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HYERS-ULAM STABILITY OF FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS WITH RANDOM IMPULSE

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Abstract. The goal of this study is to derive a class of random impulsive fractional stochastic differential equations with finite delay that are of Caputo-type. Through certain constraints, the existence of the mild solution of the aforementioned system are acquired by Kransnoselskii's fixed point theorem. Furthermore, through Ito isometry and Gronwall's inequality, the Hyers-Ulam stability of the reckoned system is evaluated using Lipschitz condition.

1. Introduction

Fractional differential equations (FDE) have replaced integer-order differential equations as a popular technique for analysing problems in modern science and technology, and also in the fields of economy and insurance see [**1, 12, 14, 15, 24**]. Ahmadova and Mahmudov [**2**] studied the wellposedness results of caputo-type fractional stochastic neutral Differential Equation systems.

Notably, stochastic disturbances are certain in practical systems due to its influence in the stability of systems. In [**25**], $\mathrm{d}\mathfrak{z}(t) = k\mathfrak{z}(t)$ is unstable when $k > 0$, but there is an increase in the stochastic feedback control $r\mathfrak{z}(t)\mathrm{d}\mathfrak{w}(t)$ to become $\mathrm{d}\mathfrak{z}(t) = k\mathfrak{z}(t) + r\mathfrak{z}(t)\mathrm{d}\mathfrak{w}(t)$, being stable if and only if $r^2 > 2k$. The above notion clearly implies a certain stochastic control term stabilizing the unstable system. It is notable and demanding to investigate stochastic stabilization of the deterministic system [**17, 26, 27**]. The existence and uniqueness of solutions have made instantaneous transformation in applied mathematics. S. Wu. and B. Zhou. [**23**] established existence and uniqueness of stochastic differential equation(SDE) with random impulse and markovian switching under Non-Lipschitz condition, the reader may also refer the monographs [**4, 5, 6, 7, 16, 18, 20**].

One among the indispensable speculation of dynamical systems in the stability concepts are taken into notice in research fields through applications. In particular, in 1940 Ulam [**19**] posted an open question for which Hyers [**10**] answered in the following year. Then, Ulam-Hyers stability was established. The evolution of the theory paved way for the creative research in stability analysis refer [**3, 8, 9,**

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11, 22]. Recently, Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delay were established by Li et al. [**13**]. Also, Anguraj et al. [**6**] investigated the existence and Hyers Ulam results of random impulsive stochastic functional integrodifferential equations.

This study, which is motivated by previous works, focuses on the existence and Hyers-Ulam stability of random impulsive fractional stochastic functional differential equations with finite delays.

Let us consider a random impulsive stochastic fractional differential equations of the form:

$$
(1)\qquad
\begin{aligned}
{}^{c}\mathcal{D}_t^{\beta}\vartheta(t) &= \mathfrak{u}(t),\vartheta_t) + \mathfrak{v}(t,\vartheta_t)\frac{\mathrm{d}\mathfrak{w}(t)}{\mathrm{d}t}, && t \neq \zeta_k,\ t \geq 0,\\
\vartheta(\zeta_k^-) &= \mathfrak{b}_k(\delta_k)\vartheta(\zeta_k^-), && k = 1,2,\ldots,\\
\vartheta(t_0) &= \zeta,
\end{aligned}
$$

where the Caputo fractional ${}^{c}\mathcal{D}_t^{\beta}$ is of order $\beta \in (0,1)$ [**13**]. A random variable δ_k is described from $\mathfrak{w}$ to $\mathcal{D}_k =^{def} (0,\mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1,2,\ldots$. Assume δ_i, δ_j to be unrestrained for $i \neq j$ as $i,j = 1,2,\ldots$. Suppose $\mathsf{T} \in (t_0,+\infty)$, $\mathfrak{u}\colon [t_0,\mathsf{T}] \times \mathbf{C} \to \mathbf{R}^{\mathfrak{d}}$, $\mathfrak{v}\colon [t_0,\mathsf{T}] \times \mathbf{C} \to \mathbf{R}^{\mathfrak{d}\times\mathfrak{m}}$, and $\mathfrak{b}_k\colon \mathcal{D}_k \to \mathbf{R}^{\mathfrak{d}\times\mathfrak{d}}$, and ϑ_t is $\mathbf{R}^{\mathfrak{d}}$-valued stochastic process $\ni \vartheta_t \in \mathbf{R}^{\mathfrak{d}}$, $\vartheta_t = \{\vartheta(t+\theta) : -\delta \leq \theta \leq 0\}$. ζ_k is the impulsive moment from a strictly increasing sequence, i.e., $\zeta_0 < \zeta_1 < \cdots < \zeta_k < \cdots < \lim\limits_{k\to\infty} \zeta_{ki} = \infty$, and $\vartheta(\zeta_k^-) = \lim_{t\to\zeta_k-0}\vartheta(t)$. Assume $\zeta_0 = t_0$ and $\zeta_k = \zeta_{k-1} + \delta_k$ as $k = 1,2,\ldots$. Evidently, $\{\zeta_k\}$ is a process with independent increments. Let $\{\mathtt{N}(t), t \geq 0\}$ be a simple counting process generated by $\{\zeta_k\}$ and $\{w(t) : t \geq 0\}$ be a given m-dimensional Wiener process. Denote $\Im_t^{(1)}$ to be the σ-algebra generated through $\{\mathtt{N}(t), t \geq 0\}$ and $\Im_t^{(2)}$ be to the σ-algebra generated by $\{w(s), s \leq t\}$, provided $\Im_\infty^{(1)}, \Im_\infty^{(2)}$, and ζ are mutually independent.

The significant contribution of this paper includes the succeeding aspects:

(i) There have not been many papers that have considered the aforementioned random impulsive stochastic fractional differential system as in (1).

(ii) The contraction principle is used to the existence results of random impulsive differential equations in [**5, 20**]. However, using Kransnoselskii's fixed point theorem, we analyse the existence findings of fractional random impulsive stochastic differential equations.

The following is a breakdown of the manuscript's structure: Section 2 contains some basic definitions and necessary assumptions. In section 3, certain needed conditions are assumed for analysing the existence and uniqueness results of the proposed stochastic system. The Hyers-Ulam stability of random impulsive stochastic fractional differential equations with finite delay is shown in Section 4.

2. Preliminaries

Let $(\Omega,\Im,\mathcal{P})$ be a probability space with filtration $\{\Im_{\mathrm{t}}\}$, $t \geq 0$ satisfying $\Im_t = \Im_t^{(1)} \vee \Im_t^{(2)}$, $\mathcal{L}^{\mathsf{p}}(\Omega,\mathbf{R}^{\mathfrak{d}})$ be the accumulation of all strongly measurable, pth integrable, $\Im_t$ measurable, ϑ be a $\mathbf{R}^{\mathfrak{d}}$-valued random variable, provided $\|\vartheta\|_{\mathcal{L}^{\mathsf{p}}} =$

$\left(\mathbf{E}\,\|\vartheta\|^{\mathsf{p}}\right)^{\frac{1}{\mathsf{p}}}$ and $\mathbf{E}\vartheta = \int_{\Omega} \vartheta \mathrm{d}\mathcal{P}$. $\delta > 0$ signifies the Banach space of entire piecewise continuous $\mathbf{R}^{\mathfrak{d}}$-valued stochastic process $\left\{\zeta(t), t \in [-\delta, 0]\right\}$ by $\mathbf{C}([-\delta, 0], \mathcal{L}^{\mathsf{p}}(\Omega, \mathbf{R}^{\mathfrak{d}}))$,

$$\|\xi\|_{\mathbf{C}} = \sup_{\theta \in [-\delta, 0]} \left(\mathbf{E}\,\|\xi(\theta)\|^{\mathsf{p}}\right)^{\frac{1}{\mathsf{p}}},$$

thereby, $\xi(\theta) \in \mathbf{C}$.

Assume $\mathsf{T} \in (t_0, +\infty)$, $\mathfrak{u}\colon [t_0, \mathsf{T}] \times \mathbf{C} \to \mathbf{R}^{\mathfrak{d}}$ along with $\mathfrak{v}\colon [t_0, \mathsf{T}] \times \mathbf{C} \to \mathbf{R}^{\mathfrak{d} \times \mathfrak{m}}$ are Borel measurable.

$$\vartheta_{t_0} = \zeta = \{\zeta(\theta) : -\delta \le \theta \le 0\} \tag{2}$$

is the initial data, where (2) is $\Im_{t_0}$ measurable, $[-\delta, 0]$ to $\mathbf{R}^{\mathfrak{d}}$-valued random variable $\ni \mathbb{E}\,\|\zeta\|^2 < \infty$.

Definition 2.1 ([**21**])**.** The fractional order integral of the function $\vartheta(t) \in \mathcal{L}^1([\mathfrak{a}, \mathfrak{b}], \mathbf{R}^n)$ of order $\beta \in \mathbf{R}^+$ is described as,

$$I_{\mathfrak{a}}^{\beta}(\vartheta(t)) = \frac{1}{\Gamma(\beta)} \int_{\mathfrak{a}}^{t} \frac{\vartheta(s)}{(t-s)^{1-\beta}} \mathrm{d}s,$$

wherein $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([**21**])**.** The β order caputo derivative of a function ϑ on the given interval $[\mathfrak{a}, \mathfrak{b}]$ is explained to be

$$({}^{c}\mathcal{D}_{\mathfrak{a},t}^{\beta}\vartheta)(t) = \frac{1}{\Gamma(n-\beta)} \int_{\mathfrak{a}}^{t} \frac{\vartheta^{(n)}(s)}{(t-s)^{\beta+1-n}} \mathrm{d}s,$$

$n = [\beta] + 1$ and $[\beta]$ indicates the integer part of β.

Definition 2.3. For a specified $\mathsf{T} \in (t_0, +\infty)$, an $\mathbf{R}^{\mathfrak{d}}$-valued stochastic process $\vartheta(t)$ on $t_0 - \delta \le t \le \mathsf{T}$ is said to be a solution to (1) along the initial data (2) if for all $t_0 \le t \le \mathsf{T}$, $\vartheta(t_0) = \phi$, $\{\vartheta_t\}_{t_0 \le t \le \mathsf{T}}$ is $\Im_t$-adapted and

$$\begin{aligned} \vartheta(t) = \sum_{k=0}^{+\infty} \Bigg[& \prod_{i=1}^{k} \mathfrak{b}_i(\delta_i)\zeta(0) + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\zeta_{i-1}}^{\zeta_i} (t-s)^{\beta-1} \mathfrak{u}(s, \vartheta_s) \mathrm{d}s \\ & + \frac{1}{\Gamma(\beta)} \int_{\zeta_k}^{t} (t-s)^{\beta-1} \mathfrak{u}(s, \vartheta_s) \mathrm{d}s \\ & + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\zeta_{i-1}}^{\zeta_i} (t-s)^{\beta-1} \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) \\ & + \frac{1}{\Gamma(\beta)} \int_{\zeta_k}^{t} (t-s)^{\beta-1} \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) \Bigg] I_{[\zeta_k, \zeta_{k+1})}(t), \qquad t \in [t_0, \mathsf{T}], \end{aligned} \tag{3}$$

where $\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) = \mathfrak{b}_k(\delta_k)\mathfrak{b}_{k-1}(\delta_{k-1}) \ldots \mathfrak{b}_i(\delta_i)$, $\prod_{j=\mathfrak{m}}^{n}(\cdot) = 1$ as $\mathfrak{m} > n$, and $I_{\mathcal{A}}(\cdot)$ is the index function, i.e.,

$$I_{\mathcal{A}}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{A}, \\ 0 & \text{if } t \notin \mathcal{A}. \end{cases}$$

Definition 2.4. Assume $\mu(t)$ is an $\mathbf{R}^{\mathfrak{d}}$-valued stochastic process. If there exist a real number $\mathfrak{C} > 0$ such that for arbitrary $\varepsilon \geq 0$, it satisfies

$$\mathbf{E}\Big\|\mu(t) - \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0) + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s$$
$$+ \frac{1}{\Gamma(\beta)} \times \int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s$$
$$+ \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)$$
$$+ \frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}} \leq \varepsilon.$$

For each $\mu(t)$ along the initial value $\mu_{t_0} = x_{t_0} = \zeta$, if there exists a solution $\vartheta(t)$ of (3) with $\mathbf{E}\,\|\mu(t) - \vartheta(t)\|^{\mathsf{p}} \leq \mathfrak{C}\varepsilon$, for all $t \in (t_0 - \delta, \mathsf{T})$. Subsequently, Hyers-Ulam Stability is found in (3).

Lemma 2.5 ([**13**]). *Assume that ϕ, φ are two functions, where $\phi, \varphi \in \mathbf{C}([a,b],\mathbf{R}^{\mathfrak{d}})$ and $\phi(t)$ is non-decreasing. If $\vartheta(t) \in \mathbf{C}([\mathfrak{a},\mathfrak{b}],\mathbf{R}^{\mathfrak{d}})$ is a solution of the following inequality*

$$\vartheta(t) \leq \phi(t) + \int_{\mathfrak{a}}^{t}\varphi(s)\vartheta(s)\mathrm{d}s, \qquad t \in [\mathfrak{a},\mathfrak{b}],$$

then, $\vartheta(t) \leq \phi(t)\exp(\int_{\mathfrak{a}}^{t}\varphi(s)\mathrm{d}s)$.

Lemma 2.6 ([**13**]). *For any $\mathsf{p} \geq 1$ and $\Theta \in \mathcal{L}^{\mathsf{p}}_{\mathfrak{d}\times\mathfrak{m}}[0,\mathsf{T}]$ a predictable process, the inequality*

$$\sup_{s\in[0,t]}\mathbf{E}\Big\|\int_0^s\Theta(\eta)\mathrm{d}\mathfrak{w}(\eta)\Big\|^{\mathsf{p}} \leq \Big(\frac{\mathsf{p}}{2}(\mathsf{p}-1)\Big)^{\mathsf{p}/2}\Big(\int_0^t(\mathbf{E}\,\|\Theta(s)\|^{\mathsf{p}})^{2/\mathsf{p}}\Big)^{\mathsf{p}/2}, \quad t \in [0,\mathsf{T}],$$

holds.

3. Main Results

Let us impose the following assumptions.

(A1) $\mathfrak{u}\colon [t_0,\mathsf{T}] \times \mathbf{C} \to \mathbf{R}^{\mathfrak{d}}$ satisfies:

(i) for all t $\in [t_0,\mathsf{T}]$, $\mathfrak{u}(t,\cdot)\colon \mathbf{C} \to \mathbf{R}^{\mathfrak{d}}$ is continuous and for all $\mathfrak{y} \in \mathbf{C}$, $\mathfrak{u}(.,\mathfrak{y})\colon [t_0],\mathsf{T}] \to \mathbf{R}^{\mathfrak{d}}$ is measurable.

(ii) There exists $\mathfrak{M} > 0$ being constant $\ni$

$$\mathbf{E}\,\|\mathfrak{u}(t,\mathfrak{y}_1) - \mathfrak{u}(t,\mathfrak{y}_2)\|^{\mathsf{p}} \leq \mathfrak{M}\left(\|\mathfrak{y}_1 - \mathfrak{y}_2\|^{\mathsf{p}}_{\mathbf{C}}\right)$$

for $\mathfrak{y}_1,\ \mathfrak{y}_2 \in \mathbf{C}$.

(iii) There exists a constant $\mathfrak{M} > 0 \ni$

$$\mathbf{E}\,\|\mathfrak{u}(t,\mathfrak{y})\|^{\mathsf{p}} \leq \mathfrak{M}\left(1 + \|\mathfrak{y}\|^{\mathsf{p}}_{\mathbf{C}}\right).$$

(A2) $\max_{i,k}\Big\{\prod_{j=i}^{k}\mathbf{E}\,\|\mathfrak{b}_j(\delta_j)\|^{\mathsf{p}}\Big\}<\infty$. There exists a constant $\mathfrak{N}>0$ such that for all for all $\delta_j\in\mathfrak{D}_j$, $(j=1,2,3,\cdots)\ni$

$$\mathbb{E}\Big(\max_{i,k}\Big\{\prod_{j=i}^{k}\mathbb{E}\,\|\mathfrak{b}_j(\delta_j)\|^{\mathsf{p}}\Big\}\Big)^{\mathsf{p}}\le\mathfrak{N}.$$

(A3) $\mathfrak{v}\colon[t_0,\mathsf{T}]\times\mathbf{C}\to\mathbf{R}^{\mathfrak{d}\times\mathfrak{m}}$ appeases:

(i) $\mathfrak{v}(t,\cdot)\colon\mathbf{C}\to\mathbf{R}^{\mathfrak{d}\times\mathfrak{m}}$ being continuous also for all $\mathfrak{y}\in\mathbf{C}, t\in[t_0,\mathsf{T}]$, where $\mathfrak{v}(.,\mathfrak{y})\colon[t_0,\mathsf{T}]\to\mathbf{R}^{\mathfrak{d}\times\mathfrak{m}}$ is measurable.

(ii) There exists $\mathfrak{S}(t)\colon[t_0,\mathsf{T}]\to[0,\infty)$ being continuous and $\mathcal{L}^{\mathfrak{q}}$ integrable, continuous, and increasing function $\Xi\colon[0,+\infty)\to[0,+\infty)\ni$

$$\mathbf{E}\,\|\mathfrak{v}(t,\mathfrak{y})\|^{\mathsf{p}}\le\mathfrak{S}(t)\Xi(\|\mathfrak{y}\|^{\mathsf{p}}_{\mathbf{C}}),$$

subjective to $(t,\mathfrak{y})\in[t_0,\mathsf{T}]\times\mathbf{C}$, $\mathfrak{S}^*=\sup_t\in[t_0,\mathsf{T}]\mathfrak{S}(t)$, and the function ϖ fulfils

$$\lim_{\delta\to\infty}\inf\frac{\varpi(\delta)}{\delta}=\alpha<\infty.$$

(A4) Let $\mathcal{F}=\max\{1,\mathfrak{N}\}\dfrac{(t-\delta)^{\beta\mathsf{p}-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}(t-t_0)^{\mathsf{p}}\mathfrak{M}<1$.

Theorem 3.1. *If the hypotheses* (A1)–(A3) *are true, the system* (3) *must have at least one mild solution*

$$3^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(t-\delta)^{\beta\mathsf{p}-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\Big[\mathfrak{M}(\mathsf{T}-t_0)^{\mathsf{p}}+\mathfrak{M}_{\mathsf{p}}(\mathsf{T}-t_0)^{\mathsf{p}/2-1}\mathfrak{S}^*\alpha\Big]\le1.$$

Proof. Let $\mathfrak{B}=\mathbf{C}\left([t_0-\delta,\mathsf{T}],\mathcal{L}^{\mathsf{p}}(\Omega,\mathbf{R}^{\mathfrak{d}})\right)$ be a space, provided

$$\|\vartheta\|^{\mathsf{p}}_{\mathfrak{B}}=\sup_{t\in[t_0,\mathsf{T}]}\|\vartheta_t\|^{\mathsf{p}}_{\mathbf{C}},$$

where $\|\vartheta_t\|^{\mathsf{p}}_{\mathfrak{B}}=\sup_{t-\delta\le s\le t}\mathbf{E}\,\|\vartheta(s)\|^{\mathsf{p}}$.

We interpret the mapping $\Psi:\mathfrak{B}\to\mathfrak{B}$ as

$$\begin{aligned}(\Psi\vartheta)(t+t_0)&=\zeta(\theta)\in\mathcal{L}^{\mathsf{p}}\left(\Omega,\mathbf{C}\right),\qquad t\in[-\delta,0],\\
(\Psi\vartheta)(t)&=\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\end{aligned}$$

$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t).$$

The problem of detecting the mild solutions for (3) is shorten to find the fixed point of Ψ.

Now, let us decompose the operator Ψ as

$$\mathfrak{P}(\vartheta)(t)=\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s$$

$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s\Big]I_{[\zeta_k,\zeta_{k+1})}(t),$$

$$\mathfrak{Q}(\vartheta)(t)=\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)$$

$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t).$$

Let the closed ball $\mathfrak{B}_{\mathfrak{r}}$ with centre ϑ and radius $\mathfrak{r}>0$ be denoted by $\mathfrak{B}_{\mathfrak{r}}=\{\vartheta\in\mathfrak{B};\|\vartheta\|_{\mathfrak{B}}^{\mathsf{p}}\leq\mathfrak{r}\}$. The subsequent steps are used to derive the proof.

Step 1:
Manifesting, $\mathfrak{P}\vartheta+\mathfrak{Q}\overline{\vartheta}\in\mathfrak{B}_{\mathfrak{r}}$, where $\mathfrak{r}>0$ and $\vartheta,\overline{\vartheta}\in\mathfrak{B}_{\mathfrak{r}}$.

Let us prove the part by confliction, i.e., for all $\mathfrak{r}>0$ and $t\in[t_0,\mathsf{T}]$, there exists $\vartheta^{\mathfrak{r}}(\cdot),\ \overline{\vartheta}^{\mathfrak{r}}(\cdot)\in\mathfrak{B}_{\mathfrak{r}}\ni$

$$\mathbf{E}\left\|\mathfrak{P}(\vartheta^{\mathfrak{r}})(t)+\mathfrak{Q}(\overline{\vartheta})^{\mathfrak{r}}(t)\right\|^{\mathsf{p}}>\mathfrak{r}.$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left\|\mathfrak{P}(\vartheta^{\mathfrak{r}})(t)+\mathfrak{Q}(\overline{\vartheta})^{\mathfrak{r}}(t)\right\|^{\mathsf{p}}\\
&\leq 3^{\mathsf{p}-1}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)\Big]\Big\|^{\mathsf{p}}\\
&\quad+3^{\mathsf{p}-1}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s^{\mathfrak{r}})\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s^{\mathfrak{r}})\mathrm{d}s\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}\\
&\quad+3^{\mathsf{p}-1}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\overline{\vartheta}_s^{\mathfrak{r}})\mathrm{d}\mathfrak{w}(s)\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\overline{\vartheta}_s^{\mathfrak{r}})\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}\\
&\leq 3^{\mathsf{p}-1}\mathfrak{N}\zeta(0)^{\mathsf{p}}+3^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathbb{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(t-t_0)^{\mathsf{p}}\mathfrak{M}(1+\|\vartheta_s^{\mathfrak{r}}\|_{\mathbf{C}}^{\mathsf{p}})\\
&\quad+3^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(t-t_0)^{\mathsf{p}/2-1}\mathfrak{M}_{\mathsf{p}}\mathfrak{S}^*\int_{t_0}^{t}\Xi(\|\overline{\vartheta}_s^{\mathfrak{r}}\|_{\mathbf{C}}^{\mathsf{p}})\mathrm{d}s.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathfrak{r} \le 3^{\mathsf{p}-1}\Big[&\mathfrak{N}\zeta(0)^{\mathsf{p}} + \max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(\mathsf{T}-t_0)^{\mathsf{p}}\mathfrak{M}\Big]\\ &+ 3^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(\mathsf{T}-t_0)^{\mathsf{p}}\mathfrak{M}\mathfrak{r}\\ &+ 3^{\mathsf{p}-1}\Big[\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(\mathsf{T}-t_0)^{\mathsf{p}/2-1}\mathfrak{M}_{\mathsf{p}}\frac{\mathfrak{S}^*}{\mathfrak{r}}\int_{t_0}^{t}\Xi(\mathfrak{r})\mathrm{d}s\Big]\mathfrak{r},\end{aligned}$$

where $\mathfrak{M}_{\mathsf{p}} = (\mathsf{p}(\mathsf{p}-1)/2)^{\mathsf{p}/2}$.

Also,

$$\sup_{t\in[t_0,\mathsf{T}]}\|\vartheta_t^{\mathfrak{r}}\|_{\mathbf{C}}^{\mathsf{p}} = \sup_{t\in[t_0-\delta,\mathsf{T}]}\|\vartheta^{\mathfrak{r}}\|^{\mathsf{p}} \le \|\vartheta^{\mathfrak{r}}(t)\|_{\mathfrak{B}}^{\mathsf{p}} \le \mathfrak{r}.$$

The aforesaid inequality is divided by $\mathfrak{r}$ and $\mathfrak{r}\to\infty$, by (A3)(ii),

$$3^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(t-\delta)^{\beta\mathsf{p}-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\Big[\mathfrak{M}(\mathsf{T}-t_0)^{\mathsf{p}} + \mathfrak{M}_{\mathsf{p}}(\mathsf{T}-t_0)^{\mathsf{p}/2-1}\mathfrak{S}^*\alpha\Big] \ge 1,$$

which conflicts our assumption. Therefore, there exists $\mathfrak{r}>0 \ni \vartheta,\overline{\vartheta}\in\mathfrak{B}_{\mathfrak{r}}$, $\mathfrak{P}\vartheta + \mathfrak{Q}\overline{\vartheta}\in\mathfrak{B}_{\mathfrak{r}}$.

Step 2:

Let $\vartheta,\overline{\vartheta}\in\mathfrak{B}_{\mathfrak{r}}$ for $t\in[t_0,\mathsf{T}]$,

$$\begin{aligned}&\mathbb{E}\left\|(\mathfrak{P}\vartheta)(t)-(\mathfrak{P}\overline{\vartheta})(t)\right\|^{\mathsf{p}}\\ &\le \mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\frac{1}{\Gamma(\beta)}\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\left[\mathfrak{u}(s,\vartheta_s)-\vartheta(s,\overline{\vartheta}_s)\right]\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\left[\mathfrak{u}(s,\vartheta_s)-\vartheta(s,\overline{\vartheta}_s)\right]\mathrm{d}s\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}\\ &\le \mathbf{E}\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|\Big\}\Big]^{\mathsf{p}}\\ &\quad\times\frac{1}{\Gamma(\beta)}\left(\int_{t_0}^{t}(t-s)^{\beta-1}\mathbb{E}\left\|\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\overline{\vartheta})_s\right\|\mathrm{d}sI_{[\zeta_k,\zeta_{k+1})}(t)\right)^{\mathsf{p}}\\ &\le \max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\beta\mathsf{p}-1)(\Gamma(\beta))^{\mathsf{p}}}(t-t_0)^{\mathsf{p}}\mathfrak{M}\left(\left\|\vartheta_t-\overline{\vartheta}_t\right\|_{\mathbf{C}}^{\mathsf{p}}\right),\end{aligned}$$

where

$$\left\|\vartheta_t-\overline{\vartheta}_t\right\|_{\mathbf{C}}^{\mathsf{p}} \le \sup_{s\in[t-\delta,t]}\mathbf{E}\left\|\vartheta(s)-\overline{\vartheta}(s)\right\|^{\mathsf{p}}.$$

Taking supremum over t and by (A4),

$$\left\|(\mathfrak{P}\vartheta)(t)-(\mathfrak{P}\overline{\vartheta})(t)\right\|_{\mathfrak{B}}^{\mathsf{p}} \le \mathcal{F}\left\|\vartheta-\overline{\vartheta}\right\|_{\mathfrak{B}}^{\mathsf{p}}$$

with $0<\mathcal{F}<1$. Hence, $\mathfrak{P}$ is a contraction on $\mathfrak{B}_{\mathfrak{r}}$.

Step 3:

Let $\{\vartheta^n\}\subset\mathfrak{B}_{\mathfrak{r}}$ with $\vartheta^n\to\vartheta$ (as $n\to\infty$). For $t\in[t_0,\mathsf{T}]$ by continuity of $\mathfrak{v}$ in

(A3)(i),

$$\begin{aligned}&\mathbf{E}\left\|(\mathfrak{Q}\vartheta^n)(t)-(\mathfrak{Q}\vartheta)(t)\right\|^{\mathsf{p}}\\&\le \max\{1,\mathfrak{N}\}\frac{(\mathrm{t}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\mathfrak{M}_{\mathsf{p}}(\mathrm{t}-\mathrm{t}_0)^{\mathsf{p}/2-1}\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbf{E}\left\|\mathfrak{v}(s,\vartheta_s^n)-\mathfrak{v}(s,\vartheta_s)\right\|^{\mathsf{p}}\mathrm{d}s\\&\longrightarrow 0 \qquad \text{as } n\to\infty.\end{aligned}$$

$\mathfrak{Q}$ is continuous $\mathfrak{B}_{\mathfrak{r}}$.

Step 4:

$\mathfrak{B}$ being a piecewise space, suppose $\zeta_k < t_1 < t_2 < \zeta_{k+1}$ ($k = 1, 2, \dots$) and $\vartheta \in \mathfrak{B}_{\mathfrak{r}}$. Then, for any fixed $\vartheta \in \mathfrak{B}_{\mathfrak{r}}$, through assumptions (A2), (A3) along with the Lemma 2.6,

$$\begin{aligned}&\mathbf{E}\left\|(\mathfrak{Q}\vartheta)(t_2)-(\mathfrak{Q}\vartheta)(t_1)\right\|^{\mathsf{p}}\\&\le 2^{\mathsf{p}-1}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\\&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t_1}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t_1)\Big\|^{\mathsf{p}}\\&\quad+2^{\mathsf{p}-1}\mathbb{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\\&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t_2}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t_2)\Big\|^{\mathsf{p}}\\&\le 2^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\,\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)I_{[\zeta_k,\zeta_{k+1})}(t_2)\Big\|^{\mathsf{p}}\\&\longrightarrow 0 \qquad \text{while} \qquad t_2\longrightarrow t_1.\end{aligned}$$

Accordingly, $\mathfrak{Q}$ maps the bounded sets $\mathfrak{B}_{\mathfrak{r}}$ into equicontinuous sets.

Step 5:

$$\begin{aligned}&\sup_{t\in[t_0,\mathsf{T}]}\mathbf{E}\left\|(\mathfrak{Q}\vartheta)(t)\right\|^{\mathsf{p}}\\&=\sup_{t\in[t_0,\mathsf{T}]}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\\&\quad+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}\\&\le\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\max\{1,\mathfrak{N}\}\mathfrak{M}_{\mathsf{p}}\left\|\mathfrak{S}^*\right\|_{\mathcal{L}^{\mathsf{q}}}\Xi(\mathfrak{r}).\end{aligned}$$

Then $\{\mathfrak{Q}(\mathfrak{B}_{\mathfrak{r}})\}$ is uniformly bounded.

Step 6:
Let $\varepsilon > 0 \ni 0 < \varepsilon < t - t_0$. For $\vartheta \in \mathfrak{B}_\mathfrak{r}$,

$$(\mathfrak{Q}\vartheta)(t) = \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\zeta_{i-1}}^{\zeta_i} (t-s)^{\beta-1} \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) + \frac{1}{\Gamma(\beta)} \int_{\zeta_k}^{t-\varepsilon} (t-s)^{\beta-1} \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) \Big] I_{[\zeta_k, \zeta_{k+1})}(t), \qquad t \in (t_0, t-\varepsilon).$$

The set $\mathcal{W}_\varepsilon(t) = \{(\mathfrak{Q}^\varepsilon \vartheta)(t) : \vartheta \in \mathfrak{B}_r\}$ is relatively compact in $\mathfrak{B}$ for all $\varepsilon \in (0, t - t_0)$. We have

$$\begin{aligned} &\mathbf{E} \, \|(\mathfrak{Q}\vartheta)(t) - (\mathfrak{Q}^\varepsilon \vartheta)(t)\|^{\mathsf{p}} \\ &\le \max\{1, \mathfrak{N}\} \frac{(\mathsf{T} - \delta)^{\mathsf{p}\beta - 1}}{(\mathsf{p}\beta - 1)(\Gamma(\beta))^{\mathsf{p}}} (\varepsilon)^{\mathsf{p}/2-1} \mathfrak{M}_{\mathsf{p}} \int_{t-\varepsilon}^{t} \mathfrak{S}^* \Xi(\mathfrak{r}) \mathrm{d}s. \end{aligned} \tag{4}$$

As $\varepsilon \to 0$, (1) tends to zero. Thus the set $\mathcal{W}(t) = \{(\mathfrak{Q}\vartheta)(t) : \vartheta \in \mathfrak{B}_\mathfrak{r}\}$ has arbitrarily precompact sets and $\mathcal{W}(t)$ is relatively compact in $\mathfrak{B}$. Therefore, $\mathfrak{Q}$ is compact and completely continuous using Arzela-Ascoli theorem.

From Kransnoselskii's Fixed point theorem, $\Phi\vartheta = \mathfrak{P}\vartheta + \mathfrak{Q}\vartheta$ has a fixed point on $\mathfrak{B}_\mathfrak{r}$. Therefore, (1) has a mild solution. Thus the proof is complete. □

Yet, existence of the solution for the system (3) can also be acquired by Banach Contraction Principle. Let us impose the following assumption.

(A3′) Let $\mathfrak{v}(t, \vartheta_t)$ be continuous, $\mathfrak{v}(t, \vartheta_t) \in \mathcal{L}^{\mathsf{p}}([t_0, \mathsf{T}] \times \mathbf{C}; \mathbf{R}^{\mathfrak{d} \times \mathsf{m}}) \ \exists \ \mathfrak{M} > 0 \ni$
$$\mathbf{E} \, \|\mathfrak{v}(t, \vartheta_1) - \mathfrak{v}(t, \vartheta_2)\|^{\mathsf{p}} \le \mathfrak{M} \, (\vartheta_1 - \vartheta_2)^{\mathsf{p}}_{\mathbf{C}}$$
for $t \in [t_0, \mathsf{T}]$, $\vartheta_1, \vartheta_2 \in \mathbf{C}$.

By assuming (A1), (A2), and (A3′) getting satisfied, let us consider the subsequent theorem.

Theorem 3.2. *If the hypotheses* (A1), (A2), *and* (A3′) *are all true, there exists a specific mild solution of* (3).

Proof. For every initial value $t_0 \ge 0$, $\vartheta_0 \in \mathfrak{B}_\mathfrak{r}$, an operator $\mathfrak{U} : \mathfrak{B} \to \mathfrak{B}$ is defined such that for $t \in [t_0, \mathsf{T}]$,

$$\begin{aligned} (\mathfrak{U}\vartheta)(t) = \sum_{k=0}^{+\infty} \Big[&\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i) \zeta(0) + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\zeta_{i-1}}^{\zeta_i} (t-s)^{\beta-1} \mathfrak{u}(s, \vartheta_s) \mathrm{d}s \\ &+ \frac{1}{\Gamma(\beta)} \int_{\zeta_k}^{t} (t-s)^{\beta-1} \mathfrak{u}(s, \vartheta_s) \mathrm{d}s + \frac{1}{\Gamma(\beta)} \sum_{i=1}^{k} \prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\zeta_{i-1}}^{\zeta_i} (t-s)^{\beta-1} \\ &\times \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) + \frac{1}{\Gamma(\beta)} \int_{\zeta_k}^{t} (t-s)^{\beta-1} \mathfrak{v}(s, \vartheta_s) \mathrm{d}\mathfrak{w}(s) \Big] I_{[\zeta_k, \zeta_{k+1})}(t). \end{aligned}$$

Subsequently,

$$\begin{aligned}
&\mathbf{E}\big\|(\mathfrak{U}\vartheta)(t)-(\mathfrak{U}\tilde{\vartheta})(t)\big\|^{\mathsf{p}}\\
&\leq 2^{\mathsf{p}-1}\mathbf{E}\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|\Big\}\Big]^{\mathsf{p}}\\
&\quad\times\Big(\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\mathbf{E}\big\|\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\tilde{\vartheta}_s)\big\|\mathrm{d}s\Big)^{\mathsf{p}}\\
&\quad+2^{\mathsf{p}-1}\mathbf{E}\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|\Big\}\Big]^{\mathsf{p}}\\
&\quad+\Big(\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}\mathbf{E}\big\|\mathfrak{v}(s,\vartheta_s)-\mathfrak{v}(s,\tilde{\vartheta}_s)\big\|\mathrm{d}\mathfrak{w}(s)\Big)^{\mathsf{p}}\\
&\leq 2^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}(t-t_0)^{\mathsf{p}}\mathfrak{M}\big\|\vartheta_s-\tilde{\vartheta}_s\big\|_{\mathbf{C}}^{\mathsf{p}}\\
&\quad+2^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\mathfrak{M}_{\mathsf{p}}(t-t_0)^{\mathsf{p}/2}\mathfrak{M}\big\|\vartheta_s-\tilde{\vartheta}_s\big\|_{\mathbf{C}}^{\mathsf{p}}.
\end{aligned}$$

Taking supremeum over t,

$$\big\|\mathfrak{U}\vartheta-\mathfrak{U}\tilde{\vartheta}\big\|_{\mathcal{B}}^{\mathsf{p}}\leq\varpi(t)\mathbf{E}\big\|\vartheta-\tilde{\vartheta}\big\|_{\mathfrak{B}}^{\mathsf{p}},$$

where $\varpi(t)=2^{\mathsf{p}-1}\max\{1,\mathfrak{N}\}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\mathfrak{M}\left[(t-t_0)^{\mathsf{p}}+\mathfrak{M}_{\mathsf{p}}(t-t_0)^{\mathsf{p}/2}\right].$

For sufficiently small $0<\mathsf{T}_1<\mathsf{T}$, $\mathcal{F}<1$. Thus $\mathfrak{U}$ is a contraction mapping.

Through Banach Contraction principle, $\mathfrak{U}\vartheta=\vartheta$ is a distinctive solution of (3). □

4. Hyers-Ulam Stability Results

Here, the Hyers-Ulam stability of system (3) is investigated presuming the hypotheses (A1), (A2), and (A3′),

Theorem 4.1. *If the assumption of* Theorem 3.2 *gets fulfilled,* (3) *has Ulam-Hyers stability.*

Proof. It is well known that $\vartheta(t)$ is the solution of (3).

$$\begin{aligned}
\vartheta(t)=\sum_{k=0}^{+\infty}\Big[&\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s\\
&+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\vartheta_s)\mathrm{d}s\\
&+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\\
&+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\vartheta_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t).
\end{aligned}$$

By the condition,

$$\mathbf{E}\Big\|\mu(t)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}\leq\varepsilon.$$

When $t\in[t_0-\delta,\mathsf{T}]$, $\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}=0$. Meanwhile for $t\in[t_0,\mathsf{T}]$,

$$\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}$$
$$\leq 2^{\mathsf{p}-1}\mathbf{E}\Big\|\mu(t)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{u}(s,\mu_s)\mathrm{d}s+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\mathfrak{v}(s,\mu_s)\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathfrak{p}}+2^{\mathsf{p}-1}\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\zeta(0)$$
$$+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\left[\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\mu_s)\right]\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\big[\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\mu_s)\big]\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\big[\mathfrak{v}(s,\vartheta_s)-\mathfrak{v}(s,\mu_s)\big]\mathrm{d}\mathfrak{w}(s)$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\left[\mathfrak{v}(s,\vartheta_(s))-\mathfrak{v}(s,\mu_s)\right]\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathfrak{p}}$$
$$\leq 2^{\mathsf{p}-1}\varepsilon+2^{\mathsf{p}-1}\mathfrak{I},$$

whereas

$$\mathfrak{I}=\mathbf{E}\Big\|\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\left[\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\mu_s)\right]\mathrm{d}s$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\left[\mathfrak{u}(s,\vartheta_s)-\mathfrak{u}(s,\mu_s)\right]\mathrm{d}s$$

$$+\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\zeta_{i-1}}^{\zeta_i}(t-s)^{\beta-1}\left[\mathfrak{v}(s,\vartheta_s)-\mathfrak{v}(s,\mu_s)\right]\mathrm{d}\mathfrak{w}(s)$$
$$+\frac{1}{\Gamma(\beta)}\int_{\zeta_k}^{t}(t-s)^{\beta-1}\left[\mathfrak{v}(s,\vartheta_(s))-\mathfrak{v}(s,\mu_s)\right]\mathrm{d}\mathfrak{w}(s)\Big]I_{[\zeta_k,\zeta_{k+1})}(t)\Big\|^{\mathsf{p}}$$
$$\times(\mathsf{T}-t_0)^{(\mathsf{p}-2)/2}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}\mathfrak{M}\int_{t_0}^{t}\|\vartheta_s-\mu_s\|_{\mathbf{C}}^{\mathsf{p}}$$
$$\leq\mathfrak{Y}_1\int_{t_0}^{t}\|\vartheta(s)-\mu(s)\|_{\mathbf{C}}^{\mathsf{p}}\,\mathrm{d}s,$$

where

$$\mathfrak{Y}_1=2^{\mathsf{p}-1}\mathfrak{M}\max\{1,\mathfrak{N}\}(\mathsf{T}-t_0)^{\mathsf{p}/2-1}\frac{(\mathsf{T}-\delta)^{\mathsf{p}\beta-1}}{(\mathsf{p}\beta-1)(\Gamma(\beta))^{\mathsf{p}}}$$
$$\times\left[(\mathsf{T}-t_0)^{\mathsf{p}/2}+(\mathsf{p}(\mathsf{p}-1)/2)^{\mathsf{p}/2}\right].$$

So,

$$\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}\leq 2^{\mathsf{p}-1}\varepsilon+2^{\mathsf{p}-1}\mathfrak{Y}_1\int_{t_0}^{t}\|\mu(s)-\vartheta(s)\|_{\mathbf{C}}^{\mathsf{p}}\,\mathrm{d}s.$$

Consider

$$\int_{t_0}^{t}\|\mu(s)-\vartheta(s)\|_{\mathbf{C}}^{\mathsf{p}}\,\mathrm{d}s=\int_{t_0}^{t}\sup_{\theta\in[-\delta,0]}\mathbf{E}\,\|\mu(s+\theta)-\vartheta(s+\theta)\|^{\mathsf{p}}\,\mathrm{d}s$$
$$=\sup_{\theta\in[-\delta,0]}\int_{t_0+\theta}^{t+\theta}\mathbf{E}\,\|\mu(\mathfrak{m})-\vartheta(\mathfrak{m})\|^{\mathsf{p}}\,\mathrm{d}\mathfrak{m}.$$

While $t\in[t_0-\delta,t_0]$, $\mathbf{E}\,\|\mu(\mathfrak{m})-\vartheta(\mathfrak{m})\|^{\mathsf{p}}=0$. Accordingly,

$$\int_{t_0}^{t}\|\mu_s-\vartheta_s\|_{\mathbf{C}}^{\mathsf{p}}\,\mathrm{d}s=\sup_{\theta\in[-\delta,0]}\int_{t_0}^{t+\theta}\mathbf{E}\,\|\mu(\mathfrak{m})-\vartheta(\mathfrak{m})\|^{\mathsf{p}}\,\mathrm{d}\mathfrak{m}$$
$$=\int_{t_0}^{t}\mathbf{E}\,\|\mu(\mathfrak{m})-\vartheta(\mathfrak{m})\|^{\mathsf{p}}\,\mathrm{d}\mathfrak{m},$$
$$\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}\leq 2^{\mathsf{p}-1}\varepsilon+2^{\mathsf{p}-1}\mathfrak{Y}_1\int_{t_0}^{t}\mathbf{E}\,\|\mu(\mathfrak{m})-\vartheta(\mathfrak{m})\|^{\mathsf{p}}\,\mathrm{d}\mathfrak{m}.$$

Through Lemma 2.5,

$$\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}\leq 2^{\mathsf{p}-1}\varepsilon\exp(2^{\mathsf{p}-1}\mathfrak{Y}_1).$$

Consequently, $\exists\ \mathfrak{C}=2^{\mathsf{p}-1}\exp(2^{\mathsf{p}-1}\mathfrak{Y}_1)\ni$

$$\mathbf{E}\,\|\mu(t)-\vartheta(t)\|^{\mathsf{p}}\leq\mathfrak{C}\varepsilon.$$

Hence the Hyers-Ulam stability of (3) is proved. □

5. Conclusion

A class of Caputo type random impulsive fractional stochastic differential equations are investigated. The existence and uniqueness of solutions have been acquired through Kransnoselskii's fixed point theorem. Hyers-Ulam stability of the aforementioned system is obtained using Lipschitz condition. The system can be further extended to g-Brownian motion with resolvent operator and Poisson jumps.

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