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Mixed sequential type pantograph fractional integro-differential equations with non-local boundary conditions

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Abstract
In this paper, we investigate the existence, uniqueness and stability of solutions to the mixed sequential pantograph fractional integro-differential equations with non-local boundary conditions. The solution of the problem is obtained and the existence and uniqueness of the solution is tested by means of Krasnosel'skii's fixed point theorem and the Banach contraction principle respectively. Moreover, the Ulam–Hyers and Ulam–Hyers–Rassias stability of the solution is determined. An example emphasising our findings is provided.

Keywords Pantograph fractional differential equation · Fixed point theory · Existence and uniqueness

Mathematics Subject Classification 34A08 · 26A33 · 47H10 · 34A12

1 Introduction

The pantograph equation, which deals with proportional delay and was developed during work on the electric current of the pantograph of an electric locomotive by Tayler and Ockendon, is one of the most prominent classes of delay differential equations (DEs) in applied sciences [20]. Delay DEs are essential for describing natural phenomena due to their reliance on historical data. It has wide applications in number theory, electrodynamics, quantum physics, control systems, and many other fields [9, 14, 18, 23]. Researchers specifically examined the existence and uniqueness of pantograph fractional DEs in [1, 4, 5, 10, 13].

In the modelling of scientific and engineering problems, the substitution of one relationship using derivatives into another led to the development of sequential fractional derivatives. The

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existence theory of solutions to sequential fractional differential equations is the most crucial and significant to understand the behaviour of dynamical systems. The articles [7, 10, 22, 26] extensively demonstrate the uses of sequential fractional derivatives in the field of fractional calculus. Also, in various branches of engineering and research, fractional integrodifferential equations play a significant role. These include microhydrodynamics [28], wind ripple in the desert [6], and drop-wise condensation [25], etc.

One among the qualitative aspects of DEs is the concept of stability. Numerous stability analyses are performed, which include exponential stability [3], Mittag-Leffler stability [17], asymptotic stability [2], Lyapunov stability [8], etc. A few of the aforementioned analyses are complex and time-consuming as well. The Ulam-Hyers and Ulam-Hyers Rassias stability, provides an accurate solution for each approximation, making it the best stability for fractional DEs that deal with non-local situations [4, 19, 24].

Motivated by the previous findings, we analyse mixed sequential pantograph fractional integro-differential equations (FIDE) with non-local boundary conditions of the form

$$\begin{cases} D_{a^+}^{u:\psi}\left({}^H D_{a^+}^{\tau,\varsigma:\psi}\eta(\xi)\right) = f\left(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^{\xi}\mathcal{P}(\xi,s,\eta(s))ds\right) \\ \qquad\qquad + \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i;\psi} g_i\left(\xi,\eta(\xi),\eta(\mu\xi)\right),\ \xi\in[a,b]=\mathcal{Z}, \\ \eta(a)=0, \quad \eta(b)=\sum_{r=1}^{p}\beta_r {}^H D_{a^+}^{\omega_r,\varsigma:\psi}\eta(\delta_r), \quad \beta_r\in\mathfrak{R}^+,\ \delta_r\in\mathcal{Z}, \end{cases} \tag{1}$$

where $D_{a^+}^{u:\psi}$ is the ψ- Riemann-Liouville (R-L) fractional derivative, ${}^H D_{a^+}^{\tau,\varsigma:\psi}$ is the ψ-Hilfer fractional derivative, $0 < u, \tau < 1, 0 \leq \varsigma \leq 1, 0 < \lambda, \mu < 1$, $\mathcal{A}_i$'s are real constants, $\mathcal{P} : \mathcal{D}\times\mathfrak{R} \longrightarrow \mathfrak{R}$ is continuous on $\mathcal{D} = \{(\xi, s) : a \leq s \leq t \leq b\}$, $f : \mathcal{Z}\times\mathfrak{R}\times\mathfrak{R}\times\mathfrak{R} \longrightarrow \mathfrak{R}$ and $g_i : \mathcal{Z}\times\mathfrak{R}\times\mathfrak{R} \longrightarrow \mathfrak{R}$, $i = 1, 2, ..., m$ are continuous functions.

2 Preliminaries

Let us recall some basic concepts and definitions relevant to our research.

$\mathcal{Q} = C(\mathcal{Z},\mathfrak{R})$ represent the space of all continuous functions from $\mathcal{Z} \longrightarrow \mathfrak{R}$ with the norm $\|\eta\| = \sup_{\xi\in\mathcal{Z}} |\eta(\xi)|$, and $AC(\mathcal{Z},\mathfrak{R})$ is the space of all absolutely continuous functions from $\mathcal{Z} \longrightarrow \mathfrak{R}$.

Definition 1 [15] Let (a, b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathfrak{R}$ and $\tau > 0$. Let $\psi(\xi) > 0$ be an increasing function on $(a, b]$, having a continuous derivative $\psi'(\xi)$ on (a, b). The ψ-R-L fractional integral $I_{a^+}^{\tau;\psi}(\cdot)$ of a function $h \in AC^n([a, b],\mathfrak{R})$ on $[a, b]$, is defined by

$$I_{a^+}^{\tau;\psi}h(\xi) = \frac{1}{\Gamma(\tau)}\int_a^{\xi}\psi'(s)(\psi(\xi)-\psi(s))^{\tau-1}h(s)ds, \quad \xi > a > 0,$$

where $\Gamma(.)$ represents the Gamma function.

Definition 2 [15] Let $n \in \mathbb{N}$ and $\psi'(\xi) \neq 0$. The ψ-R-L fractional derivative of order $\tau > 0$ of a function $h \in AC^n([a,b],\mathfrak{R})$ with respect to another function ψ is defined by

$$\begin{aligned} D_{a^+}^{u;\psi} h(\xi) =& \Big(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\Big)^n I_{a^+}^{n-u;\psi} h(\xi) \\ =& \frac{1}{\Gamma(n-u)}\Big(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\Big)^n \int_a^\xi \psi'(s)(\psi(\xi)-\psi(s))^{n-u-1} h(s)ds, \end{aligned}$$

where $n = [u]+1,\ [u]$ represents the integer part of the real number u.

Definition 3 [27] Let $[a,b]$ be the interval such that $-\infty \leq a < b \leq \infty, n \in \mathbb{N}, n-1 < \tau < n$ and $h, \psi \in C^n([a,b],\mathfrak{R})$ are two functions such that $\psi(\xi)$ is increasing and $\psi'(\xi) \neq 0$, for all $\xi \in [a,b]$. The ψ-Hilfer fractional derivative ${}^H D_{a^+}^{\tau,\varsigma;\psi}(\cdot)$ of a function h of order τ and type $0 \leq \varsigma \leq 1$, is defined by

$${}^H D_{a^+}^{\tau,\varsigma;\psi} h(\xi) = I_{a^+}^{\varsigma(n-\tau);\psi}\Big(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\Big)^n I_{a^+}^{(1-\varsigma)(n-\tau);\psi} h(\xi),$$

$n = [\tau]+1,\ [\tau]$ represents the integer part of the real number τ with $\gamma = \tau + \varsigma(n-\tau)$.

Lemma 1 [15] *For $\tau, \alpha > 0$, we have the following semigroup property given by*

$$I_{a^+}^{\tau;\psi} I_{a^+}^{\alpha;\psi} h(\xi) = I_{a^+}^{\tau+\alpha;\psi} h(\xi),\ \xi > a.$$

Lemma 2 [15] *If $h \in C^n([a,b],\mathfrak{R}),\ n-1 < u < n$ then*

$$I_{a^+}^{u;\psi} D_{a^+}^{u;\psi} h(\xi) = h(\xi) - \sum_{k=1}^{n} \frac{h^{[k-1]}(a^+)}{\Gamma(u-k)}(\psi(\xi)-\psi(a))^{u-k+1},$$

for all $\xi \in [a,b]$, where $h_\psi^{[n]} h(\xi) = \big(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\big)^k h(\xi)$.

Lemma 3 [27] *If $h \in C^n([a,b],\mathfrak{R}),\ n-1 < \tau < n$ and $0 \leq \varsigma \leq 1$ and $\gamma = \tau + \varsigma(n-\tau)$ then*

$$I_{a^+}^{\tau;\psi}\ {}^H D_{a^+}^{\tau,\varsigma;\psi} h(\xi) = h(\xi) - \sum_{k=1}^{n} \frac{(\psi(\xi)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} h_\psi^{[n-k]} I_{a^+}^{(1-\varsigma)(n-\tau);\psi} h(a),$$

for all $\xi \in [a,b]$, where $h_\psi^{[n]} h(\xi) = \big(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\big)^n h(\xi)$.

Proposition 1 [15, 27] *Let $\tau \geq 0$, $l > 0$ and $\xi > a$. Then the ψ-fractional integral and derivative of a power function are given by*

1. $I_{a^+}^{\tau;\psi}(\psi(\xi)-\psi(a))^{l-1} = \frac{\Gamma(l)}{\Gamma(l+\tau)}(\psi(\xi)-\psi(a))^{l+\tau-1}$,
2. $D_{a^+}^{u;\psi}(\psi(\xi)-\psi(a))^{l-1} = \frac{\Gamma(l)}{\Gamma(l-\tau)}(\psi(\xi)-\psi(a))^{l-u-1}$,
3. ${}^H D_{a^+}^{\tau,\varsigma;\psi}(\psi(\xi)-\psi(a))^{l-1} = \frac{\Gamma(l)}{\Gamma(l-\tau)}(\psi(\xi)-\psi(a))^{l-\tau-1},\ l > \gamma = \tau + \varsigma(n-\tau)$.

Lemma 4 (Banach contraction principle) [11] *If D is a closed non-empty subset of a Banach space B then any contraction mapping $\mathcal{G} : D \longrightarrow D$ has a unique fixed point.*

Theorem 2 (Krasnosel'skii's fixed point theorem) [16] *Let D be a closed, bounded, convex and non-empty subset of a Banach space $(B, \|\cdot\|)$. Suppose that $\mathcal{G}_1, \mathcal{G}_2$ are operators from D to D such that*

1. $\mathcal{G}_1\eta_1 + \mathcal{G}_2\eta_2 \in D,\ \forall\, \eta_1, \eta_2 \in D,$
2. $\mathcal{G}_1$ *is continuous and compact,*
3. $\mathcal{G}_2$ *is a contraction mapping.*

Then there exist a $\eta_3 \in D$ *such that* $\eta_3 = \mathcal{G}_1\eta_3 + \mathcal{G}_2\eta_3$.

To establish stability results such as Ulam-Hyers ($\mathcal{UH}$) and Ulam-Hyers-Rassias ($\mathcal{UHR}$) stability, consider the following:
For $\kappa \in \mathcal{R}^+$, let $\Theta : \mathcal{Z} \longrightarrow \mathcal{R}^+$ be a continuous function and

$$\Big| D^{u:\psi}_{a^+}\Big({}^H D^{\tau,\varsigma:\psi}_{a^+} z(\xi)\Big) - f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\Big) - \sum_{i=1}^m \mathcal{A}_i I^{\phi_i;\psi}_{a^+} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big)\Big| \le \kappa, \tag{2}$$

$$\Big| D^{u:\psi}_{a^+}\Big({}^H D^{\tau,\varsigma:\psi}_{a^+} z(\xi)\Big) - f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\Big) - \sum_{i=1}^m \mathcal{A}_i I^{\phi_i;\psi}_{a^+} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big)\Big| \le \kappa\ \Theta(\xi). \tag{3}$$

Definition 4 [21] The system (1) is $\mathcal{UH}$ stable if there exists a real number $\mathcal{M}_{f,g_i} > 0$ such that for each $\kappa > 0$ and each solution $z \in \mathcal{Q}$ of the inequality (2), there exists a solution $\eta \in \mathcal{Q}$ of (1) with

$$|z(\xi) - \eta(\xi)| \le \mathcal{M}_{f,g_i}\ \kappa,\ \ \xi \in \mathcal{Z},\ \ i = 1, 2, ..., m. \tag{4}$$

Definition 5 [21] The system (1) is $\mathcal{UHR}$ stable with respect to $\Theta \in C(\mathcal{Z}, \mathcal{R}^+)$ if there exists a real number $\mathcal{M}_{f,g_i,\Theta} > 0$ such that for each $\kappa > 0$ and each solution $z \in \mathcal{Q}$ of the inequality (3), there exists a solution $\eta \in \mathcal{Q}$ of (1) with

$$|z(\xi) - \eta(\xi)| \le \mathcal{M}_{f,g_i,\Theta}\ \kappa\ \Theta(\xi),\ \ \xi \in \mathcal{Z},\ \ i = 1, 2, ..., m. \tag{5}$$

Remark 1 A function $z \in \mathcal{Q}$ is a solution of (2) if and only if there exits a function $w \in \mathcal{Q}$ such that $\forall\, \xi \in \mathcal{Z}$

(i) $|w(\xi)| \le \kappa$, and

(ii) $D^{u:\psi}_{a^+}\Big({}^H D^{\tau,\varsigma:\psi}_{a^+} z(\xi)\Big) = f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\Big) + \sum_{i=1}^m \mathcal{A}_i I^{\phi_i;\psi}_{a^+} \times g_i\Big(\xi, z(\xi), z(\mu\xi)\Big) + w(\xi).$

Remark 2 A function $z \in \mathcal{Q}$ is a solution of (3) if and only if there exits a function $v \in \mathcal{Q}$ such that $\forall\, \xi \in \mathcal{Z}$

(i) $|v(\xi)| \le \kappa\ \Theta(\xi)$, and

(ii) $D^{u:\psi}_{a^+}\Big({}^H D^{\tau,\varsigma:\psi}_{a^+} \eta(\xi)\Big) = f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big) + \sum_{i=1}^m \mathcal{A}_i I^{\phi_i;\psi}_{a^+} \times g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big) + v(\xi).$

Furthermore, we consider the following notation: $\Xi(g, h) = \frac{(\psi(g)-\psi(a))^h}{\Gamma(h+1)}$.

3 An auxiliary result

The solution of the system (1) is obtained in the following lemma.

Lemma 5 *Let* $0 < u, \tau < 1$, $0 \leq \varsigma \leq 1$, $\gamma = \tau + \varsigma(1-\tau)$, *a* ≥ 0, *and* $\Delta \neq 0$. *Then for* $f : \mathcal{Z} \times \Re \times \Re \times \Re \longrightarrow \Re$ *and* $g_i : \mathcal{Z} \times \Re \times \Re \longrightarrow \Re$, *the solution of the system (1) is given by*

$$\begin{aligned}\eta(\xi) =& I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big) + \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big)\\ &+ \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^r \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, \eta(\delta_r), \eta(\lambda\delta_r), \int_0^\xi \mathcal{P}(\delta_r, s, \eta(s))\Big)\\ &- I_{a^+}^{u+\tau;\psi} f\Big(b, \eta(b), \eta(\lambda b), \int_0^\xi \mathcal{P}(b, s, \eta(s))\Big) + \sum_{r=1}^p\sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi}\\ &\times g_i\Big(\delta_r, \eta(\delta_r), \eta(\mu\delta_r)\Big) - \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(b, \eta(b), \eta(\mu b)\Big)\Bigg], \end{aligned} \tag{6}$$

where $\Delta = \Xi(b, u+\tau-1) - \sum_{r=1}^p \beta_r \Xi(b, u+\tau-\omega_r-1)$.

Proof By applying operators $I_{a^+}^{u;\psi}$ and $I_{a^+}^{\tau;\psi}$ on both sides of (1), from Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}{}^H D_{a^+}^{\tau,\varsigma;\psi}\eta(\xi) &= I_{a^+}^{u;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big)\\ &+ \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big) + c_1\Xi(\xi, u-1).\\ \eta(\xi) &= I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big)\\ &+ \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big)\\ &+ c_1\Xi(\xi, u+\tau-1) + c_2\Xi(\xi, \gamma-1).\end{aligned}$$

Since $\gamma = \tau + \varsigma(n-\tau) < 1$, $\eta(a) = 0$ implies $c_2 = 0$. The above equation reduces to

$$\begin{aligned}\eta(\xi) =& I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big)\\ &+ \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big) + c_1\Xi(\xi, u+\tau-1).\end{aligned} \tag{7}$$

Using $\eta(b) = \sum_{r=1}^{p} \beta_r {}^H D_{a^+}^{\omega_r,\varsigma:\psi} \eta(\delta_r)$, we derive

$$\begin{aligned} c_1 =& \frac{1}{\Delta}\Bigg[\sum_{r=1}^{p} \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, \eta(\delta_r), \eta(\lambda\delta_r), \int_0^{\xi} \mathcal{P}(\delta_r, s, \eta(s))\Big) \\ & - I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^{\xi} \mathcal{P}(\xi, s, \eta(s))ds\Big) + \sum_{r=1}^{p}\sum_{i=1}^{m} \beta_r \mathcal{A}_i \\ & \times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} g_i\Big(\delta_r, \eta(\delta_r), \eta(\mu\delta_r)\Big) - \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big)\Bigg]. \end{aligned}$$

By substituting c_1 in (7) we obtain (6).
Conversely, by direct calculation we verify that (6) satisfies (1). □

4 Existence and uniqueness results

In this section, we establish the existence and uniqueness results.
Let us define an operator $\mathcal{G} : \mathcal{Q} \longrightarrow \mathcal{Q}$ by

$$\begin{aligned} \mathcal{G}\eta(\xi) =& I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^{\xi} \mathcal{P}(\xi, s, \eta(s))ds\Big) + \sum_{i=1}^{m} \mathcal{A}_i \\ & \times I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big) + \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r} \beta_r \\ & \times I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, \eta(\delta_r), \eta(\lambda\delta_r), \int_0^{\xi} \mathcal{P}(\delta_r, s, \eta(s))\Big) \\ & - I_{a^+}^{u+\tau;\psi} f\Big(b, \eta(b), \eta(\lambda b), \int_0^{\xi} \mathcal{P}(b, s, \eta(s))\Big) + \sum_{r=1}^{p}\sum_{i=1}^{m} \mathcal{A}_i \\ & \times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} g_i\Big(\delta_r, \eta(\delta_r), \eta(\mu\delta_r)\Big) - \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(b, \eta(b), \eta(\mu b)\Big)\Bigg]. \end{aligned} \tag{8}$$

and assume the following hypothesis:

($\mathbf{H_1}$) Let $f : \mathcal{Z} \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ and $g_i : \mathcal{Z} \times \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ be continuous functions, and there exists a constant $\mathcal{K}_i > 0$ $(i = 1, 2, ..., m+1)$ such that for all $\xi \in \mathcal{Z}$ and $\eta_1, \bar{\eta_1}, \eta_2, \bar{\eta_2}, \eta_3, \bar{\eta_3} \in \mathfrak{R}$,

$$\begin{aligned} |f(\xi, \eta_1(\xi), \eta_2(\xi), \eta_3(\xi)) - f(\xi, \bar{\eta_1}(\xi), \bar{\eta_2}(\xi), \bar{\eta_3}(\xi))| \leq& \ \mathcal{K}_1\big(|\eta_1 - \bar{\eta_1}| + |\eta_2 - \bar{\eta_2}| \\ & + |\eta_3 - \bar{\eta_3}|\big), \\ |g_i(\xi, \eta_1(\xi), \eta_2(\xi)) - g_i(\xi, \bar{\eta_1}(\xi), \bar{\eta_2}(\xi))| \leq& \ \mathcal{K}_{i+1}\big(|\eta_1 - \bar{\eta_1}| + |\eta_2 - \bar{\eta_2}|\big), \\ & i = 1, 2, ..., m. \end{aligned}$$

($\mathbf{H_2}$) Let $\mathcal{P} : \mathcal{D} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ be a continuous function on $\mathcal{D} = \big\{(\xi, s) : a \leq s \leq t \leq b\big\}$, and there exists constant $\mathcal{N} > 0$ such that for all $\xi \in \mathcal{Z}$ and $\eta, \bar{\eta} \in \mathfrak{R}$,

$$|\mathcal{P}(\xi, s, \eta(\xi)) - \mathcal{P}(\xi, s, \bar{\eta}(\xi))| \leq \mathcal{N}|\eta - \bar{\eta}|,$$

($\mathbf{H_3}$) Let $f : \mathcal{Z} \times \Re \times \Re \times \Re \longrightarrow \Re$ and $g_i : \mathcal{Z} \times \Re \times \Re \longrightarrow \Re$ be continuous functions, and there exists functions $\sigma, \nu_i > 0,\ i = 1, 2, ..., m$ such that for all $\xi \in \mathcal{Z}$ and $\eta_1, \eta_2, \eta_3 \in \Re$,

$$|f(\xi, \eta_1(\xi), \eta_2(\xi), \eta_3(\xi))| \leq \sigma(\xi),$$
$$|g_i(\xi, \eta_1(\xi), \eta_2(\xi))| \leq \nu_i(\xi),\ i = 1, 2, ..., m.$$

To simplify the process, let us introduce some notations.

$$\mathcal{V}_1 = \Xi(\xi, u+\tau) + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\left[\sum_{p=1}^{r} \beta_r \Xi(\delta_r, u+\tau-\omega_r) + \Xi(b, u+\tau)\right], \tag{9}$$

$$\mathcal{V}_2 = \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(\xi, \phi_i + u + \tau) + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|} \left[\sum_{p=1}^{r}\sum_{i=1}^{m} \beta_r |\mathcal{A}_i| \Xi(\delta_r, \phi_i + u + \tau - \omega_r) + \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(b, \phi_i + u + \tau)\right]. \tag{10}$$

Theorem 3 *Assume that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are satisfied. Suppose that* $\mathcal{K}\big(\mathcal{V}_1(2+b\mathcal{N})+2\mathcal{V}_2\big) < 1$, *where* $\mathcal{K} = \max\{\mathcal{K}_i,\ i = 1, 2, ..., m+1\}$ *and* $\mathcal{N}$ *are constants,* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are given by (9) and (10) respectively. Then, the system (1) has a unique solution on* $\mathcal{Z}$.

Proof Consider the operator $\mathcal{G}\eta(\xi)$ defined in (8). Suppose that $\mathcal{L} = \max\{\mathcal{L}_i,\ i = 1, 2, ..., m+1\}$, $\mathcal{L}_i$ are finite numbers given by $\mathcal{L}_1 = \sup_{\xi\in\mathcal{Z}} |f(\xi, 0, 0, 0)|$ and $\mathcal{L}_{i+1} = \sup_{\xi\in\mathcal{Z}} |g_i(\xi, 0, 0)|$ and $\mathcal{Q}_r = \{\eta \in \mathcal{Q} : |\eta| \leq r\}$ with $r \geq \frac{\mathcal{V}_1+V_2}{1-\mathcal{K}\left(\mathcal{V}_1(2+b\mathcal{N})+2V_2\right)}$. Clearly, $\mathcal{Q}_r$ is a bounded, closed and convex subset of $\mathcal{Q}$.

Step 1: To prove $\mathcal{G}\mathcal{Q}_r \subset \mathcal{Q}_r$.
For any $\eta \in \mathcal{Q}_r$, $\xi \in \mathcal{Z}$, using ($\mathbf{H_1}$) and ($\mathbf{H_2}$), we obtain

$$\left|f\left(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\right)\right| \leq \mathcal{K}_1(2 + b\mathcal{N})|\eta| + \mathcal{L}_1,$$
$$\left|g_i\left(\xi, \eta(\xi), \eta(\mu\xi)\right)\right| \leq 2\,\mathcal{K}_{i+1}|\eta| + \mathcal{L}_{i+1}.$$

Then,

$$\begin{aligned}|\mathcal{G}\eta(\xi)| &\leq I_{a^+}^{u+\tau;\psi}\left|f\left(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\right)\right| \\ &+ \sum_{i=1}^{m} |\mathcal{A}_i| I_{a^+}^{\phi_i+u+\tau;\psi}\left|g_i\left(\xi, \eta(\xi), \eta(\mu\xi)\right)\right| \\ &+ \frac{\Xi(\xi, u+\tau-1)}{\Delta}\left[\sum_{p=1}^{r} \beta_r I_{a^+}^{u+\tau-\omega_r;\psi}\left|f\left(\delta_r, \eta(\delta_r), \eta(\lambda\delta_r), \int_0^\xi \mathcal{P}(\delta_r, s, \eta(s))\right)\right|\right.\end{aligned}$$

$$+ I_{a^+}^{u+\tau;\psi}\left|f\Big(b,\eta(b),\eta(\lambda b),\int_0^{\xi}\mathcal{P}(b,s,\eta(s))\Big)\right| + \sum_{r=1}^{p}\sum_{i=1}^{m}|\mathcal{A}_i|$$
$$\times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi}\left|g_i\Big(\delta_r,\eta(\delta_r),\eta(\mu\delta_r)\Big)\right| + \sum_{i=1}^{m}|\mathcal{A}_i| I_{a^+}^{\phi_i+u+\tau;\psi}\left|g_i\Big(b,\eta(b),\eta(\mu b)\Big)\right|\Bigg]$$
$$\leq \ \Xi(\xi,u+\tau)\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta|+\mathcal{L}_1\Big] + \sum_{i=1}^{m}|\mathcal{A}_i|\Xi(\xi,\phi_i+u+\tau)\Big[2\,\mathcal{K}_{i+1}|\eta|+\mathcal{L}_{i+1}\Big]$$
$$+ \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r,u+\tau-\omega_r)\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta|+\mathcal{L}_1\Big] + \Xi(b,u+\tau)$$
$$\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta|+\mathcal{L}_1\Big] + \sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|\Xi(\delta_r,\phi_i+u+\tau-\omega_r)\Big[2\mathcal{K}_{i+1}|\eta|+\mathcal{L}_{i+1}\Big]$$
$$+ \sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b,\phi_i+u+\tau)\Big[2\mathcal{K}_{i+1}|\eta|+\mathcal{L}_{i+1}\Big]\Bigg]$$
$$\leq \Bigg(\Xi(\xi,u+\tau) + \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r,u+\tau-\omega_r)+\Xi(b,u+\tau)\Bigg]\Bigg)$$
$$\times\mathcal{K}_1(2+b\mathcal{N})|\eta| + \Bigg(\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(\xi,\phi_i+u+\tau) + \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|$$
$$\times\,\Xi(\delta_r,\phi_i+u+\tau-\omega_r) + \sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b,\phi_i+u+\tau)\Bigg]\Bigg)2\mathcal{K}_{i+1}|\eta| + \Bigg(\Xi(\xi,u+\tau)$$
$$+ \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r,u+\tau-\omega_r)+\Xi(b,u+\tau)\Bigg]\Bigg)\mathcal{L}_1$$
$$+ \Bigg(\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(\xi,\phi_i+u+\tau) + \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}$$
$$\Bigg[\sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|\Xi(\delta_r,\phi_i+u+\tau-\omega_r) + \sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b,\phi_i+u+\tau)\Bigg]\Bigg)\mathcal{L}_{i+1}$$
$$\leq \Big(\mathcal{V}_1\mathcal{K}(2+b\mathcal{N})+2\mathcal{V}_2\mathcal{K}\Big)r + \Big(\mathcal{V}_1+\mathcal{V}_2\Big)\mathcal{L}.$$

Thus, $\|\mathcal{G}\eta\| \leq r$.
This implies $\mathcal{G}\mathcal{Q}_r \subset \mathcal{Q}_r$.

Step 2: To prove $\mathcal{G}$ is a contraction.
For any $\eta_1,\ \eta_2 \in \mathcal{Q}_r$, and for each $\xi \in \mathcal{Z}$, using $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we have

$$|\mathcal{G}\eta_1(\xi)-\mathcal{G}\eta_2(\xi)|$$
$$\leq \Xi(\xi,u+\tau)\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|\Big] + \sum_{i=1}^{m}|\mathcal{A}_i|\Xi(\xi,\phi_i+u+\tau)\Big[2\,\mathcal{K}_{i+1}|\eta_1-\eta_2|\Big]$$
$$+ \frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r,u+\tau-\omega_r)\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|\Big] + \Xi(b,u+\tau)$$

$$\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|\Big]+\sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|\Xi(\delta_r,\phi_i+u+\tau-\omega_r)\Big[2\mathcal{K}_{i+1}|\eta_1-\eta_2|\Big]$$
$$+\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b,\phi_i+u+\tau)\Big[2\mathcal{K}_{i+1}|\eta_1-\eta_2|\Big]\Big]$$
$$\leq\Bigg(\Xi(\xi,u+\tau)+\frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\Xi(\delta_r,u+\tau-\omega_r)+\Xi(b,u+\tau)\Bigg]\Bigg)$$
$$\times\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|+\Bigg(\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(\xi,\phi_i+u+\tau)+\frac{\Xi(\xi,u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r$$
$$\times|\mathcal{A}_i|\Xi(\delta_r,\phi_i+u+\tau-\omega_r)+\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b,\phi_i+u+\tau)\Bigg]\Bigg)2\mathcal{K}_{i+1}|\eta_1-\eta_2|.$$

Thus, $\|\mathcal{G}\eta_1-\mathcal{G}\eta_2\|\leq\mathcal{K}\Big(\mathcal{V}_1(2+b\mathcal{N})+2\mathcal{V}_2\Big)\|\eta_1-\eta_2\|$.
Since $\mathcal{K}\Big(\mathcal{V}_1(2+b\mathcal{N})+2\mathcal{V}_2\Big)<1$, the operator $\mathcal{G}$ is a contraction. Therefore, by Lemma 4 we conclude that $\mathcal{G}$ has a unique fixed point which is the unique solution of (1) on $\mathcal{Z}$. □

Theorem 4 *Assume that* $(\mathbf{H_1})-(\mathbf{H_3})$ *are satisfied. Suppose that* $\big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2-\sum_{i=1}^{m}|\mathcal{A}_i|\,\Xi(\xi,\phi_i+u+\tau)\big]\big)<1$, *where* $\mathcal{K}=\max\{\mathcal{K}_i,\ i=1,2,...,m+1\}$ *and* $\mathcal{N}$ *are constants,* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are given by (9) and (10) respectively. Then, the system (1) has at least one solution on* $\mathcal{Z}$.

Proof Let $\|\sigma\|=\sup_{\xi\in\mathcal{J}}|\sigma(\xi)|,\ \|\nu_i\|=\sup_{\xi\in\mathcal{J}}|\nu_i(\xi)|$.
Define a bounded subset $\mathcal{Q}_\rho=\{\eta\in\mathcal{Q}:|\eta|\leq\ \rho\}$ of $\mathcal{Q}$ with $\rho\geq\mathcal{V}_1\|\sigma\|+\mathcal{V}_2\|\nu\|$.

Let us split the operator $\mathcal{G}$ into $\mathcal{G}_1$ and $\mathcal{G}_2$ which is defined on $\mathcal{Q}_\rho$ for all $\xi\in\mathcal{Z}$, where

$$\mathcal{G}_1\eta(\xi)=I_{a^+}^{u+\tau;\psi}f\Big(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^{\xi}\mathcal{P}(\xi,s,\eta(s))ds\Big)$$
$$+\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big),$$
$$\mathcal{G}_2\eta(\xi)=\frac{\Xi(\xi,u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r}\beta_rI_{a^+}^{u+\tau-\omega_r;\psi}f\Big(\delta_r,\eta(\delta_r),\eta(\lambda\delta_r),\int_0^{\xi}\mathcal{P}(\delta_r,s,\eta(s))\Big)$$
$$-I_{a^+}^{u+\tau;\psi}f\Big(b,\eta(b),\eta(\lambda b),\int_0^{\xi}\mathcal{P}(b,s,\eta(s))\Big)+\sum_{r=1}^{p}\sum_{i=1}^{m}\mathcal{A}_i$$
$$\times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi}g_i\Big(\delta_r,\eta(\delta_r),\eta(\mu\delta_r)\Big)-\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\Big(b,\eta(b),\eta(\mu b)\Big)\Bigg].$$

Step 1: To prove $\mathcal{Q}_\rho$ is bounded.

For any $\eta_1, \eta_2 \in \mathcal{Q}_\rho, \xi \in \mathcal{Z}$, we consider

$$
\begin{aligned}
&|\mathcal{G}_1\eta_1(\xi) + \mathcal{G}_2\eta_2(\xi)| \\
&\quad \leq \left(\Xi(\xi, u+\tau) + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|} \left[\sum_{p=1}^{r} \beta_r \Xi(\delta_r, u+\tau-\omega_r) + \Xi(b, u+\tau) \right] \right) |\sigma| \\
&\quad + \left(\sum_{i=1}^{m} |\mathcal{A}_i| \Xi(\xi, \phi_i + u + \tau) + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|} \right. \\
&\quad \times \left. \left[\sum_{p=1}^{r} \sum_{i=1}^{m} \beta_r |\mathcal{A}_i| \Xi(\delta_r, \phi_i + u + \tau - \omega_r) + \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(b, \phi_i + u + \tau) \right] \right) |\nu_i|.
\end{aligned}
$$

Thus, $\|\mathcal{G}_1\eta_1 + \mathcal{G}_2\eta_2\| \leq \mathcal{V}_1\|\sigma\| + \mathcal{V}_2\|\nu_i\| \leq \rho$.
$\Longrightarrow \mathcal{Q}_\rho$ is bounded.

Step 2: To prove $\mathcal{G}_1$ is completely continuous.
ie. to show that $\mathcal{G}_1$ is continuous and compact on $\mathcal{Q}_\rho$.
Let η_n be a sequence and $\eta_n \longrightarrow \eta$ as $n \longrightarrow \infty$ in $\mathcal{Q}_\rho$. Then for $\xi \in \mathcal{Z}$, we have

$$
\begin{aligned}
&|\mathcal{G}_1\eta_n(\xi) - \mathcal{G}_1\eta(\xi)| \\
&\quad \leq (2 + b\mathcal{N}\mathcal{K}\Xi(b, u+\tau))|\eta_n - \eta| + 2\mathcal{K} \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(b, \phi_i + u + \tau)|\eta_n - \eta| \\
&\quad \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Thus, $\|\mathcal{G}_1\eta_n(\xi) - \mathcal{G}_1\eta(\xi)\| \longrightarrow 0$ as $n \longrightarrow \infty$ which implies $\mathcal{G}_1$ is continuous.
Now, consider

$$
\begin{aligned}
&|\mathcal{G}_1\eta(\xi_2) - \mathcal{G}_1\eta(\xi_1)| \\
&\quad = \Bigg| \int_{\xi_1}^{\xi_2} \left(\frac{1}{\Gamma(u+\tau)} \psi'(v)(\psi(\xi_2) - \psi(v))^{u+\tau-1} f\left(v, \eta(v), \eta(\lambda v), \int_0^v \mathcal{P}(v, s, \eta(s))\right) \right. \\
&\quad - \left. \frac{\sum_{i=1}^{m} |\mathcal{A}_i|}{\Gamma(\phi_i + u + \tau)} \psi'(v)(\psi(\xi_2) - \psi(v))^{\phi_i + u + \tau - 1} g_i\left(v, \eta(v), \eta(\mu v)\right) dv \right. \\
&\quad + \int_a^{\xi_1} \left(\frac{1}{\Gamma(u+\tau)} \psi'(v) \left[(\psi(\xi_2) - \psi(v))^{u+\tau-1} - (\psi(\xi_1) - \psi(v))^{u+\tau-1} \right] \right. \\
&\quad \times f\left(v, \eta(v), \eta(\lambda v), \int_0^v \mathcal{P}(v, s, \eta(s))\right) - \frac{\sum_{i=1}^{m} |\mathcal{A}_i|}{\Gamma(\phi_i + u + \tau)} \psi'(v) \left[(\psi(\xi_2) - \psi(v))^{\phi_i + u + \tau - 1} \right. \\
&\quad - \left. (\psi(\xi_1) - \psi(v))^{\phi_i + u + \tau - 1} \right] g_i\left(v, \eta(v), \eta(\mu v)\right) dv \Bigg| \\
&\quad \leq \frac{1}{\Gamma(u+\tau+1)} \left[2(\psi(\xi_2) - \psi(\xi_1))^{u+\tau} + \psi(\xi_2) - \psi(a))^{u+\tau} - \psi(\xi_1) - \psi(a))^{u+\tau} \right] \\
&\quad \times \left| f\left(v, \eta(v), \eta(\lambda v), \int_0^v \mathcal{P}(v, s, \eta(s))\right) \right| - \frac{1}{\Gamma(\phi_i + u + \tau + 1)} \left[2(\psi(\xi_2) - \psi(\xi_1))^{\phi_i + u + \tau} \right. \\
&\quad + \left. \psi(\xi_2) - \psi(a))^{\phi_i + u + \tau} - \psi(\xi_1) - \psi(a))^{\phi_i + u + \tau} \right] \left| g_i\left(v, \eta(v), \eta(\mu v)\right) \right| \\
&\quad \longrightarrow 0 \text{ as } \xi_2 \longrightarrow \xi_1.
\end{aligned}
$$

Thus, $\|\mathcal{G}_1\eta(\xi_2) - \mathcal{G}_1\eta(\xi_1)\| \longrightarrow 0$ as $\xi_2 \longrightarrow \xi_1$.
i.e $\mathcal{G}_1\mathcal{Q}_\rho$ is equicontinuous.
Hence $\mathcal{G}_1$ is completely continuous on $\mathcal{Q}_\rho$, according to the Arzela-Ascoli theorem [12].

Step 3: To prove $\mathcal{G}_2$ is a contraction.
For any $\eta_1, \eta_2 \in \mathcal{Q}_\rho$, and for each $\xi \in \mathcal{Z}$, using $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we have

$$\begin{aligned}
&\|\mathcal{G}_2\eta_1(\xi) - \mathcal{G}_2\eta_2(\xi)\| \\
&\quad\leq \frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r, u+\tau-\omega_r)\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|\Big] + \Xi(b, u+\tau) \\
&\quad\Big[\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2|\Big] + \sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|\Xi(\delta_r, \phi_i+u+\tau-\omega_r)\Big[2\mathcal{K}_{i+1}|\eta_1-\eta_2|\Big] \\
&\quad+\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b, \phi_i+u+\tau)\Big[2\mathcal{K}_{i+1}|\eta_1-\eta_2|\Big]\Bigg] \\
&\quad\leq \Bigg(\frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\beta_r\,\Xi(\delta_r, u+\tau-\omega_r) + \Xi(b, u+\tau)\Bigg]\Bigg)\mathcal{K}_1(2+b\mathcal{N})|\eta_1-\eta_2| \\
&\quad+\Bigg(\frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r}\sum_{i=1}^{m}\beta_r|\mathcal{A}_i|\Xi(\delta_r, \phi_i+u+\tau-\omega_r) \\
&\quad+\sum_{i=1}^{m}|\mathcal{A}_i|\Xi(b, \phi_i+u+\tau)\Bigg]\Bigg)2\mathcal{K}_{i+1}|\eta_1-\eta_2|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathcal{G}_2\eta_1 - \mathcal{G}_2\eta_2\| \leq \Big(&\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\big] + 2\mathcal{K}\big[\mathcal{V}_2 \\
&-\sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi, \phi_i+u+\tau)\big]\Big)\|\eta_1-\eta_2\|.
\end{aligned}$$

Since $\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\big] + 2\mathcal{K}\big[\mathcal{V}_2 - \sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi, \phi_i+u+\tau)\big]\Big) < 1$, the operator $\mathcal{G}_2$ is a contraction. Therefore, by Theorem 2, we conclude that the BVP (1) has at least one solution on $\mathcal{Z}$. □

5 Stability results

We prove the following lemma which is a prerequisite for the proof of $\mathcal{UH}$ stability.

Lemma 6 *Let $u, \tau \in (0, 1)$, $\varsigma \in [0, 1]$. If $z \in \mathcal{Q}$ is a solution of the inequality (2), then z is a solution of the following inequality*

$$\begin{aligned}
&\Bigg|z(\xi) - R_z - I_{a^+}^{u+\tau;\psi} f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^{\xi}\mathcal{P}(\xi, s, z(s))ds\Big) \\
&\quad-\sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big)\Bigg| \leq (\mathcal{V}_1+\mathcal{V}_2)\,\kappa,
\end{aligned}\tag{11}$$

where

$$
\begin{aligned}
R_z =& \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r}\beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, z(\delta_r), z(\lambda\delta_r), \int_0^{\xi}\mathcal{P}(\delta_r, s, z(s))\Big)\\
&- I_{a^+}^{u+\tau;\psi} f\Big(b, z(b), z(\lambda b), \int_0^{\xi}\mathcal{P}(b, s, z(s))\Big) + \sum_{r=1}^{p}\sum_{i=1}^{m}\mathcal{A}_i\\
&\times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} g_i\Big(\delta_r, z(\delta_r), z(\mu\delta_r)\Big) - \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(b, z(b), z(\mu b)\Big)\Bigg].
\end{aligned}
$$

Proof Let z be a solution of the inequality (2). Using Lemma 5, we obtain that the solution of the system

$$
\begin{cases}
D_{a^+}^{u;\psi}\Big({}^{H}D_{a^+}^{\tau,\varsigma;\psi} z(\xi)\Big) = f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^{\xi}\mathcal{P}(\xi, s, z(s))\Big)\\
\qquad\qquad + \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i;\psi} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big),\ \xi \in \mathcal{Z},\\
z(a) = 0, \qquad z(b) = \sum_{r=1}^{p}\beta_r\, {}^{H}D_{a^+}^{\omega_r,\varsigma:\psi} z(\delta_r), \quad \beta_r \in \mathfrak{R}^+,\ \delta_r \in \mathcal{Z},
\end{cases}
\tag{12}
$$

is of the form

$$
\begin{aligned}
z(\xi) =& I_{a^+}^{u+\tau;\psi} f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^{\xi}\mathcal{P}(\xi, s, z(s))\Big) + \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big)\\
&+ \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r}\beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, z(\delta_r), z(\lambda\delta_r), \int_0^{\xi}\mathcal{P}(\delta_r, s, z(s))\Big)\\
&- I_{a^+}^{u+\tau;\psi} f\Big(b, z(b), z(\lambda b), \int_0^{\xi}\mathcal{P}(b, s, z(s))\Big) + \sum_{r=1}^{p}\sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi}\\
&\times g_i\Big(\delta_r, z(\delta_r), z(\mu\delta_r)\Big) - \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(b, z(b), z(\mu b)\Big)\Bigg] + I_{a^+}^{u+\tau;\psi} w(\xi)\\
&+ \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} w(\xi) + \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r}\beta_r I_{a^+}^{u+\tau-\omega_r;\psi} w(\delta_r)\\
&- I_{a^+}^{u+\tau;\psi} w(b) + \sum_{r=1}^{p}\sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} w(\delta_r) - \sum_{i=1}^{m}\mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} w(b)\Bigg].
\end{aligned}
\tag{13}
$$

Now, using Remark 1, it follows that

$$\begin{aligned}
&\Bigg|z(\xi) - R_z - I_{a^+}^{u+\tau;\psi} f\Big(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\Big) \\
&\quad - \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, z(\xi), z(\mu\xi)\Big)\Bigg| \\
&\quad = \Bigg| I_{a^+}^{u+\tau;\psi} w(\xi) + \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} w(\xi) + \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r} \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} w(\delta_r) \\
&\quad - I_{a^+}^{u+\tau;\psi} w(b) + \sum_{r=1}^{p}\sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} w(\delta_r) - \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} w(b)\Bigg]\Bigg| \\
&\quad \le \Bigg(\Xi(\xi, u+\tau) + \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(\xi, \phi_i + u + \tau) + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\Bigg[\sum_{p=1}^{r} \beta_r \Xi(\delta_r, u+\tau-\omega_r) \\
&\quad + \Xi(b, u+\tau) + \sum_{p=1}^{r}\sum_{i=1}^{m} \beta_r |\mathcal{A}_i| \Xi(\delta_r, \phi_i + u + \tau - \omega_r) + \sum_{i=1}^{m} |\mathcal{A}_i| \Xi(b, \phi_i + u + \tau)\Bigg]\Bigg)\kappa \\
&\quad \le (\mathcal{V}_1 + \mathcal{V}_2)\,\kappa.
\end{aligned}$$

Thus, (11) is obtained. □

Theorem 5 *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *holds with* $\big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\big] + 2\mathcal{K}\big[\mathcal{V}_2 - \sum_{i=1}^{m} |\mathcal{A}_i|\ \Xi(\xi, \phi_i + u + \tau)\big]\big) < 1$. *Then the system (1) is* $\mathcal{UH}$ *stable on* $\mathcal{Z}$.

Proof Let $\kappa > 0$ and $z \in \mathcal{Q}$ be any solution of the inequality (2). Let $\eta \in \mathcal{Q}$ be the unique solution of (1). Using Lemma 5, we obtain

$$\begin{aligned}
\eta(\xi) =& R_\eta + I_{a^+}^{u+\tau;\psi} f\Big(\xi, \eta(\xi), \eta(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, \eta(s))ds\Big) \\
&+ \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi, \eta(\xi), \eta(\mu\xi)\Big),
\end{aligned}$$

where

$$\begin{aligned}
R_\eta =& \frac{\Xi(\xi, u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r} \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\Big(\delta_r, \eta(\delta_r), \eta(\lambda\delta_r), \int_0^\xi \mathcal{P}(\delta_r, s, \eta(s))\Big) \\
&- I_{a^+}^{u+\tau;\psi} f\Big(b, \eta(b), \eta(\lambda b), \int_0^\xi \mathcal{P}(b, s, \eta(s))\Big) + \sum_{r=1}^{p}\sum_{i=1}^{m} \mathcal{A}_i \\
&\times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} g_i\Big(\delta_r, \eta(\delta_r), \eta(\mu\delta_r)\Big) - \sum_{i=1}^{m} \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(b, \eta(b), \eta(\mu b)\Big)\Bigg].
\end{aligned}$$

On the otherhand, if $\eta(a) = z(a),\ {}^{H}D_{a^+}^{\omega_r,\varsigma:\psi}\eta(\delta_r) = {}^{H}D_{a^+}^{\omega_r,\varsigma:\psi} z(\delta_r),\ \eta(b) = z(b)$, then

$$\big|R_\eta - R_z\big| = 0 \text{ which implies } R_\eta = R_z.$$

Now, by applying triangle inequality and Lemma 6, for any $\xi \in \mathcal{Z}$, we have

$$
\begin{aligned}
&|z(\xi)-\eta(\xi)|\\
&\leq \Big|z(\xi)-R_\eta - I_{a^+}^{u+\tau;\psi} f\Big(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^\xi \mathcal{P}(\xi,s,\eta(s))ds\Big)\\
&\quad -\sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big)\Big|\\
&\leq \Big|z(\xi)-R_z - I_{a^+}^{u+\tau;\psi} f\Big(\xi,z(\xi),z(\lambda\xi),\int_0^\xi \mathcal{P}(\xi,s,z(s))\Big)\\
&\quad -\sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\Big(\xi,z(\xi),z(\mu\xi)\Big)\Big| + I_{a^+}^{u+\tau;\psi}\Big|f\Big(\xi,z(\xi),z(\lambda\xi),\int_0^\xi \mathcal{P}(\xi,s,z(s))\Big)\\
&\quad - f\Big(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^\xi \mathcal{P}(\xi,s,\eta(s))ds\Big)\Big| + \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi}\Big|g_i\Big(\xi,z(\xi),z(\mu\xi)\Big)\\
&\quad - g_i\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big)\Big| + \big|R_z-R_\eta\big|\\
&\leq (\mathcal{V}_1+\mathcal{V}_2)\kappa + \Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2\\
&\quad -\sum_{i=1}^m |\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\big]\Big)|z(\xi)-\eta(\xi)|
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|z(\xi)-\eta(\xi)|\\
&\quad\leq \frac{\mathcal{V}_1+\mathcal{V}_2}{1-\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2-\sum_{i=1}^m|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\big]\Big)}\,\kappa
\end{aligned}
$$

By setting

$$
\mathcal{M}_{f,g_i}=\frac{\mathcal{V}_1+\mathcal{V}_2}{1-\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2-\sum_{i=1}^m|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\big]\Big)},
$$

we obtain

$$
|z(\xi)-\eta(\xi)|\leq \mathcal{M}_{f,g_i}\ \kappa.
$$

Thus, the system (1) is $\mathcal{UH}$ stable on $\mathcal{Z}$. □

Next, we prove a lemma which is a prerequisite for the proof of $\mathcal{UHR}$ stability. Consider the following:
(**H₄**) Let $\Theta\in C(\mathcal{Z},\mathfrak{R}^+)$ be an increasing function, and there exists $n_\Theta>0$ such that for any $\xi\in\mathcal{Z}$,

$$
I^{u+\tau;\psi}\ \Theta(\xi)\leq n_\Theta\ \Theta(\xi).
$$

Lemma 7 *Let $u, \tau \in (0,1)$, $\varsigma \in [0,1]$. If $z \in \mathcal{Q}$ is a solution of the inequality (3), then z is a solution of the following inequality*

$$\begin{aligned} &\left| z(\xi) - R_z - I_{a^+}^{u+\tau;\psi} f\left(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\right) \right. \\ &\left. - \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\left(\xi, z(\xi), z(\mu\xi)\right) \right| \le \mathcal{V}_3\, \kappa\, n_\Theta\, \Theta(\xi). \end{aligned} \tag{14}$$

Proof Let z be a solution of the inequality (3). Using Lemma 5, we obtain that the solution of the system (12) is of the form

$$\begin{aligned} z(\xi) =& I_{a^+}^{u+\tau;\psi} f\left(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))\right) + \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\left(\xi, z(\xi), z(\mu\xi)\right) \\ &+ \frac{\Xi(\xi, u+\tau-1)}{\Delta}\left[\sum_{p=1}^r \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} f\left(\delta_r, z(\delta_r), z(\lambda\delta_r), \int_0^\xi \mathcal{P}(\delta_r, s, z(s))\right)\right. \\ &- I_{a^+}^{u+\tau;\psi} f\left(b, z(b), z(\lambda b), \int_0^\xi \mathcal{P}(b, s, z(s))\right) + \sum_{r=1}^p \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} \\ &\times g_i\left(\delta_r, z(\delta_r), z(\mu\delta_r)\right) - \left.\sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\left(b, z(b), z(\mu b)\right)\right] + I_{a^+}^{u+\tau;\psi} v(\xi) \\ &+ \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} v(\xi) + \frac{\Xi(\xi, u+\tau-1)}{\Delta}\left[\sum_{p=1}^r \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} v(\delta_r)\right. \\ &- \left. I_{a^+}^{u+\tau;\psi} v(b) + \sum_{r=1}^p \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} v(\delta_r) - \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} v(b)\right]. \end{aligned} \tag{15}$$

Now, using Remark 2 and $(\mathbf{H_4})$, it follows that

$$\begin{aligned} &\left| z(\xi) - R_z - I_{a^+}^{u+\tau;\psi} f\left(\xi, z(\xi), z(\lambda\xi), \int_0^\xi \mathcal{P}(\xi, s, z(s))ds\right) \right. \\ &\quad \left. - \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} g_i\left(\xi, z(\xi), z(\mu\xi)\right) \right| \\ &= \left| I_{a^+}^{u+\tau;\psi} v(\xi) + \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} v(\xi) + \frac{\Xi(\xi, u+\tau-1)}{\Delta}\left[\sum_{p=1}^r \beta_r I_{a^+}^{u+\tau-\omega_r;\psi} v(\delta_r)\right.\right. \\ &\quad \left.\left. - I_{a^+}^{u+\tau;\psi} v(b) + \sum_{r=1}^p \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi} v(\delta_r) - \sum_{i=1}^m \mathcal{A}_i I_{a^+}^{\phi_i+u+\tau;\psi} v(b)\right]\right| \\ &\le \left(1 + \sum_{i=1}^m |\mathcal{A}_i| + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\left[\sum_{p=1}^r \beta_r + 1 + \sum_{p=1}^r \sum_{i=1}^m \beta_r |\mathcal{A}_i| + \sum_{i=1}^m |\mathcal{A}_i|\right]\right)\kappa\, n_\Theta \Theta(\xi) \\ &\le \left[1 + \sum_{i=1}^m |\mathcal{A}_i| + \frac{\Xi(\xi, u+\tau-1)}{|\Delta|}\left(\sum_{p=1}^r \beta_r + 1\right)\left(|\mathcal{A}_i| + 1\right)\right]\kappa\, n_\Theta \Theta(\xi) \\ &\le \mathcal{V}_3\, \kappa\, n_\Theta\, \Theta(\xi). \end{aligned}$$

Thus, (14) is obtained. □

Theorem 6 *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *holds with* $\left(\mathcal{K}(2+b\mathcal{N})\left[\mathcal{V}_1-\Xi(\xi,u+\tau)\right]+2\mathcal{K}\left[\mathcal{V}_2-\sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\right]\right)<1$. *Then the system (1) is* $\mathcal{UHR}$ *stable on* $\mathcal{Z}$.

Proof Let $\kappa>0$ and $z\in\mathcal{Q}$ be any solution of the inequality (3). Let $\eta\in\mathcal{Q}$ be the unique solution of (1). Using Lemma 5, we obtain

$$\begin{aligned}\eta(\xi)=&R_\eta+I_{a^+}^{u+\tau;\psi}f\left(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^\xi\mathcal{P}(\xi,s,\eta(s))ds\right)\\&+\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\left(\xi,\eta(\xi),\eta(\mu\xi)\right)\end{aligned}$$

where

$$\begin{aligned}R_\eta=&\frac{\Xi(\xi,u+\tau-1)}{\Delta}\Bigg[\sum_{p=1}^{r}\beta_rI_{a^+}^{u+\tau-\omega_r;\psi}f\left(\delta_r,\eta(\delta_r),\eta(\lambda\delta_r),\int_0^\xi\mathcal{P}(\delta_r,s,\eta(s))\right)\\&-I_{a^+}^{u+\tau;\psi}f\left(b,\eta(b),\eta(\lambda b),\int_0^\xi\mathcal{P}(b,s,\eta(s))\right)+\sum_{r=1}^{p}\sum_{i=1}^{m}\mathcal{A}_i\\&\times I_{a^+}^{\phi_i+u+\tau-\omega_r;\psi}g_i\left(\delta_r,\eta(\delta_r),\eta(\mu\delta_r)\right)-\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\left(b,\eta(b),\eta(\mu b)\right)\Bigg].\end{aligned}$$

On the otherhand, if $\eta(a)=z(a),\ {}^HD_{a^+}^{\omega_r,\varsigma:\psi}\eta(\delta_r)={}^HD_{a^+}^{\omega_r,\varsigma:\psi}z(\delta_r),\ \eta(b)=z(b)$, then

$$\left|R_\eta-R_z\right|=0\text{ which implies }R_\eta=R_z.$$

Now, by applying triangle inequality and Lemma 6, for any $\xi\in\mathcal{Z}$, we have

$$\begin{aligned}&|z(\xi)-\eta(\xi)|\\&\le\Bigg|z(\xi)-R_\eta-I_{a^+}^{u+\tau;\psi}f\left(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^\xi\mathcal{P}(\xi,s,\eta(s))ds\right)\\&\quad-\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\left(\xi,\eta(\xi),\eta(\mu\xi)\right)\Bigg|\\&\le\Bigg|z(\xi)-R_z-I_{a^+}^{u+\tau;\psi}f\left(\xi,z(\xi),z(\lambda\xi),\int_0^\xi\mathcal{P}(\xi,s,z(s))\right)\\&\quad-\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}g_i\left(\xi,z(\xi),z(\mu\xi)\right)\Bigg|+I_{a^+}^{u+\tau;\psi}\Bigg|f\left(\xi,z(\xi),z(\lambda\xi),\int_0^\xi\mathcal{P}(\xi,s,z(s))\right)\\&\quad-f\left(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^\xi\mathcal{P}(\xi,s,\eta(s))ds\right)\Bigg|+\sum_{i=1}^{m}\mathcal{A}_iI_{a^+}^{\phi_i+u+\tau;\psi}\Big|g_i\left(\xi,z(\xi),z(\mu\xi)\right)\\&\quad-g_i\left(\xi,\eta(\xi),\eta(\mu\xi)\right)\Big|+\left|R_z-R_\eta\right|\\&\le\left(\mathcal{K}(2+b\mathcal{N})\left[\mathcal{V}_1-\Xi(\xi,u+\tau)\right]+2\mathcal{K}\left[\mathcal{V}_2-\sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\right]\right)|z(\xi)-\eta(\xi)|\\&\quad+\mathcal{V}_3\,\kappa\,n_\Theta\,\Theta(\xi)\end{aligned}$$

This implies

$$|z(\xi)-\eta(\xi)| \le \frac{\mathcal{V}_3\, n_\Theta}{1-\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2-\sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\big]\Big)}\,\kappa\ \Theta(\xi)$$

By setting

$$\mathcal{M}_{f,g_i,\Theta}=\frac{\mathcal{V}_3\, n_\Theta}{1-\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1-\Xi(\xi,u+\tau)\big]+2\mathcal{K}\big[\mathcal{V}_2-\sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi,\phi_i+u+\tau)\big]\Big)},$$

we obtain

$$|z(\xi)-\eta(\xi)| \le \mathcal{M}_{f,g_i,\Theta}\ \kappa\ \Theta(\xi).$$

Thus, the system (1) is $\mathcal{UHR}$ stable on $\mathcal{Z}$. □

6 Application

This section contains an example which demonstrates the significance and reliability of our findings.

Example 1 Consider the mixed sequential pantograph FIDE with non-local boundary condition

$$\begin{cases} D_{0^+}^{\frac{3}{4}:e^{\frac{\xi}{3}}}\Big({}^{H}D_{0^+}^{\frac{1}{2},\frac{1}{4}:e^{\frac{\xi}{3}}}\ \eta(\xi)\Big)=f\Big(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^{\xi}\mathcal{P}(\xi,s,\eta(s))ds\Big) \\ \qquad\qquad +\sum_{i=1}^{2}\frac{i}{5(i+1)}I_{0^+}^{\frac{2i}{5};\psi}\,g_i\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big),\ \xi\in[0,1] \\ \eta(0)=0,\qquad \eta(1)=\sum_{r=1}^{3}\frac{r^2}{r+7}\,{}^{H}D_{0^+}^{\frac{r+1}{6},\frac{1}{2}:e^{\frac{\xi}{3}}}\ \eta\Big(\frac{r}{5}\Big), \end{cases} \tag{16}$$

where

$$f\Big(\xi,\eta(\xi),\eta(\lambda\xi),\int_0^{\xi}\mathcal{P}(\xi,s,\eta(s))ds\Big)=\frac{(\xi^2+1)\sin|\eta(\xi)|}{14}+\frac{e^{-\xi}\ |\eta(\frac{1}{4}\xi)|}{2\xi+7}+\int_0^{\xi}e^{\frac{-1}{2}\eta(s)ds},$$

$$g_1\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big)=\frac{\sqrt{3\xi+6}\ \cos|\eta(\xi)|}{e^{\xi}+15}+\frac{1}{\xi+1}\ \frac{|\eta(\frac{2}{3}\xi)|}{5+|\eta(\frac{2}{3}\xi)|},$$

$$g_2\Big(\xi,\eta(\xi),\eta(\mu\xi)\Big)=\frac{2-\sin^2\pi\xi}{\xi^5+9}\ |\eta(\xi)|+\frac{\xi^2+1}{18}\ \Big|\eta\Big(\frac{2}{3}\xi\Big)\Big|.$$

Comparing the system with BVP (1) we observe that

$$u=\frac{3}{4},\ \tau=\frac{5}{9},\ \varsigma=\frac{1}{2},\ a=0,\ b=1,\ i=2,\ r=3,\ \lambda=\frac{1}{4},\ \mu=\frac{2}{3},$$

$$\mathcal{A}_1=\frac{1}{10},\ \mathcal{A}_2=\frac{2}{15},\ \phi_i=\frac{2i}{5},\ \beta_r=\frac{r^2}{r+7},\ \omega_r=\frac{r+1}{6}\ \delta_r=\frac{r}{5},\ \psi(\xi)=e^{\frac{\xi}{3}}.$$

Applying these we get,

$$\Delta \approx 0.9666 \neq 0,\ \mathcal{V}_1 \approx 1.5197,\ \ \mathcal{V}_2 \approx 0.2199,$$
$$\Xi(1, u+\tau) \approx 0.2769 \text{ and } \sum_{i=1}^{m} |\mathcal{A}_i|\ \Xi(\xi, \phi_i + u + \tau) \approx 0.0241.$$

Let $\mathcal{B}\eta(\xi) = \int_0^{\xi} e^{\frac{-1}{2}\eta(s)} ds.$

(i) Uniquessness of Solution
For all $\xi \in [0, 1]$ and $\eta_1, \eta_2 \in \mathfrak{R}$,

$$|\mathcal{B}\eta_1(\xi) - \mathcal{B}\eta_2(\xi)| \leq \frac{1}{2}\big(|\eta_1 - \eta_2|\big),$$
$$|f\big(\xi, \eta_1(\xi), \eta(\lambda\xi), \mathcal{B}\eta_1(\xi)\big) - f\big(\xi, \eta_2(\xi), \eta(\lambda\xi), \mathcal{B}\eta_2(\xi)\big)| \leq \frac{1}{7}\big(|\eta_1 - \eta_2| + \frac{1}{2}|\eta_1 - \eta_2|\big)$$
$$\leq \frac{1}{7}\big(|\eta_1 - \eta_2|\big),$$
$$|g_1(\xi, \eta_1(\xi), \eta_1(\mu\xi)) - g_1(\xi, \eta_2(\xi), \eta_2(\mu\xi))| \leq \frac{1}{5}\big(|\eta_1 - \eta_2|\big),$$
$$|g_2(\xi, \eta_1(\xi), \eta_1(\mu\xi)) - g_2(\xi, \eta_2(\xi), \eta_2(\mu\xi))| \leq \frac{1}{9}\big(|\eta_1 - \eta_2|\big).$$

$\Longrightarrow \mathcal{K}_1 = \frac{1}{7},\ \mathcal{K}_2 = \frac{1}{5},\ \ \mathcal{K}_3 = \frac{1}{9},\ \mathcal{N} = \frac{1}{5}.$
Therefore, $\mathcal{K}\Big(\mathcal{V}_1(2 + b\mathcal{N}) + 2\mathcal{V}_2\Big) \approx 0.8478 < 1.$
Thus, the hypothesis of Theorem 3 is satisfied and hence the system (16) has a unique solution on [0, 1].

(ii) Existence of solution
For all $\xi \in [0, 1]$ and $\eta_1, \eta_2 \in \mathfrak{R}$,

$$|f\big(\xi, \eta(\xi), \eta(\lambda\xi), \mathcal{B}\eta(\xi)\big)| \leq \frac{(\xi^2+1)}{14} + \frac{|e^{-\xi}|}{2\xi + 7} + \frac{1}{2},$$
$$|g_1(\xi, \eta(\xi), \eta(\mu\xi))| \leq \frac{\sqrt{3\xi + 6}}{e^{\xi} + 15} + \frac{1}{5(\xi + 1)},$$
$$|g_2(\xi, \eta(\xi), \eta(\mu\xi))| \leq \frac{1}{\xi^5 + 9} + \frac{\xi^2 + 1}{18}.$$

Therefore,

$$\left(\mathcal{K}\Big(2 + b\mathcal{N}\Big)\Big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\Big] + 2\mathcal{K}\Big[\mathcal{V}_2 - \sum_{i=1}^{m} |\mathcal{A}_i|\ \Xi(\xi, \phi_i + u + \tau)\Big]\right) \approx 0.6997 < 1.$$

Thus, the hypothesis of Theorem 4 is satisfied and hence the system (16) has at least one solution on [0, 1].

(iii) Stability
We compute that

$$\mathcal{M}_{f,g_i} = \frac{\mathcal{V}_1 + \mathcal{V}_2}{1 - \Big(\mathcal{K}(2 + b\mathcal{N})\big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\big] + 2\mathcal{K}\big[\mathcal{V}_2 - \sum_{i=1}^{m} |\mathcal{A}_i|\ \Xi(\xi, \phi_i + u + \tau)\big]\Big)}$$
$$\approx 5.7927 > 0.$$

Therefore, by Theorem 5, the system (16) is $\mathcal{UH}$ stable.
In addition, let $\Theta(\xi) = \psi(\xi) - \psi(0)$.
Using Proposition 1 we calculate that

$$I^{u+\tau;\psi}\ \Theta(\xi) \leq \frac{64(e^{\frac{\xi}{3}}-1)^{\frac{5}{4}}}{45\ \Gamma(\frac{1}{4})}\Theta(\xi).$$

Thus, using ($\mathbf{H_4}$) we observe that $n_\Theta = \frac{64(e^{\frac{\xi}{3}}-1)^{\frac{5}{4}}}{45\ \Gamma(\frac{1}{4})}\Theta(\xi) = 0.1231 > 0$.
It follows that

$$\mathcal{M}_{f,g_i,\Theta} = \frac{\mathcal{V}_3\ n_\Theta}{1-\Big(\mathcal{K}\big(2+b\mathcal{N}\big)\big[\mathcal{V}_1 - \Xi(\xi, u+\tau)\big] + 2\mathcal{K}\big[\mathcal{V}_2 - \sum_{i=1}^{m}|\mathcal{A}_i|\ \Xi(\xi, \phi_i + u + \tau)\big]\Big)}$$
$$\approx 1.6353 > 0.$$

Therefore, by Theorem 6, the system (16) is $\mathcal{UHR}$ stable.

7 Conclusion

In this paper, we have considered a new class of mixed sequential pantograph fractional integro-differential equations involving the ψ-R-L and the ψ-Hilfer fractional derivatives with non-local boundary conditions. The existence and uniqueness results are acquired by the Krasnosel'skii's fixed point theorem and Banach contraction principle, respectively. Additionally, the system is subjected to a stability analysis, followed by an illustration, to validate our findings.

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