The Averaging principle of Hilfer fractional stochastic pantograph equations with non-lipschitz conditions

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Abstract

The purpose of this study is to present an averaging principle for Hilfer fractional stochastic differential pantograph equations (FSDPE). Under appropriate non-Lipschitz conditions, the mean-square sense and probability of solutions to averaged stochastic systems can be used to approximate solutions to HFSDPE. Finally, an example is provided to demonstrate the results' viability. Furthermore, our findings have greatly broadened prior findings.

Keywords: Hilfer fractional stochastic differential system, Averaging principle, Pantograph equations.

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1 Introduction

Fractional differential equations are the generalization of classical ordinary differential equations (ODEs) with arbitrary order see [1, 2]. Pantograph equations (PEs) [15] are a class of unbounded delay equations that have been utilized to express a variety of applications, including those in biology, electrodynamics, finance, and other nonlinear dynamical systems,(see [16, 17]). Numerous academics have extensively researched the existence, uniqueness, and stability of various types of pantograph equations based on these fundamental aspects. Naturally, a few outstanding and significant papers have also appeared in our view (see [18, 19, 20] and references therein). The nature of solutions for fractional stochastic differential pantograph equations (FSDPEs) in Euclidean space n-dimensional $\mathbb{R}^n$ [3, 4], is particularly interesting in practical applications. Non-linear FSDPEs solutions are extremely challenging to solve. As a result, we used symmetrical methodologies and tactics across the board. It has had a significant impact on the evolution of partial calculus [5, 7].

On the other hand, Khasminiskii [8] focused on analysing the convergence of idle systems on the drag time scale $\epsilon \to 0$, in order to resolve uncertain issues. A typical approach that is highly successful for investigating the use of fractional stochastic differential equations (FSDE) in many interesting disciplines is the AP in stochastic fractional dynamical systems. The averaging method proposes a potent tool for achieving equilibrium between complex models and simple systems (see [9, 10, 11, 13]). For instance, few authors have addressed the AP of FSDEs, for example Abouagwa and Ji Li looked into the approximation properties of solutions to Itô-Doob FSDEs with non-Lipschitz coefficients [13]. The averaging principle for SDEs with Caputo fractional derivative was investigated by Wenjing et al. [14]. This work is motivated by the fact that the averaging concept for Hilfer FSDPEs has not yet been addressed in the literature. The highlights and major contributions of this paper are reflected in the subsequent key aspects:

(i) Initially, the effort to study the characteristic of solutions for a class of Hilfer FSDPEs via the averaging principle (AP) under non-Lipschitz conditions is made. Comparing with literatures [20, 21], the corresponding conditions are required to satisfy the Lipschitz condition or the local Lipschitz condition. The Lipschitz condition usually fails in several practical situations, though. Therefore, in our study, the non-Lipschitz conditions which are significantly weaker than the Lipschitz condition will take the place of the Lipschitz condition.

(ii) The impact of delay terms on the AP for the relevant Hilfer stochastic system has not been taken into consideration in the prior literature [10, 14]. The delay effects, however, do exist in certain Hilfer stochastic differential systems. As a result, in this study, we look at delay Hilfer stochastic differential equations with a linear delay $\tau(\iota) = \theta\iota$ with $0 < \theta < 1$.

The purpose of this study is to establish the averaging principle for Hilfer FSDPEs in the following form

$$\begin{aligned} \mathcal{D}_{0^+}^{\psi,\mathfrak{n}}\mathfrak{w}(\iota) &= \mathfrak{l}\left(\iota, \mathfrak{w}(\iota), \mathfrak{w}(\theta\iota)\right) + \mathfrak{m}\left(\iota, \mathfrak{w}(\iota), \mathfrak{w}(\theta\iota)\right)\frac{d\mathfrak{B}(\iota)}{d\iota} \\ \mathcal{I}_{0^+}^{(1-\psi)(1-\mathfrak{n})}\mathfrak{w}(0) &= \mathfrak{w}_0, \end{aligned} \tag{1.1}$$

where $\iota \in [0, \mathcal{T}]$, $\mathfrak{w}_0 \in \mathscr{R}^n$ is the initial value, which is $\Im_0-$measurable on $\mathscr{R}^n$ and satisfies $\mathbb{E}|\mathfrak{w}_0|^2 < \infty$, $0 < \theta < 1$, $\mathcal{D}_{0^+}^{\psi,\mathfrak{n}}$ is the Hilfer fractional derivative with $0 \leq \psi \leq 1$, $\frac{1}{2} < \mathfrak{n} < 1$, $\mathfrak{l} : [0, \mathcal{T}] \times \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^n$,

$\mathfrak{m} : [0, \mathcal{T}] \times \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^{n\times m}$, and $\mathfrak{B}(\iota)$ is a m-dimensional Brownian motion on the complete probability space $(\Omega, \Im, \mathcal{P})$.

2 Preliminaries

Definition 2.1. *[6] The fractional integral operator of order $\mathfrak{n} > 0$ is expressed as*

$$\mathcal{I}^{\mathfrak{n}}\mathfrak{l}(\iota) = \frac{1}{\Gamma(\mathfrak{n})}\int_0^{\iota} \frac{\mathfrak{l}(s)}{(\iota - s)^{1-\mathfrak{n}}}ds, \qquad \iota > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *[6] The hilfer fractional derivative of order $0 < \psi < 1$ and $0 < \mathfrak{n} < 1$ is interpreted as*

$$\mathcal{D}_{0^+}^{\psi,\mathfrak{n}}\mathfrak{l}(\iota) = \mathcal{I}_{0^+}^{\psi(1-\mathfrak{n})}\frac{d}{d\iota}\mathcal{I}_{0^+}^{(1-\psi)(1-\mathfrak{n})}\mathfrak{l}(\iota),$$

where $\mathcal{D} = \frac{d}{d\iota}$.

Definition 2.3. *An $\mathscr{R}^n-$valued stochastic process $\{\mathfrak{w}(\iota)\}_{0\leq\iota\leq\mathcal{T}}$ is said to be a unique solution of (1.1) if,*

(i) $\{\mathfrak{w}(\iota)\}$ *is continuous with respect to ι and $\Im_\iota-$ adapted.*

(ii) $\mathfrak{l}(\iota, \mathfrak{w}(\iota), \mathfrak{w}(\theta\iota)) \in \mathscr{L}^1([0, \mathcal{T}]; \mathscr{R}^n)$ *and* $\mathfrak{m}(\iota, \mathfrak{w}(\iota), \mathfrak{w}(\theta\iota)) \in \mathscr{L}^2([0, \mathcal{T}]; \mathscr{R}^{n\times m})$.

(iii) For all $\iota \in [0, \mathcal{T}]$, we have

$$\begin{aligned} \mathfrak{w}(\iota) \quad &= \quad \frac{\iota^{(\psi-1)(1-\mathfrak{n})}}{\Gamma(\psi(1-\mathfrak{n})+\mathfrak{n})}\mathfrak{w}_0 + \frac{1}{\Gamma(\mathfrak{n})}\int_0^{\iota}(\iota - s)^{\mathfrak{n}-1}\mathfrak{l}(s, \mathfrak{w}(s), \mathfrak{w}(\theta s))ds \\ &+ \quad \frac{1}{\Gamma(\mathfrak{n})}\int_0^{\iota}(\iota - s)^{\mathfrak{n}-1}\mathfrak{m}(s, \mathfrak{w}(s), \mathfrak{w}(\theta s))d\mathfrak{B}(s). \end{aligned} \tag{2.2}$$

(iv) For $\mathfrak{w}^*(\iota)$, we have $P\{\mathfrak{w}(\iota) = \mathfrak{w}^*(\iota),\ \forall\ \iota \in [0, \mathcal{T}]\} = 1$.

We impose the subsequent conditions:

(H1) There exist a function $\Phi(\cdot)$ such that, for any fixed $\iota \geq 0$ and $\mathfrak{w}_i, \mathsf{y}_i \in \mathscr{R}^n$, $i = 1, 2$, we have

$$|\mathfrak{l}(\iota, \mathfrak{w}_1, \mathsf{y}_1) - \mathfrak{l}(\iota, \mathfrak{w}_2, \mathsf{y}_2)|^2 \vee |\mathfrak{m}(\iota, \mathfrak{w}_1, \mathsf{y}_1) - \mathfrak{m}(\iota, \mathfrak{w}_2, \mathsf{y}_2)|^2 \leq \Phi\left(\iota, |\mathfrak{w}_1 - \mathfrak{w}_2|^2, |\mathsf{y}_1 - \mathsf{y}_2|^2\right), \tag{2.3}$$

where $\Phi(\cdot)$ satisfies $\Phi(\iota, 0, 0) = 0$ and define $\Omega = [0, +\infty) \times [0, +\infty)$ and $\int_\Omega \frac{1}{\Phi(\iota,\mathsf{u},\mathsf{v})}d\mathsf{u}\mathsf{v} = \infty$ and there exist non-negative functions $\lambda_i(\iota)$, $i = 1, 2, 3$ such that for $\mathsf{u}, \mathsf{v} \geq 0$, $\Phi(\iota, \mathsf{u}, \mathsf{v}) \leq \lambda_1(\iota) + \lambda_2(\iota)\mathsf{u} + \lambda_3(\iota)\mathsf{v}$ and $\int_0^{\mathcal{T}} \lambda_i(\iota)d\iota < \infty,\ i = 1, 2, 3.$

3 Main Results

Initially, let us consider the standard form of the system (1.1).

$$\begin{aligned}\mathfrak{w}_\varepsilon(\iota) &= \frac{\iota^{(\psi-1)(1-\mathfrak{n})}}{\Gamma(\psi(1-\mathfrak{n})+\mathfrak{n})}\mathfrak{w}_0 + \frac{\varepsilon}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\mathfrak{l}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)ds \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)d\mathfrak{B}(s), \end{aligned} \tag{3.1}$$

where $\varepsilon \in (0,\varepsilon_0]$ is a positive small parameter and ε_0 is a given fixed number.

Before concluding with the AP for SPE, there exist measurable function $\mathfrak{l}^* : \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^n$ and $\mathfrak{m}^* : \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^{n\times m}$ such that the following holds.

(H2) For $\mathcal{T}_1 \in [0,\mathcal{T}]$, $\mathfrak{w}, \mathsf{y} \in \mathscr{R}^n$, there exist positive bounded functions $\Psi_i(\mathcal{T}_1)$, $i=1,2$ such that

$$\begin{aligned}\frac{1}{\mathcal{T}_1}\int_0^{\mathcal{T}_1} |\mathfrak{l}\left(s,\mathfrak{w},\mathsf{y}\right)-\mathfrak{l}^*\left(\mathfrak{w},\mathsf{y}\right)|^2 ds &\le \varpi_1(\mathcal{T}_1)\left(|\mathfrak{w}|^2+|\mathsf{y}|^2\right), \\ \frac{1}{\mathcal{T}_1}\int_0^{\mathcal{T}_1} |\mathfrak{m}\left(s,\mathfrak{w},\mathsf{y}\right)-\mathfrak{m}^*\left(\mathfrak{w},\mathsf{y}\right)|^2 ds &\le \varpi_1(\mathcal{T}_1)\left(|\mathfrak{w}|^2+|\mathsf{y}|^2\right), \end{aligned} \tag{3.2}$$

where $\lim\limits_{\mathcal{T}_1\to\infty}\varpi_i(\mathcal{T}_1)=0$. The solution $\mathfrak{w}_\varepsilon(\iota)$ converges, as $\varepsilon \to 0$ to the solution $\mathfrak{w}^*_\varepsilon(\iota)$ of the averaged system:

$$\begin{aligned}\mathfrak{w}^*_\varepsilon(\iota) &= \frac{\iota^{(\psi-1)(1-\mathfrak{n})}}{\Gamma(\psi(1-\mathfrak{n})+\mathfrak{n})}\mathfrak{w}_0 + \frac{\varepsilon}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)ds \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\mathfrak{m}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)d\mathfrak{B}(s), \end{aligned} \tag{3.3}$$

as $\varepsilon \to 0$ and $\iota \in [0,\mathcal{T}]$.

Theorem 3.1. *Assume that the conditions (H1),(H2) hold. Then, for a given arbitrarily small number $\varrho_1 > 0$, there exist $\mathscr{M} > 0$, $\varepsilon \in (0,\varepsilon_1]$ and $\tau \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon_1]$, we have*

$$\mathbb{E}\left(\sup_{\iota\in[-\tau,\mathscr{M}\varepsilon^{-\tau}]}|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}^*_\varepsilon(\iota)|^2\right) \le \varrho_1.$$

Proof. From (3.1) and (3.3), we may obtain for $\iota \in [0,\mathsf{u}] \subset [0,\mathcal{T}]$, we get

$$\begin{aligned}\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}^*_\varepsilon(\iota) &= \frac{\varepsilon}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathsf{h}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds \\ &+ \frac{\sqrt{\varepsilon}}{\Gamma(\mathfrak{n})}\int_0^\iota (\iota-s)^{\mathsf{h}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{m}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathfrak{B}(s). \end{aligned}$$

Applying the elementary inequality,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le\iota\le\mathsf{u}}|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}^*_\varepsilon(\iota)|^2\right) &\le \frac{2\varepsilon^2}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\le\iota\le\mathsf{u}}\left|\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds\right|^2 \\ &+ \frac{2\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\le\iota\le\mathsf{u}}\left|\int_0^\iota (\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{m}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathcal{B}(s)\right|^2 \\ &= \mathscr{I}_1+\mathscr{I}_2. \end{aligned}$$

Using the elementary inequality,

$$\begin{aligned}\mathscr{I}_1 &\leq \frac{4\varepsilon^2}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds\right|^2\\ &+ \frac{4\varepsilon^2}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds\right|^2\\ &= \mathscr{I}_{11}+\mathscr{I}_{12}.\end{aligned}$$

Using Cauchy Schwartz inequality,

$$\begin{aligned}\mathscr{I}_{11} &\leq \frac{4\varepsilon^2}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{l}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds\right|^2\\ &\leq \frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\int_0^\iota\Phi\left(s,\left|\mathfrak{w}_\varepsilon(s)-\mathfrak{w}^*_\varepsilon(s)\right|^2,\left|\mathfrak{w}_\varepsilon(\theta s)-\mathfrak{w}^*_\varepsilon(\theta s)\right|^2\right)ds\\ &\leq \frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\int_0^\iota\left[\lambda_1(s)+\lambda_2(s)|\mathfrak{w}_\varepsilon(s)-\mathfrak{w}^*_\varepsilon(s)|^2+\lambda_3(s)|\mathfrak{w}_\varepsilon(\theta s)-\mathfrak{w}^*_\varepsilon(\theta s)|^2\right]ds\\ &\leq \frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sup_{0\leq\iota\leq\mathfrak{u}}\lambda_1(\iota)+4\frac{\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sum_{i=2}^3\sup_{0\leq\iota\leq\mathfrak{u}}\lambda_i(\iota)\int_0^{\mathsf{u}}\mathbb{E}\sup_{0\leq s_1\leq s}|\mathfrak{w}_\varepsilon(s_1)-\mathfrak{w}^*_\varepsilon(s_1)|^2ds.\end{aligned}$$

Also, by (H2) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}\mathscr{I}_{12} &= \frac{4\varepsilon^2}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{l}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]ds\right|^2\\ &\leq \frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\frac{1}{\iota}\int_0^\iota\left|\mathfrak{l}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)-\mathfrak{l}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right|^2ds\\ &\leq \frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\left(\sup_{0\leq\iota\leq\mathfrak{u}}\varpi_1(\iota)\right)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathfrak{u}}|\mathfrak{w}^*_\varepsilon(\iota)|^2\right].\end{aligned}$$

Using elementary inequality,

$$\begin{aligned}\mathscr{I}_2 &\leq \frac{2\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{m}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathfrak{B}(s)\right|^2\\ &\leq \frac{4\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{m}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathfrak{B}(s)\right|^2\\ &+ \frac{4\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)-\mathfrak{m}^*\left(\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathfrak{B}(s)\right|^2\\ &= \mathscr{I}_{21}+\mathscr{I}_{22}.\end{aligned}$$

From (H1) and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathscr{I}_{21} &\leq \frac{4\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon(s),\mathfrak{w}_\varepsilon(\theta s)\right)-\mathfrak{m}\left(s,\mathfrak{w}^*_\varepsilon(s),\mathfrak{w}^*_\varepsilon(\theta s)\right)\right]d\mathfrak{B}(s)\right|^2\\ &\leq \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\int_0^\iota\Phi\left(s,\left|\mathfrak{w}_\varepsilon(s)-\mathfrak{w}^*_\varepsilon(s)\right|^2,\left|\mathfrak{w}_\varepsilon(\theta s)-\mathfrak{w}^*_\varepsilon(\theta s)\right|^2\right)ds\\ &\leq \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathfrak{u}}\int_0^\iota\left[\lambda_1(s)+\lambda_2(s)\left|\mathfrak{w}_\varepsilon(s)-\mathfrak{w}^*_\varepsilon(s)\right|^2+\lambda_3(s)\left|\mathfrak{w}_\varepsilon(\theta s)-\mathfrak{w}^*_\varepsilon(\theta s)\right|^2\right]ds\\ &\leq \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sup_{0\leq\iota\leq\mathfrak{u}}\lambda_1(\iota)+16\frac{\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sum_{i=2}^3\sup_{0\leq\iota\leq\mathfrak{u}}\lambda_i(\iota)\int_0^{\mathsf{u}}\mathbb{E}\sup_{0\leq s_1\leq s}\left|\mathfrak{w}_\varepsilon(s_1)-\mathfrak{w}^*_\varepsilon(s_1)\right|^2ds.\end{aligned}$$

From the hypotheses (H2) and Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
\mathscr{I}_{22} &\leq \frac{4\varepsilon}{[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathsf{u}}\left|\int_0^\iota(\iota-s)^{\mathfrak{n}-1}\left[\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon^*(s),\mathfrak{w}_\varepsilon^*(\theta s)\right)-\mathfrak{m}^*\left(\mathfrak{w}_\varepsilon^*(s),\mathfrak{w}_\varepsilon^*(\theta s)\right)\right]d\mathfrak{B}(s)\right|^2 \\
&\leq \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\mathbb{E}\sup_{0\leq\iota\leq\mathsf{u}}\frac{1}{\iota}\int_0^\iota\left|\mathfrak{m}\left(s,\mathfrak{w}_\varepsilon^*(s),\mathfrak{w}_{\theta s}^*\right)-\mathfrak{m}^*\left(\mathfrak{w}_\varepsilon^*(s),\mathfrak{w}_\varepsilon^*(\theta s)\right)\right|^2ds \\
&\leq \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\left(\sup_{0\leq\iota\leq\mathsf{u}}\varpi_2(\iota)\right)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon^*(\iota)\right|^2+\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon^*(\theta\iota)\right|^2\right] \\
&\leq \frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\left(\sup_{0\leq\iota\leq\mathsf{u}}\varpi_2(\iota)\right)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}|\mathfrak{w}_\varepsilon^*(\iota)|^2\right].
\end{aligned}
$$

By the estimation of $\mathscr{I}_{11},\mathscr{I}_{12},\mathscr{I}_{21}$ and $\mathscr{I}_{22}$, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}_\varepsilon^*(\iota)\right|^2\right] \quad &\leq \quad \frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sup_{0\leq\iota\leq\mathsf{u}}\lambda_1(\iota)+\frac{4\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sum_{i=2}^{3}\sup_{0\leq\iota\leq\mathsf{u}}\lambda_i(\iota) \\
&\times \quad \int_0^{\mathsf{u}}\mathbb{E}\sup_{0\leq s_1\leq s}|\mathfrak{w}_\varepsilon(s_1)-\mathfrak{w}_\varepsilon^*(s_1)|^2ds+\frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\left(\sup_{0\leq\iota\leq\mathsf{u}}\varpi_1(\iota)\right) \\
&\times \quad \mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}|\mathfrak{w}_\varepsilon^*(\iota)|^2\right]+\frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sup_{0\leq\iota\leq\mathsf{u}}\lambda_1(\iota) \\
&+ \quad \frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\sum_{i=2}^{3}\sup_{0\leq\iota\leq\mathsf{u}}\lambda_i(\iota)\int_0^{\mathsf{u}}\mathbb{E}\sup_{0\leq s_1\leq s}\left|\mathfrak{w}_\varepsilon(s_1)-\mathfrak{w}_\varepsilon^*(s_1)\right|^2ds \\
&+ \quad \frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\left(\sup_{0\leq\iota\leq\mathsf{u}}\varpi_2(\iota)\right)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}|\mathfrak{w}_\varepsilon^*(\iota)|^2\right] \\
&\leq \quad \frac{4\mathscr{M}_\lambda\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}}\mathscr{M}_\lambda}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}}\mathscr{M}_\varpi}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}\mathscr{M}_\varpi}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2} \\
&+ \quad \left[\frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\right]\int_0^{\mathsf{u}}\mathbb{E}\sup_{0\leq s_1\leq s}\left|\mathfrak{w}_\varepsilon(s_1)-\mathfrak{w}_\varepsilon^*(s_1)\right|^2ds,
\end{aligned}
$$

where,

$$
\left\{\sup_{0\leq\iota\leq\mathsf{u}}\varpi_1(\iota)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon^*(\iota)\right|^2,\right],\sup_{0\leq\iota\leq\mathsf{u}}\varpi_2(\iota)\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon^*(\iota)\right|^2,\right]\right\}=\mathscr{M}_\varpi,\left\{\sup_{0\leq\iota\leq\mathsf{u}}\lambda_i(\iota),\ i=1,2,3\right\}=\mathscr{M}_\lambda.
$$

By Gronwall-Bellman inequality, we get

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq\iota\leq\mathsf{u}}\left|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}_\varepsilon^*(\iota)\right|^2\right] \quad &\leq \quad \left[\frac{4\mathscr{M}_\lambda\varepsilon^2\mathsf{u}^{2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{16\varepsilon\mathsf{u}^{2\mathfrak{n}}\mathscr{M}_\lambda}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}}\mathscr{M}_\varpi}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}\mathscr{M}_\varpi}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\right] \\
&\times \quad e^{\left[\frac{8\varepsilon^2\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\varepsilon\mathsf{u}^{2\mathfrak{n}-1}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\right]},
\end{aligned}
$$

which implies $\mathscr{M}>0$ and $\tau\in(0,1)\ni$ for $\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]\subset[0,\mathcal{T}]$ having

$$
\mathbb{E}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]}\left|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}_\varepsilon^*(\iota)\right|^2\right)\leq\mathbb{K}\varepsilon^{1-\tau}.
$$

where,

$$
\begin{aligned}
\mathbb{K} \quad &= \quad \left[\frac{4\mathscr{M}_\lambda\mathscr{M}^{2\mathfrak{n}}\varepsilon^{1+\tau-2\tau\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{16\mathscr{M}_\lambda\mathscr{M}^{2\mathfrak{n}}\varepsilon^{\tau-2\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{8\mathscr{M}_\varpi\mathscr{M}^{2\mathfrak{n}}\varepsilon^{1+\tau-2\tau\mathfrak{n}}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\mathscr{M}_\varpi\mathscr{M}^{2\mathfrak{n}-1}\varepsilon^{2\tau(1-\mathfrak{n})}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\right] \\
&\times \quad e^{\left[\frac{8\mathscr{M}^{2\mathfrak{n}-1}\varepsilon^{1+2\tau(1-\mathfrak{n})}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}+\frac{32\mathscr{M}^{2\mathfrak{n}-1}\varepsilon^{2\tau(1-\mathfrak{n})}}{(2\mathfrak{n}-1)[\Gamma(\mathfrak{n})]^2}\right]},
\end{aligned}
$$

is a constant. For arbitrary given number δ_1, $\exists\ \varepsilon_1 \in (0, \varepsilon_0]$ such that, for every $\varepsilon \in (0, \varepsilon_1]$ and $\iota \in [0, \mathscr{M}\varepsilon^{-\tau}]$, we have

$$\mathbb{E}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]} |\mathfrak{w}_\varepsilon(\iota) - \mathfrak{w}^*_\varepsilon(\iota)|^2\right) \leq \delta_1.$$

This completes the proof.

□

Theorem 3.2. *Assuming the conditions of the Theorem 3.1 hold, then for a given arbitrarily small number* $\delta_2 > 0$*, there exists* $\mathscr{M} > 0$, $\varepsilon_1 \in (0, \varepsilon_0]$ *and* $\tau \in (0, 1)$, $\ni$ *for* $\varepsilon \in (0, \varepsilon_1]$*, we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]} |\mathfrak{w}_\varepsilon(\iota) - \mathfrak{w}^*_\varepsilon(\iota)| > \delta_2\right) = 0.$$

Proof. Given $\delta_2 > 0$, by Chebyshev's inequality and Theorem 3.1, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]} |\mathfrak{w}_\varepsilon(\iota) - \mathfrak{w}^*_\varepsilon(\iota)| > \delta_2\right) &\leq \frac{1}{\delta_2^2}\mathbb{E}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]} |\mathfrak{w}_\varepsilon(\iota) - \mathfrak{w}^*_\varepsilon(\iota)|^2\right)\\ &\leq \frac{1}{\delta_2^2}\mathbb{K}\varepsilon^{1-\tau} \to 0 \ \text{ as } \ \varepsilon \to 0.\end{aligned}$$

Thus the proof is complete.

□

4 Illustration

Consider, the hilfer FSPDEs of the form:

$$\begin{aligned}\mathcal{D}^{\frac{1}{2},\frac{3}{4}}\mathfrak{w}_\varepsilon &= \varepsilon\left(2\mathfrak{w}_\varepsilon\cos^2(\iota) - \mathfrak{w}_\varepsilon\sin^2\left(\frac{1}{2}\iota\right)\right)d\iota + \sqrt{\varepsilon}d\mathfrak{B}(\iota)\\ \mathcal{I}^{\frac{1}{8}}_{0+}\mathfrak{w}_\varepsilon(0) &= \mathfrak{w}_0, \qquad \iota \in [0, \pi], \qquad (4.1)\\ \mathfrak{l}\left(\iota, \mathfrak{w}_\varepsilon(\iota), \mathfrak{w}_\varepsilon(\theta\iota)\right) &= 2\mathfrak{w}_\varepsilon\cos^2(\iota) - \mathfrak{w}_\varepsilon\sin^2\left(\frac{1}{2}\iota\right),\\ \mathfrak{m}\left(\iota, \mathfrak{w}_\varepsilon(\iota), \mathfrak{w}_\varepsilon(\theta\iota)\right) &= 1. \qquad (4.2)\end{aligned}$$

Let,

$$\begin{aligned}\mathfrak{l}^*\left(\iota, \mathfrak{w}_\varepsilon(\iota), \mathfrak{w}_\varepsilon(\theta\iota)\right) &= \frac{1}{\pi}\int_0^\pi \mathfrak{l}\left(\iota, \mathfrak{w}_\varepsilon(\iota), \mathfrak{w}_\varepsilon(\theta\iota)\right) = \frac{\mathfrak{w}_\varepsilon}{2},\\ \mathfrak{m}^*\left(\iota, \mathfrak{w}_\varepsilon(\iota), \mathfrak{w}_\varepsilon(\theta\iota)\right) &= 1. \qquad (4.3)\end{aligned}$$

Therefore, the simplified stochastic PEs 4.1 can be defined as

$$\mathcal{D}^{\frac{1}{2},\frac{3}{4}}\mathfrak{w}^*_\varepsilon(\iota) = \epsilon\frac{\mathfrak{w}^*_\varepsilon}{2}d\iota + \sqrt{\epsilon}d\mathfrak{B}(\iota), \qquad (4.4)$$

Evidently, the conditions of Theorem 3.1 and Theorem 3.2 gets satisfied for the functions defined in (4.1)-(4.4). Therefore we obtain,

$$\mathbb{E}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]}|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}^*_\varepsilon(\iota)|^2\right) \leq \delta_1,$$

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{\iota\in[0,\mathscr{M}\varepsilon^{-\tau}]}|\mathfrak{w}_\varepsilon(\iota)-\mathfrak{w}^*_\varepsilon(\iota)|>\delta_2\right)=0.$$

5 Conclusion

This works concerns with the AP for hilfer FSPDEs which is new to the literature. We obtain solutions for SPEs that can be approximated by solutions to averaged stochastic systems in the mean-square sense and probability under suitable non-Lipschitz conditions. The AP for FSPDEs with Levy noise and time delays, as well as G-Brownian motion with non-Lipschitz conditions, would be an intriguing expansion of our research in the future.

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