{k}}+\left[5\mathcal{M}^2\hat{N}_{\mathfrak{f}}(1+\tilde{\mathfrak{m}}_1\mathfrak{b}^2)+5\mathcal{M}^2\hat{N}_{\mathfrak{h}}Tr(\mathcal{Q})(1+\tilde{\mathfrak{m}}_2\mathfrak{b}^2)\right]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}<1.$$

Proof Let us define $\eta:\mathfrak{G}'_{\mathcal{J}}\to\mathfrak{G}'_{\mathcal{J}}$ as

$$\eta\mathfrak{x}(\mathfrak{t})=\begin{cases}\zeta(\mathfrak{t}), & \mathfrak{t}\in(-\infty,0],\\ \mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)[\zeta(0)-\mathsf{k}(0,\zeta)]+\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\\ \mathfrak{f}\left(s,\mathfrak{x}_s,\int_0^s\mathfrak{g}\left(s,\rho,\mathfrak{x}_\rho\right)d\rho\right)ds+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{x}_s,\int_0^s\tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{x}_\rho\right)d\rho\right)d\omega(s), & \mathfrak{t}>0.\end{cases}$$

Consider $\overline{\zeta}$ as

$$\overline{\zeta}(\mathfrak{t})=\begin{cases}\zeta(\mathfrak{t}), & \mathfrak{t}\in(-\infty,0]\\ \zeta(0), & \mathfrak{t}\in\mathcal{E}\end{cases}\ \zeta\in\mathfrak{G}_{\mathcal{J}};$$

then $\overline{\zeta}\in\mathfrak{G}'_{\mathcal{J}}$. Let $\mathfrak{x}(\mathfrak{t})=\mathfrak{w}(\mathfrak{t})+\overline{\zeta}(\mathfrak{t})$, $-\infty<\mathfrak{t}\leq\mathfrak{b}$. Clearly $\mathfrak{x}$ is satisfied if and only if $\mathfrak{w}$ fulfills $\mathfrak{w}_0=0$.

$$\begin{aligned}\mathfrak{w}(\mathfrak{t}) = &- \mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}\Big)\mathsf{k}(0,\zeta) + \mathsf{k}\Big(\mathfrak{t},\mathfrak{w}_\mathfrak{t} + \overline{\zeta}(\mathfrak{t})\Big) + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds \\ &+ \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big) \\ &\times \mathfrak{f}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\Big(s,\rho,\mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)ds + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big) \\ &\times \mathfrak{h}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s,\rho,\mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)d\omega(s).\end{aligned}$$

For any $\mathfrak{w} \in \mathfrak{G}''_\mathcal{J} = \{\mathfrak{w} \in \mathfrak{G}'_\mathcal{J}; \mathfrak{w}_0 = 0 \in \mathfrak{G}_\mathcal{J}\}$,

$$\|\mathfrak{w}\|_\mathfrak{v} = \|\mathfrak{w}_0\|_{\mathfrak{G}_\mathcal{J}} + \sup_{0\le s\le \mathfrak{v}} \big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big)^{1/2} = \sup_{0\le s\le \mathfrak{v}} \big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big)^{1/2}.$$

Eventually, $\Big(\mathfrak{G}''_\mathcal{J}, \|.\|_\mathfrak{v}\Big)$ is a Banach space. We consider $\mathcal{B}_\mathfrak{p} = \{\mathfrak{w} \in \mathfrak{G}''_\mathcal{J} : \|\mathfrak{w}\|^2_\mathfrak{v} \le \mathfrak{p}\}$; For some $\mathfrak{p} > 0$, then $\mathcal{B}_\mathfrak{p} \subseteq \mathfrak{G}''_\mathcal{J}$ is uniformly bounded, for each $\mathfrak{p}$. For $\mathfrak{w} \in \mathcal{B}_\mathfrak{p}$ and by the virtue of Lemma 2.1

$$\begin{aligned}\|\mathfrak{w}_\mathfrak{t} + \overline{\zeta}_\mathfrak{t}\|^2_{\mathfrak{G}_\mathcal{J}} &\le 2\Big(\|\mathfrak{w}_\mathfrak{t}\|^2_{\mathfrak{G}_\mathcal{J}} + \|\overline{\zeta}_\mathfrak{t}\|^2_{\mathfrak{G}_\mathcal{J}}\Big) \\ &\le 4\Big(\|\mathfrak{w}_0\|^2_{\mathfrak{G}_\mathcal{J}} + \mathfrak{l}^2 \sup_{s\in[0,\mathfrak{t}]}\big(\mathbb{E}\|\mathfrak{w}(s)\|^2\big) + \|\overline{\zeta}_0\|^2_{\mathfrak{G}_\mathcal{J}} + \mathfrak{l}^2 \sup_{s\in[0,\mathfrak{t}]}\Big(\mathbb{E}\|\overline{\zeta}(s)\|^2\Big)\Big) \\ &\le 4\Big(0 + \mathfrak{l}^2\|\mathfrak{w}\|^2_\mathfrak{v} + \|\zeta\|^2_{\mathfrak{G}_\mathcal{J}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\Big) \\ &\le 4\Big(\mathfrak{l}^2\mathfrak{p} + \|\zeta\|^2_{\mathfrak{G}_\mathcal{J}} + \mathfrak{l}^2\mathbb{E}\|\zeta(0)\|^2\Big) = \mathfrak{p}'.\end{aligned} \tag{4.2}$$

Now we may define $\tilde{\Psi} : \mathfrak{G}''_\mathcal{J} \to \mathfrak{G}''_\mathcal{J}$ as

$$\tilde{\Psi}\mathfrak{w}(\mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \in (-\infty, 0] \\ \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{z}(s)ds & \\ + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{f}\big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\big(s,\rho,\mathfrak{w}_\rho + \overline{\zeta}_\rho\big)d\rho\big)ds & \\ + \int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\mathfrak{h}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s,\rho,\mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)d\omega(s), & \mathfrak{t} \in \mathcal{E}, \end{cases} \tag{4.3}$$

implies that $\tilde{\Psi}$ has a unique fixed point. Let us divide the proof into several steps.

Step 1: $\overline{\eta}(\mathcal{B}_\mathfrak{p}) \subset \mathcal{B}_\mathfrak{p}$ for $\mathfrak{p} > 0$.

Assuming the contrary, for each $\mathfrak{p} > 0$ there exists $\mathfrak{w}(.) \in \mathcal{B}_\mathfrak{p}$ and $\overline{\eta}(\mathfrak{w}) \notin \mathcal{B}_\mathfrak{p}$ (i.e), $\mathbb{E}\big\|(\overline{\eta}\mathfrak{w})(\mathfrak{t})\big\|^2 > \mathfrak{p}$ for some $\mathfrak{t} \in \mathcal{E}$.

$$
\begin{aligned}
\mathfrak{p} \leq & \mathbb{E}\left\|(\overline{\eta}\mathfrak{w})(\mathfrak{t})\right\|^2 \\
\leq & 5\mathbb{E}\left\|\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\mathsf{k}(0,\zeta)\right\|^2 + 5\mathbb{E}\left\|\mathsf{k}\left(\mathfrak{t},\mathfrak{w}_\mathfrak{t}+\overline{\zeta}_\mathfrak{t}\right)\right\|^2 + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2 \\
& + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2 \\
& + 5\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)d\omega(s)\right\|^2 \\
\leq & \mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_3+\mathfrak{S}_4+\mathfrak{S}_5,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{S}_1 =& \mathbb{E}\|\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}\right)\mathsf{k}(0,\zeta)\|^2 \leq \mathcal{M}^2\hat{N}_\mathsf{k}\left(1+\|\zeta\|^2_{\mathfrak{G}_\mathcal{J}}\right), \\
\mathfrak{S}_2 =& \mathbb{E}\left\|\mathsf{k}\left(\mathfrak{t},\mathfrak{w}_\mathfrak{t}+\overline{\zeta}_\mathfrak{t}\right)\right\|^2 \leq \hat{N}_\mathsf{k}\left(1+\|\mathfrak{w}_\mathfrak{t}+\overline{\zeta}_\mathfrak{t}\|^2\right) \leq \hat{N}_\mathsf{k}(1+\mathfrak{p}_1), \\
\mathfrak{S}_3 =& \mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{z}(s)ds\right\|^2 \\
\leq & \mathcal{M}^2\|\mathbb{B}\|^2_\infty\mathbb{E}\left[\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\|\mathfrak{z}(s)\|^2ds\right] \leq \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^\mathfrak{q}(\mathcal{E},\mathcal{K})}, \\
\mathfrak{S}_4 \leq & \mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{f}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2 \\
\leq & \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_\mathcal{J}}+\mathbb{E}\left\|\int_0^s \mathfrak{g}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right\|^2\right) \\
\leq & \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_\mathcal{J}}+\overline{\mathfrak{m}}_1\mathfrak{v}^2\left(1+\|\mathfrak{w}_\rho+\overline{\zeta}_\rho\|^2_{\mathfrak{G}_\mathcal{J}}\right)\right) \\
\leq & \mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{f}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}_1)\right), \\
\mathfrak{S}_5 =& \mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{s}^{\alpha-1}\mathcal{T}\left(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)d\omega(s)\right\|^2 \\
\leq & \mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left\|\int_0^\mathfrak{t} \mathfrak{h}\left(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s \tilde{\mathfrak{g}}\left(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\right)d\rho\right)ds\right\|^2 \\
\leq & \mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{h}\left(1+\|\mathfrak{w}_s+\overline{\zeta}_s\|^2_{\mathfrak{G}_\mathcal{J}}+\overline{\mathfrak{m}}_2\mathfrak{v}^2\left(1+\left\|\mathfrak{w}_\rho+\overline{\zeta}_\rho\right\|^2_{\mathfrak{G}_\mathcal{J}}\right)\right) \\
\leq & \mathcal{M}^2Tr(Q)\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_\mathfrak{h}\left(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}_1)\right).
\end{aligned}
$$

Therefore,

$$\begin{aligned}\mathfrak{p} \leq & 5\mathcal{M}^2\hat{N}_{\mathsf{k}}\Big(1+\|\zeta\|^2_{\mathfrak{G}_{\mathcal{J}}}\Big)+5\hat{N}_{\mathsf{k}}(1+\mathfrak{p}_1)+5\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\|\mathfrak{z}\|^2_{\mathcal{L}^{\mathfrak{q}}(\mathcal{E},\mathcal{K})}\\ &+5\mathcal{M}^2\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{f}}\big(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_1\mathfrak{v}^2(1+\mathfrak{p}_1)\big)\\ &+5\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\hat{N}_{\mathfrak{h}}\big(1+\mathfrak{p}_1+\overline{\mathfrak{m}}_2\mathfrak{v}^2(1+\mathfrak{p}_1)\big).\end{aligned}$$

Dividing by $\mathfrak{p}$ throughout and let $\mathfrak{p}\to\infty$, we obtain

$$1\leq 5\hat{N}_{\mathsf{k}}+5\mathcal{M}^2\hat{N}_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\overline{\mathfrak{m}}_1\mathfrak{v}^2)+5\mathcal{M}^2Tr(\mathcal{Q})\hat{N}_{\mathfrak{h}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}(1+\overline{\mathfrak{m}}_2\mathfrak{v}^2).$$

It opposes our theory (4.1). Therefor $\overline{\eta}(\mathcal{B}_{\mathfrak{p}})\subset\mathcal{B}_{\mathfrak{p}}$, $\mathfrak{p}>0$.

Step 2: $\overline{\eta}$ is a contraction on $\mathcal{B}_{\mathfrak{p}}$.

Using $\mathfrak{w},\hat{\mathfrak{w}}\in\mathcal{B}_{\mathfrak{p}}$,

$$\begin{aligned}&\mathbb{E}\big\|\overline{\eta}\mathfrak{w}(\mathfrak{t})-\overline{\eta}\hat{\mathfrak{w}}(\mathfrak{t})\big\|^2\\ =&\mathbb{E}\Bigg\|\Big[\mathsf{k}\Big(\mathfrak{t},\mathfrak{w}_{\mathfrak{t}}\\ &+\overline{\zeta}_{\mathfrak{t}}\Big)-\mathsf{k}\Big(\mathfrak{t},\hat{\mathfrak{w}}_{\mathfrak{t}}+\overline{\zeta}_{\mathfrak{t}}\Big)\Big]+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\Big[\mathfrak{f}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\\ &-\mathfrak{f}\Big(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\mathfrak{g}\Big(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\Big]ds+\int_0^{\mathfrak{t}}\mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\Big)\\ &\times\Bigg[\mathfrak{h}\Big(s,\mathfrak{w}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\Big(s,\rho,\mathfrak{w}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\\ &-\mathfrak{h}\Big(s,\hat{\mathfrak{w}}_s+\overline{\zeta}_s,\int_0^s\tilde{\mathfrak{g}}\Big(s,\rho,\hat{\mathfrak{w}}_\rho+\overline{\zeta}_\rho\Big)d\rho\Big)\Bigg]ds\Bigg\|^2\\ \leq&3N_{\mathsf{k}}\|\mathfrak{w}_{\mathfrak{t}}-\hat{\mathfrak{w}}_{\mathfrak{t}}\|^2_{\mathfrak{G}_{\mathcal{J}}}+3\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\big(1+\mathfrak{m}_1\mathfrak{v}^2\big)\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\ &+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\big(1+\mathfrak{m}_2\mathfrak{v}^2\big)\\ &\times\|\mathfrak{w}_s-\hat{\mathfrak{w}}_s\|^2_{\mathfrak{G}_{\mathcal{J}}}\\ \leq&\Bigg[3N_{\mathsf{k}}+3\mathcal{M}^2N_{\mathfrak{f}}\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\big(1+\mathfrak{m}_1\mathfrak{v}^2\big)+3\mathcal{M}^2Tr(\mathcal{Q})\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}N_{\mathfrak{h}}\big(1+\mathfrak{m}_2\mathfrak{v}^2\big)\Bigg]\\ &\times\Bigg(\mathfrak{l}^2\sup_{s\in\mathcal{E}}\mathbb{E}\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\|^2+\|\mathfrak{w}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}+\|\hat{\mathfrak{w}}_0\|^2_{\mathfrak{G}_{\mathcal{J}}}\Bigg)\\ \leq&\mathfrak{l}^2\Big[3N_{\mathsf{k}}\\ &+\big[3\mathcal{M}^2N_{\mathfrak{f}}(1+\mathfrak{m}_1\mathfrak{v}^2)+3\mathcal{M}^2Tr(\mathcal{Q})N_{\mathfrak{h}}\big(1+\mathfrak{m}_2\mathfrak{v}^2\big)\big]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\Big]\sup_{s\in\mathcal{E}}\mathbb{E}\big\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\big\|^2\\ \leq&\mathscr{Q}^*\sup_{s\in\mathcal{E}}\mathbb{E}\big\|\mathfrak{w}(s)-\hat{\mathfrak{w}}(s)\big\|^2,\end{aligned}$$

with

$$\mathscr{Q}^* = \mathfrak{l}^2\left[3N_{\mathsf{k}} + \left[3\mathcal{M}^2 N_{\mathfrak{f}}(1+\mathfrak{m}_1\mathfrak{v}^2) + 3\mathcal{M}^2 Tr(\mathcal{Q})N_{\mathfrak{h}}\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)\right]\frac{\mathfrak{t}^{2\alpha-1}}{2\alpha-1}\right].$$

By taking supremum over $\mathfrak{t}$,

$$\|\overline{\eta}\mathfrak{w} - \overline{\eta}\hat{\mathfrak{w}}\|_{\mathfrak{v}}^2 \le \mathscr{Q}^*\|\mathfrak{w} - \hat{\mathfrak{w}}\|_{\mathfrak{v}}^2.$$

Thus $\overline{\eta}$ is a contraction. As a result, the mild solution of (4.1) has a unique fixed point $\mathfrak{w}(.) \in \mathcal{B}_{\mathfrak{p}}$ for the function $\overline{\eta}$. □

Trajectory Controllability

In this section, Gronwall's inequality is used to establish the T-Controllability of the proposed conformable fractional stochastic neutral differential equations with infinite delay.

Theorem 5.1 *If* (*A*1)-(*A*5) *hold*, *the proposed system* (4.1) *is T- controllable*.

Proof Consider the feedback control $\mathfrak{z}(\mathfrak{t})$ with the trajectory $\lambda(\mathfrak{t})$ on $\mathcal{E}$ as,

$$\begin{aligned}\mathfrak{z}(\mathfrak{t}) =& \mathcal{D}^\alpha\left[\lambda(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right] - \mathfrak{A}\left[\lambda(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right] - \mathfrak{f}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\lambda_s\right)ds\right)\\ &- \mathfrak{h}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\lambda_s\right)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}, \quad \alpha \in (0,1).\end{aligned}$$

Thus [4.1] implies,

$$\begin{aligned}\mathcal{D}^\alpha\left[\mathfrak{x}(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right] =& \mathfrak{A}\left[\mathfrak{x}(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}})\right] + \Bigg[\mathcal{D}^\alpha\left[\lambda(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right] - \mathfrak{A}\left[\lambda(\mathfrak{t}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right]\\ &- \mathfrak{f}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\lambda_s\right)ds\right) - \mathfrak{h}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\lambda_s\right)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}\Bigg]\\ &+ \mathfrak{f}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right) + \mathfrak{h}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right)\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}.\end{aligned}$$

Let $\Phi(\mathfrak{t}) = \mathfrak{x}(\mathfrak{t}) - \lambda(\mathfrak{t})$, we obtain

$$\begin{aligned}&\mathcal{D}^\alpha\left[\Phi(\mathfrak{t}) - \left[\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right]\right] = \mathfrak{A}\left[\Phi(\mathfrak{t}) - \left[\mathsf{k}(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}}) - \mathsf{k}(\mathfrak{t},\lambda_{\mathfrak{t}})\right]\right]\\ &\quad+ \mathfrak{f}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\mathfrak{x}_s\right)ds\right) - \mathfrak{f}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\mathfrak{g}\left(\mathfrak{t},s,\lambda_s\right)ds\right)\\ &\quad+ \left[\mathfrak{h}\left(\mathfrak{t},\mathfrak{x}_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}(\mathfrak{t},s,\mathfrak{x}_s)ds\right) - \mathfrak{h}\left(\mathfrak{t},\lambda_{\mathfrak{t}},\int_0^{\mathfrak{t}}\tilde{\mathfrak{g}}\left(\mathfrak{t},s,\lambda_s\right)ds\right)\right]\frac{d\omega(\mathfrak{t})}{d\mathfrak{t}}, \ \mathfrak{t} \in \mathcal{E}.\\ &\mathscr{I}\Phi(\mathfrak{t}) = \mathscr{I}\phi - \mathscr{I}\phi = 0, \quad \mathfrak{t} \in (-\infty, 0].\end{aligned}$$

Therefore the mild solution becomes

$$\Phi(\mathfrak{t}) = \Big[\mathsf{k}\Big(\mathfrak{t}, \mathfrak{w}_{\mathfrak{t}} + \overline{\zeta}_{\mathfrak{t}}\Big) - \mathsf{k}\Big(\mathfrak{t}, \tilde{\mathfrak{w}}_{\mathfrak{t}} + \overline{\zeta}_{\mathfrak{t}}\Big)\Big] + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\Big[\mathfrak{f}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)$$
$$- \mathfrak{f}\Big(s, \tilde{\mathfrak{w}}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)\Big]ds + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)$$
$$\times \Big[\mathfrak{h}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)$$
$$- \mathfrak{h}\Big(s, \tilde{\mathfrak{w}}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)\Big]d\omega(s), \ \mathfrak{t} \in \mathcal{E}.$$

Hence for $\mathfrak{t} \in \mathcal{E}$, the initial data is to be zero. We obtain $\eta(\mathfrak{t}) = 0, \ \mathfrak{t} \in \mathcal{E}$.

Thus $\mathfrak{x}_{\mathfrak{t}} = \eta_{\mathfrak{t}} + \mathfrak{x}_{\mathfrak{t}} = \mathfrak{x}_{\mathfrak{t}}$ and $\lambda_{\mathfrak{t}} = \lambda_{\mathfrak{t}} + \eta_{\mathfrak{t}} = \lambda_{\mathfrak{t}}$. Now,

$$\mathbb{E}\|\Phi(\mathfrak{t})\|^2 \leq 3\mathbb{E}\Big\|\mathsf{k}\Big(\mathfrak{t}, \mathfrak{w}_{\mathfrak{t}} + \overline{\zeta}_{\mathfrak{t}}\Big) - \mathsf{k}\Big(\mathfrak{t}, \tilde{\mathfrak{w}}_{\mathfrak{t}} + \overline{\zeta}_{\mathfrak{t}}\Big)\Big\|^2 + 3\mathbb{E}\Big\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)$$
$$\times \Big[\mathfrak{f}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)$$
$$-\mathfrak{f}\Big(s, \tilde{\mathfrak{w}}_s + \overline{\zeta}_s, \int_0^s \mathfrak{g}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)\Big]ds\Big\|^2$$
$$+ 3\mathbb{E}\Big\|\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathcal{T}\Big(\frac{\mathfrak{t}^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\Big)\Big[\mathfrak{h}\Big(s, \mathfrak{w}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)$$
$$- \mathfrak{h}\Big(s, \tilde{\mathfrak{w}}_s + \overline{\zeta}_s, \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big)\Big]d\omega(s)\Big\|^2$$
$$\leq 3N_{\mathsf{k}}\mathbb{E}\|\Phi(\mathfrak{t})\|^2 + 3N_{\mathfrak{f}}\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\Big(\Big\|(\mathfrak{w}_s + \overline{\zeta}_s) - (\tilde{\mathfrak{w}}_s + \overline{\zeta}_s)\Big\|^2_{\mathfrak{G}_{\mathcal{J}}}$$
$$+ \mathbb{E}\Big\|\int_0^s \mathfrak{g}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho$$
$$- \int_0^s \mathfrak{g}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big\|^2\Big)$$
$$+ 3N_{\mathfrak{h}}Tr(\mathcal{Q})\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\Big(\Big\|(\mathfrak{w}_s + \overline{\zeta}_s) - (\tilde{\mathfrak{w}}_s + \overline{\zeta}_s)\Big\|^2_{\mathfrak{G}_{\mathcal{J}}}$$
$$+ \mathbb{E}\Big\|\int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \mathfrak{w}_\rho + \overline{\zeta}_\rho\Big)d\rho - \int_0^s \tilde{\mathfrak{g}}\Big(s, \rho, \tilde{\mathfrak{w}}_\rho + \overline{\zeta}_\rho\Big)d\rho\Big\|^2\Big)$$
$$\leq 3N_{\mathsf{k}}\mathbb{E}\|\Phi(\mathfrak{t})\|^2 + \int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\big[3N_{\mathfrak{f}}\big(1 + \mathfrak{m}_1\mathfrak{v}^2\big)$$
$$+3N_{\mathfrak{h}}Tr(\mathcal{Q})\big(1 + \mathfrak{m}_2\mathfrak{v}^2\big)\big]\mathbb{E}\|\Phi(s)\|^2ds.$$

Thus,

$$\mathbb{E}\|\Phi(\mathfrak{t})\|^2 \le \frac{3N_{\mathfrak{f}}\left(1+\mathfrak{m}_1\mathfrak{v}^2\right)+3N_{\mathfrak{h}}Tr(\mathcal{Q})\left(1+\mathfrak{m}_2\mathfrak{v}^2\right)}{1-3N_{\mathrm{k}}}\int_0^{\mathfrak{t}} \mathfrak{s}^{\alpha-1}\mathbb{E}\|\Phi(s)\|^2 ds.$$

Therefore by generalized Gronwall's inequality, $\mathbb{E}\|\Phi(\mathfrak{t})\| = 0$, (i.e) $\mathfrak{x}_{\mathfrak{t}} = \lambda_{\mathfrak{t}}$. Thus system (4.1) is T-Controllable on $\mathcal{E}$. □

Illustration

Let us consider a conformable fractional stochastic neutral control system of the form:

$$\begin{cases} \mathcal{D}^{\alpha}\left[\mathfrak{x}(\mathfrak{t},\varepsilon)-\int_{-\infty}^{0}\mathfrak{b}(\mathfrak{t},\varepsilon)\mathfrak{x}(\mathfrak{t},\varepsilon)d\varepsilon\right]=\frac{\partial^2}{\partial\varepsilon^2}\mathfrak{x}(\mathfrak{t},\varepsilon)+\mathfrak{z}(\mathfrak{t},\varepsilon) \\ +\Xi\left(\mathfrak{t},\int_{-\infty}^{\mathfrak{t}}\Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds,\int_0^{\mathfrak{t}}\int_{-\infty}^{0}\Xi_2(s,\varepsilon,\epsilon-s)\mathfrak{x}(\epsilon,\varepsilon)d\epsilon ds\right) \\ +\Upsilon\left(\mathfrak{t},\int_{-\infty}^{\mathfrak{t}}\Xi_1(s-\mathfrak{t})\mathfrak{x}(s,\varepsilon)ds,\int_0^{\mathfrak{t}}\int_{-\infty}^{0}\Xi_3(s,\varepsilon,\epsilon-s)\mathfrak{x}(\epsilon,\varepsilon)d\epsilon ds\right)\frac{d\omega(\mathfrak{t})}{dt}, \quad \mathfrak{t}\in\mathcal{E}' \\ \mathfrak{x}(\mathfrak{t},0)=\mathfrak{x}(\mathfrak{t},\pi)=0, \mathfrak{t}\ge 0, \\ \mathfrak{x}(\mathfrak{t},\varepsilon)=\phi(\mathfrak{t},\varepsilon),\ \varepsilon\in[0,\pi], -\infty<\mathfrak{t}<0, \end{cases} \tag{6.1}$$

where $\mathcal{D}^{\alpha}$ is the conformable derivative, and the neutral function be $\mathfrak{b}(\mathfrak{t},\varepsilon)$. $(\Omega,\mathfrak{F},\mathcal{P})$ is the filtered probability space with one-dimensional Wiener process $\omega(\mathfrak{t})\in\mathcal{Y}$. The generator $\phi(\mathfrak{t},\varepsilon), \Xi_1, \Upsilon, \Xi_2$, and Ξ_3 are continuous.

Let us equalize the considered Hilbert spaces, (i.e) $\left(\mathcal{K}=\mathcal{Y}=\mathcal{L}^2([0,\pi])\right)$. The generator $\mathfrak{A}:\mathscr{D}(\mathfrak{A})\subset\mathcal{Y}\to\mathcal{Y}$ generates a continuous, compact, analytic, and self adjoint semigroup $\mathfrak{G}(\mathfrak{t})_{\mathfrak{t}\ge 0}$, where $\mathfrak{A}\mathfrak{u}=\mathfrak{u}''\in\mathscr{D}(\mathfrak{A})$ with

$$\mathscr{D}(\mathfrak{A})=\{\mathfrak{u}\in\mathcal{Y}:\ \mathfrak{u},\mathfrak{u}' \text{ are absolutely continuous, } \mathfrak{u}''\in\mathcal{Y}, \mathfrak{u}(0)=\mathfrak{u}(\pi)=0\}.$$

Let $-v^2, v\in\mathbb{N}$ are the eigenvalues and normalized eigenvectors corresponding to v are

$$\mathfrak{w}_v(\varepsilon)=\left(\frac{2}{\pi}\right)^{1/2}\sin(v\varepsilon),\ v=1,2,\cdots.$$

Let us consider these presumptions:

(i) $\forall\ \mathfrak{u}\in\mathscr{D}(\mathfrak{A}), \mathfrak{A}\mathfrak{u}=\sum_{v=1}^{\infty}\langle\mathfrak{u},\mathfrak{w}_v\rangle\mathfrak{w}_v$.

(ii) $\forall\ \mathfrak{u}\in\mathcal{Y}, \mathfrak{Q}=\sum_{v=1}^{\infty}\frac{1}{v}\langle\mathfrak{u},\mathfrak{w}_v\rangle\mathfrak{w}_v$ in the space $\mathscr{D}(\mathfrak{Q})=\{\mathfrak{u}\in\mathcal{Y},\sum_{v=1}^{\infty}v\langle\mathfrak{u},\mathfrak{w}_v\rangle\mathfrak{w}_v\}$, $\mathfrak{Q}=1-\mathfrak{A}$.

Consider $\mathcal{J}(s)=e^{2s}, s<0$, then $\mathfrak{l}=\int_{-\infty}^{0}\mathcal{J}(s)ds=\frac{1}{2}$. The phase space $\mathfrak{G}_{\mathcal{J}}$ equipped with

$$\|\zeta\|_{\mathfrak{G}_{\mathcal{J}}}=\int_{-\infty}^{0}\mathcal{J}(s)\sup_{s\le\eta\le 0}\left(\mathbb{E}\|\zeta(\eta)\|^2\right)^{1/2}ds.$$

Then, $\left(\mathfrak{G}_{\mathcal{J}},\|.\|_{\mathfrak{G}_{\mathcal{J}}}\right)$ is a Banach space.

For $(\mathfrak{t},\zeta)\in[0,\mathfrak{v}]\times\mathfrak{G}_{\mathcal{J}}$, where $\zeta(.,\varepsilon)=\phi(.,\varepsilon)\in(-\infty,0]\times[0,\pi]$, we consider

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t})(\varepsilon) &= \mathfrak{x}(\mathfrak{t}, \varepsilon),\\
\mathfrak{g}(\mathfrak{t}, \zeta)(\varepsilon) &= \int_{-\infty}^{0} \Xi_2(\mathfrak{t}, \varepsilon, s)\zeta(s)(\varepsilon)ds,\\
\mathfrak{f}\left(\mathfrak{t}, \zeta, \int_0^{\mathfrak{t}} \mathfrak{g}(s, \zeta)\right)(\varepsilon) &= \Xi\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Xi_1(s-\mathfrak{t})\mathfrak{x}(s, \varepsilon)ds, \int_0^{\mathfrak{t}} \mathfrak{g}(\mathfrak{t}, \zeta)(\varepsilon)ds\right),\\
\tilde{\mathfrak{g}}(\mathfrak{t}, \zeta)(\varepsilon) &= \int_{-\infty}^{0} \Xi_3(\mathfrak{t}, \varepsilon, s)\zeta(s)(\varepsilon)ds,\\
\mathfrak{h}\left(\mathfrak{t}, \zeta, \int_0^{\mathfrak{t}} \tilde{\mathfrak{g}}(s, \zeta)\right)(\varepsilon) &= \Upsilon\left(\mathfrak{t}, \int_{-\infty}^{\mathfrak{t}} \Xi_1(s-\mathfrak{t})\mathfrak{x}(s, \varepsilon)ds, \int_0^{\mathfrak{t}} \tilde{\mathfrak{g}}(\mathfrak{t}, \zeta)(\varepsilon)ds\right),\\
\mathsf{k}(\mathfrak{t}, \zeta)(\varepsilon) &= \int_{-\infty}^{0} \mathfrak{b}(\mathfrak{t}, \varepsilon)\zeta(\mathfrak{t}, \varepsilon)d\varepsilon.
\end{aligned}$$

It is clear that the functions $\mathfrak{f}$, $\mathfrak{g}$, $\tilde{\mathfrak{g}}$, $\mathfrak{h}$, and k meets hypothesis (A1)-(A7).

The control $\mathfrak{z} : \mathfrak{G}\mathfrak{x}([0, \pi]) \to \mathbb{R}$ where $\mathfrak{z} \in \mathcal{L}^2(\mathfrak{G}\mathfrak{x}([0, \pi]))$, $\mathfrak{t} \to \mathfrak{z}(\mathfrak{t})$ is measurable. Let us assume that $\mathfrak{A} = \{\mathfrak{z} \in \mathcal{K} : \|\mathfrak{z}\|_{\mathcal{K}} \le \varpi\}$ where $\varpi \in \mathcal{L}^2(\mathcal{E}, \mathbb{R}^+)$. Thus (4.1) is T-controllable on (0, 1] by Theorem 5.1.

Numerical Simulation

In this section we will show numerical simulations of the application system (7.17.27.37.4)-(7.4). In terms of approximating a numerical solution this example has a variety of interesting and challenging components. If it only had the parabolic partial differential equation (PDE) component then we could solve it with a variety of software suites and the methods described in [34, 35]. If it was only the neutral delay term then we could use the methods described in [36]. This equation has all of these components in addition to an integral term.

To the author's knowledge there is no existing software suite built to handle all of these components in the example NSIDE with infinite delay and mixed fBm so we have built one in Matlab and the code is available in the supplemental materials. All of the specifics of the simulation are available within the code but the primary techniques are as follows. A forward finite difference was used in the time derivative as to make it an explicit method. A centered difference approximation was used for all spatial (θ) derivatives. The trapezoid rule as implemented by the Matlab function trapz.m was used to approximate the integral term at each discretization point. The Brownian term dB is normally distributed with mean zero and variance $d\imath$ where $d\imath$ is the time discretization step. the fractional Brownian term dZ is normally distributed with mean zero and variance dt^{2h} where h is the fBm parameter. The delayed derivative terms were approximated by difference derivatives on the mesh as well.

In the included simulations (Figures 1, 2) the following functions and parameters have been used. We have used $n = 2$ and 20 points in each spatial dimension for a total of 400 spatial points at each timestep. We have used 5000 time-steps therefore $d\imath = 0.0002$. The fBm parameter $h = 0.7$. The functions are $\phi(\imath, \theta) = \imath^2 + \sum_i^n \theta_i^2$, $\Lambda(s) = cos(s)$, and $c(\imath, \theta) = \frac{1}{\imath^2 + \sum_i^n \theta_i^2}$. Figure 1 shows the function at the beginning at $\imath = 0$ and a third way through the simulation. Figure 2 shows the simulation at roughly two thirds through the simulation and at the end. Figure 3 shows the simulation at roughly third eight the simulation and at the end.

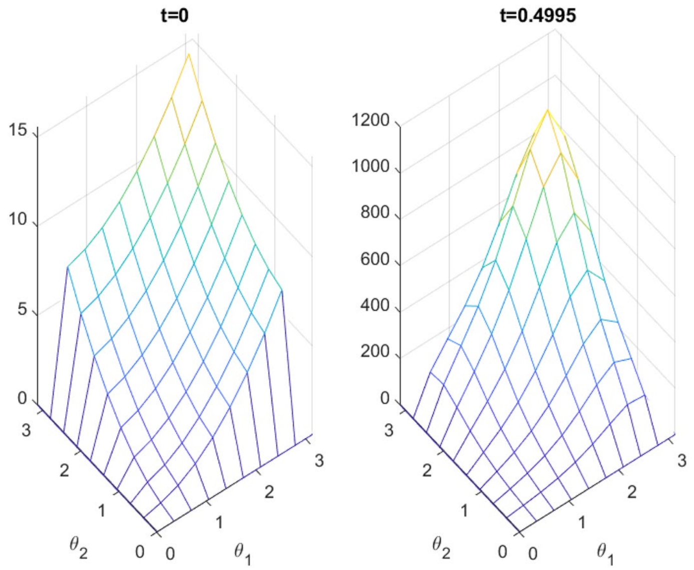

Fig. 1 Graph of $\vartheta(\imath = 0, \theta)$ and $\vartheta(\imath = 0.333, \theta)$

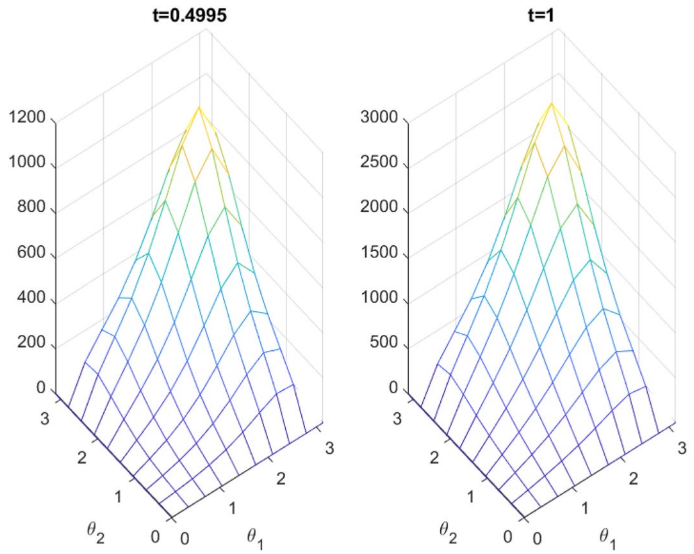

Fig. 2 Graph of $\vartheta(\imath = 0.4995, \theta)$ and $\vartheta(\imath = 1, \theta)$

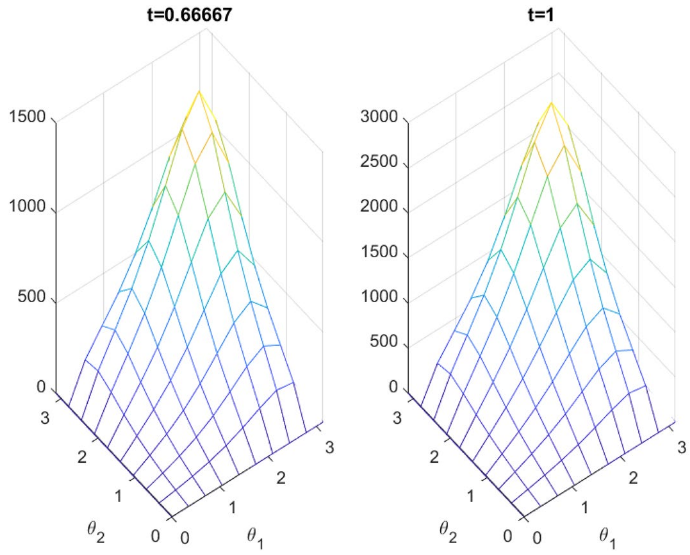

Fig. 3 Graph of $\vartheta(\iota = 0.667, \theta)$ and $\vartheta(\iota = 1, \theta)$

$$\frac{\partial^\alpha}{\partial \iota^\alpha}\left[\vartheta(\iota,\theta) + \frac{\iota^2 + e^\iota |\vartheta(\iota-\varphi,\theta)|^2}{18}\right] \tag{7.1}$$

$$\begin{aligned} =&\frac{\partial^2}{\partial \theta^2}\left[\vartheta(\iota,\theta) + \frac{\iota^2 + e^\iota |\vartheta(\iota-\varphi,\theta)|^2}{18}\right] \\ +&\frac{e^\iota \vartheta(\iota-\varphi,\theta)/\sqrt{2} + \vartheta(\iota, \sin\iota|\vartheta(\iota,\theta)|)/\sqrt{2}}{9} \end{aligned} \tag{7.2}$$

$$\begin{aligned} &+e^\iota \int_{-\infty}^{t} \sin(\iota-\zeta) d\iota \int_0^t \int_{-\infty}^{0} \left[\frac{\iota \sin\theta}{8\pi} + \frac{e^\iota |\vartheta(\iota-\varphi,\theta)|}{2 + |\vartheta(\iota,\theta)|}\right] d\iota d\theta \\ &+\left[e^\iota \int_{-\infty}^{t} \sin(\iota-\zeta) d\iota \int_0^t \int_{-\infty}^{0} \left[\frac{\iota \sin\theta}{8\pi} + \frac{e^\iota |\vartheta(\iota-\varphi,\theta)|}{2 + |\vartheta(\iota,\theta)|}\right] d\iota d\theta\right] \frac{d\omega(\mathrm{t})}{d\mathrm{t}} \end{aligned} \tag{7.3}$$

$$\vartheta(\iota,\ \theta)\ =\ \varphi\ (\iota,\ \theta),\ \theta \in [0,\pi],\ \iota \in (-\infty,\ 0]. \tag{7.4}$$

Conclusion

In this article, we discussed the trajectory controllability of a conformable fractional neutral stochastic integro-differential system. This equation has infinite delays and takes place in a separable Hilbert space. In addition, using Banach fixed point theorem, the existence results and some conclusions concerning the trajectory controllability are obtained. An illustration of the theory that has been presented is provided as the concluding part.

This work can be further extended to optimal controllability and stability of conformable fractional stochastic differential equations. We developed a numerical scheme to justify the theory. The code contains all of the details of the simulation for the finite difference method in Matlab. This work is a unique combination of theoretical proof with numerical estimations. The following are the future aspects of this research work:

- One can extend the same idea for the fractional order/hybrid fractional order system with deviated arguments using Riemann-Liouville(R-L) and Caputo derivatives.
- One can consider the Hilfer (or Ψ-Hilfer), Hadamard, Hilfer-Katugampola fractional system with state-dependent delay and non-instantaneous impulses.
- Trajectory controllability of the system will be the new work with the numerical simulation.
- The technique used in this paper can be replaced by the method of Measure of Non-compactness. Also, monotone operator theory can be effectively used to study the same system and to study all kinds of fractional order SIDEs.
- Lipschitz continuity on the nonlinear operators can be weakened by the method of Integral Contractor with Regularity.

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Declarations

Conflict of Interest The authors declare no competing interests.

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