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# Existence results for coupled sequential $\psi$-Hilfer fractional impulsive BVPs: topological degree theory approach

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**Abstract**

In this paper, the coupled system of sequential $\psi$-Hilfer fractional boundary value problems with non-instantaneous impulses is investigated. The existence results of the system are proved by means of topological degree theory. An example is constructed to demonstrate our results. Additionally, a graphical analysis is performed to verify our results.

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## 1 Introduction

The concept of impulsive differential equations has gathered a lot of attention in recent times, considering its significance for the precise mathematical modelling of a variety of real-world situations, including epidemics, population ecology, optimal control, mechanical engineering, astronomy, and others [1, 2]. In particular, impulsive fractional differential equations (FDEs) have proven to be an efficient tool for describing the hereditary and memory properties of evolutionary systems characterised by sudden changes in their state at various instants. A non-instantaneous impulse refers to a force or an impulse exerted on a system over a finite duration. Hernandez and O'Regan [3] introduced the theory of non-instantaneous impulsive differential equations. We refer to [4–6] for recent studies on the non-instantaneous impulsive FDEs.

Sequential FDEs can capture the combined effects of multiple fractional derivatives. The sequential FDEs have several applications in real-world problems and can be used to model complex dynamics. For example, the Langevin equation, developed by Langevin, explains the progress of physical processes in changing environments [7]. An essential feature to be examined is the existence of such physical problems [8–14].

The study of qualitative aspects of the solutions to mathematical problems is crucial in various fields due to its implications for the understanding, analysis, and application of mathematical models [15, 16]. For investigating the existence of solutions, several methods, including fixed point theory, monotone iterative technique, topological degree theory

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(TDT), etc., are employed. The advantages of topological degree theory in investigating FDEs lie in its abstract framework, adaptability to infinitedimensional spaces, ability to handle nonlinearity, and applicability to global and qualitative analysis of solutions. It offers a powerful and unified approach to study the qualitative aspects of solutions. Atta Ullah et al. [17] investigated the impulsive Caputo FDE under Robin boundary conditions using the TDT. J. Xie and L. Duan [18] established the existence results via TDT for a coupled system of Caputo FDEs.

While there are several fractional derivatives available to investigate the FDEs, the $\psi$-Hilfer fractional derivative proposed by Sousa and Oliveira in [19] has the advantage of a choice of the differential operator. Only a few researchers have used degree theory to study the FDEs [20]. There are numerous applications for FDEs with integral boundary conditions in viscoelasticity, optimisation theory, fluid mechanics, etc. [21–23]. Motivated by the above works, we consider the coupled system of $\psi$-Hilfer sequential fractional boundary value problems with non-instantaneous impulses

$$\begin{cases} {}^HD_{a^+}^{\delta_1,\alpha_1:\psi}\left({}^HD_{a^+}^{\delta_2,\alpha_2:\psi}+d_1\right)\varphi(\varepsilon)=f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)),\ \varepsilon\in[\varepsilon_j,\mathfrak{s}_j],\ j=0,1,..,r,\\ {}^HD_{a^+}^{\tau_1,\beta_1:\psi}\left({}^HD_{a^+}^{\tau_2,\beta_2:\psi}+d_2\right)\rho(\varepsilon)=g(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)),\ \varepsilon\in[\varepsilon_j,\mathfrak{s}_j],\ j=0,1,..,r,\\ \varphi(\varepsilon)=\mathcal{M}_j(\varepsilon,\varphi(\varepsilon)),\quad \rho(\varepsilon)=\mathcal{N}_j(\varepsilon,\rho(\varepsilon)),\quad \varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j],\ j=1,2,..,r,\\ \varphi(a)=0,\ \varphi(\mathfrak{s}_j)=\sum_{m=1}^{p}\lambda_m I_{a^+}^{\theta_m:\psi}\varphi(\zeta_m),\\ \rho(a)=0,\ \rho(\mathfrak{s}_j)=\sum_{n=1}^{q}\mu_n I_{a^+}^{\eta_n:\psi}\rho(\xi_n),\quad a<\zeta_m,\xi_n<\mathfrak{s}_j,\ j=0,1,..,r, \end{cases} \tag{1}$$

where ${}^HD_{a^+}^{\delta_1,\alpha_1:\psi}$, ${}^HD_{a^+}^{\delta_2,\alpha_2:\psi}$, ${}^HD_{a^+}^{\tau_1,\beta_1;\psi}$ and ${}^HD_{a^+}^{\tau_2,\beta_2;\psi}$ are the $\psi$-Hilfer fractional derivatives of order $\delta_1$, $\delta_2$, $\tau_1$ and $\tau_2$, respectively, with $0<\delta_1,\delta_2,\tau_1,\ \tau_2<1$ and type $0\le\alpha_1,\alpha_2,\beta_1,\ \beta_2\le 1$ such that $1<\delta_1+\delta_2<2$, $1<\tau_1+\tau_2<2$. $I_{a^+}^{\theta_m:\psi}$ and $I_{a^+}^{\eta_n:\psi}$ are the $\psi$-Riemann–Liouville (RL) fractional integrals of order $\theta_m$ and $\eta_n$, respectively. $a=\varepsilon_0<\mathfrak{s}_0<\varepsilon_1<\mathfrak{s}_1<\cdots<\varepsilon_r<\mathfrak{s}_r=b$, $d_1,d_2\in\mathcal{R}\backslash\{0\}$ and $\lambda_n,\mu_n\in\mathcal{R}^+$. Also, $\zeta_m,\eta_n\in[a,b]$. The state variables $\varphi,\rho:[a,b]\to\mathcal{R}$, and the functions $f,g:[a,b]\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ and $\mathcal{M}_j,\mathcal{N}_j:[\mathfrak{s}_{j-1},\varepsilon_j]\times\mathcal{R}\to\mathcal{R}$ are continuous for all $j=1,2,\dots,r$.

We highlight that the present study is novel and makes notable contributions to the existing literature on the topic. The proposed problem includes:

- A coupled system of fractional differential equations in which fractional derivatives appear sequentially.
- $\psi$-Hilfer fractional boundary value problems with non-instantaneous impulses.
- Investigation of the existence of a solution to the coupled system with non-instantaneous impulses via topological degree theory.

## 2 Preliminaries

The fundamental ideas, theorems, and lemmas that influence our analysis are stated below:

Let $\mathcal{T}=[a,b]$. Denote by $C(\mathcal{T},\mathcal{R})$ the space of all continuous functions, and by $AC(\mathcal{T},\mathcal{R})$ the space of all absolutely continuous functions. Let $X$ be a Banach space, and let $\mathcal{B}\subset P(X)$ denote the family of all bounded subsets of $X$. Also, let $\psi\in C(\mathcal{T},\mathcal{R})$ be an increasing function such that $\psi'(\varepsilon)>0$ for all $\varepsilon\in\mathcal{T}$.

**Definition 1** [24] Let $F : \mathcal{K} \longrightarrow X$ be a bounded continuous map, where $\mathcal{K} \subseteq X$. Then $F$ is

1. $\sigma$-Lipschitz if there exists $k \geq 0$ such that $\sigma(F(S)) \leq k\sigma(S)$ for all bounded subsets $S \subseteq \mathcal{K}$;
2. a strict $\sigma$-contraction if there exists $0 \leq k < 1$ such that $\sigma(F(S)) \leq k\sigma(S)$;
3. $\sigma$-condensing if $\sigma(F(S)) < \sigma(S)$ for all bounded subsets $S \subseteq \mathcal{K}$ with $\sigma(S) > 0$. In other words, $\sigma(F(S)) \geq \sigma(S)$ implies $\sigma(S) = 0$,

where $\sigma$ is the Kuratowski measure of non-compactness.

**Proposition 1** [25] *If $F, G : \mathcal{K} \longrightarrow X$ are $\sigma$-Lipschitz with respective constants $a_1$ and $a_2$, then $F + G$ is $\sigma$-lipschitz with constant $a_1 + a_2$.*

**Proposition 2** [25] *If $F : \mathcal{K} \longrightarrow X$ is Lipschitz with constant $a$, then $F$ is $\sigma$-lipschitz with the same constant $a$.*

**Proposition 3** [25] *If $F : \mathcal{K} \longrightarrow X$ is compact, then $F$ is $\sigma$-lipschitz with constant $a = 0$.*

Let $\curlywedge = \{(I - F, \mathcal{K}, y) : \mathcal{K} \subseteq X$ open and bounded, $F \in C_\sigma(\bar{\mathcal{K}})$, $y \in X \backslash (I - F)(\partial \mathcal{K})\}$ be the family of admissible triplets, where $C_\sigma(\bar{\mathcal{K}})$ represents the class of all $\sigma$-condensing maps $F : \bar{\mathcal{K}} \to X$.

**Theorem 4** [25] *Let $F : \mathcal{K} \longrightarrow X$ is $\sigma$-condensing and*

$$\curlywedge = \{\varphi \in X : \text{there exists } 0 \leq \omega \leq 1 \text{ such that } \varphi = \omega F \varphi\}.$$

*If $\curlywedge$ is a bounded set in $X$, then there exists $r > 0$ such that $\curlywedge \subset B_i(0)$ and*

$$D(I - \omega F, B_i(0), 0) = 1 \text{ for all } \omega \in [0, 1].$$

*Thus, $F$ has at least one fixed point, and the set of all fixed points of $F$ lies in $B_i(0)$.*

**Definition 2** [26] Let $(a, b)$ $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathcal{R}$ and $\vartheta > 0$. The $\psi$-RL fractional integral $I_{a^+}^{\vartheta;\psi}(\cdot)$ of a function $h \in AC^n([a, b], \mathbb{R})$ on $[a, b]$ is defined by

$$I_{a^+}^{\vartheta;\psi} h(\varepsilon) = \frac{1}{\Gamma(\vartheta)} \int_a^\varepsilon \psi'(s)(\psi(\varepsilon) - \psi(s))^{\vartheta - 1} h(s) ds, \quad \varepsilon > a > 0, \tag{2}$$

where $\Gamma(\vartheta) = \int\limits_0^\infty t^{\vartheta - 1} e^{-t} dt$ represents the Gamma function.

**Definition 3** [19] Let $[a, b]$ be the interval such that $-\infty \leq a < b \leq \infty$, $n \in \mathbb{N}$, $n - 1 < \vartheta < n$ and $h \in C^n([a, b]$. The $\psi$-Hilfer fractional derivative ${}^H D_{a^+}^{\vartheta, \kappa; \psi}(\cdot)$ of a function $h$ of order $\vartheta$ and type $0 \leq \kappa \leq 1$ is defined by

$${}^H D_{a^+}^{\vartheta, \kappa; \psi} h(\varepsilon) = I_{a^+}^{\kappa(n - \vartheta); \psi} \left( \frac{1}{\psi'(\varepsilon)} \frac{d}{d\varepsilon} \right)^n I_{a^+}^{(1 - \kappa)(n - \vartheta); \psi} h(\varepsilon), \tag{3}$$

where $n = [\vartheta] + 1$ and $[\vartheta]$ is the integer part of the real number $\vartheta$ with $\gamma = \vartheta + \kappa(n - \vartheta)$.

**Lemma 1** [26] *For $\vartheta, \tau > 0$, we have the following semigroup property*:

$$I_{a^+}^{\vartheta;\psi} I_{a^+}^{\tau;\psi} h(\varepsilon) = I_{a^+}^{\vartheta+\tau;\psi} h(\varepsilon), \ \varepsilon > a. \tag{4}$$

**Lemma 2** [19] *If $h \in C^n([a,b],\mathcal{R})$, $n-1 < \vartheta < n$ and $0 \le \kappa \le 1$ and $\gamma = \vartheta + \kappa(n-\vartheta)$, then*

$$I_{a^+}^{\vartheta;\psi}\, {}^{H}D_{a^+}^{\vartheta,\kappa;\psi} h(\varepsilon) = h(\varepsilon) - \sum_{k=1}^{n} \frac{(\psi(\varepsilon)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} h_{\psi}^{[n-k]} I_{a^+}^{(1-\kappa)(n-\vartheta);\psi} h(a), \tag{5}$$

*for all $\varepsilon \in [a,b]$, where $h_{\psi}^{[n]} h(\varepsilon) = \left(\frac{1}{\psi'(\varepsilon)}\frac{d}{dt}\right)^n h(\varepsilon)$.*

**Proposition 5** [19, 26] *Let $\vartheta \ge 0$, $l > 0$ and $\varepsilon > a$. Then the $\psi$-fractional integral and derivative of a power function are given by*:

1. $I_{a^+}^{\vartheta;\psi}(\psi(\varepsilon)-\psi(a))^{l-1}(\varepsilon) = \frac{\Gamma(l)}{\Gamma(l+\vartheta)}(\psi(\varepsilon)-\psi(a))^{l+\vartheta-1}$.
2. ${}^{H}D_{a^+}^{\vartheta,\kappa;\psi}(\psi(\varepsilon)-\psi(a))^{l-1}(\varepsilon) = \frac{\Gamma(l)}{\Gamma(l-\vartheta)}(\psi(\varepsilon)-\psi(a))^{l-\vartheta-1}$, $l > \gamma = \vartheta + \kappa(n-\vartheta)$.

## 3 Solution framework

The solution of the BVPs (1) is derived in this section.

Define $\mathcal{Z} = PC(\mathcal{T},\mathcal{R})$, the space of all piece-wise continuous functions, by

$$\mathcal{Z} = \Big\{\varphi : \mathcal{T} \longrightarrow \mathcal{R};\ \varphi \in C(\varepsilon_j, \varepsilon_{j+1}),\ j = 1,2,\ldots,r,\ \varphi(\varepsilon_j^+) \text{ and } \varphi(\varepsilon_j^-) \text{ exist with } \varphi(\varepsilon_j^-) = \varphi(\varepsilon_j)\Big\},$$

with the norm $\|\varphi\|_{\mathcal{Z}} = \sup_{\varepsilon\in\mathcal{T}}\{|\varphi(\varepsilon)| : \varepsilon \in \mathcal{T}\}$. Under this norm, $\mathcal{Z}$ is a Banach space. Consequently, $\mathcal{Z}\times\mathcal{Z}$ is a Banach space with the norm $\|(\varphi,\rho)\|_{\mathcal{Z}} = \|\varphi\|_{\mathcal{Z}} + \|\rho\|_{\mathcal{Z}}$, $(\varphi,\rho) \in \mathcal{Z}\times\mathcal{Z}$.

To demonstrate the existence and uniqueness of (1), it is essential to prove the following lemma.

We consider the following notation throughout the paper: $\mathfrak{S}(l,s) = \frac{(\psi(l)-\psi(a))^s}{\Gamma(s+1)}$.

**Lemma 3** *Let $\gamma_1 = \delta_1 + \alpha_1(1-\delta_1)$, $\gamma_2 = \delta_2 + \alpha_2(1-\delta_2)$, $\bar{\gamma_1} = \tau_1 + \beta_1(1-\tau_1)$, and $\bar{\gamma_2} = \tau_2 + \beta_2(1-\tau_2)$. Let $f, g : \mathcal{T}\times\mathcal{R}\times\mathcal{R} \to \mathcal{R}$, and $\Lambda_{1_j}, \Lambda_{2_j}, \Lambda_{3_j}, \Lambda_{4_j} \neq 0$. Then the solution of the coupled sequential impulsive fractional BVPs (1) is given by*

$$\varphi(\varepsilon) = \begin{cases} I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon) + \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m \\ \times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_0,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0)) \\ -d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m) + d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_0)\Big), & \varepsilon \in [a,\mathfrak{s}_0], \\ I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon) \\ +\Big(\frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\varepsilon,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\Big(\sum_{m=1}^{p}\lambda_m \\ \times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_j,\rho(\mathfrak{s}_j),\varphi(\mathfrak{s}_j)) \\ -d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m) + d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_j)\Big) + \Big(\Lambda_{1_j}\mathfrak{S}(\varepsilon,\gamma_2-1) \\ -\Lambda_{3_j}\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\Big)\Big(\mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon_j,\rho(\varepsilon_j),\varphi(\varepsilon_j)) \\ +d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon_j)\Big), & \varepsilon \in (\varepsilon_j,\mathfrak{s}_j], \\ \mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j)), & \varepsilon \in (\mathfrak{s}_{j-1},\varepsilon_j],\ j = 1,2,\ldots,r, \end{cases} \tag{6}$$

$$\rho(\varepsilon)=\begin{cases} I_{a^+}^{\tau_1+\tau_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_2I_{a^+}^{\tau_2;\psi}\rho(\varepsilon)+\frac{\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)}{\Lambda_{2_0}}\Big(\sum_{n=1}^{q}\mu_n \\ \times I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi}g(\xi_n,\varphi(\xi_n),\rho(\xi_n))-I_{a^+}^{\tau_1+\tau_2;\psi}g(\mathfrak{s}_0,\varphi(\mathfrak{s}_0),\rho(\mathfrak{s}_0)) \\ -d_2\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_2+\eta_n;\psi}\rho(\xi_n)+d_2I_{a^+}^{\tau_2;\psi}\rho(\mathfrak{s}_0)\Big), & \varepsilon\in[a,\mathfrak{s}_0], \\ I_{a^+}^{\tau_1+\tau_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_2I_{a^+}^{\tau_2;\psi}\rho(\varepsilon) \\ +\Big(\frac{\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma}_2-1)-\mathfrak{S}(\varepsilon,\bar{\gamma}_2-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma}_1+\tau_2-1)}{\Lambda_{2_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma}_2-1)-\Lambda_{4_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma}_1+\tau_2-1)}\Big)\Big(\sum_{n=1}^{q}\mu_n \\ \times I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi}g(\xi_n,\varphi(\xi_n),\rho(\xi_n))-I_{a^+}^{\tau_1+\tau_2;\psi}g(\mathfrak{s}_j,\varphi(\mathfrak{s}_j),\rho(\mathfrak{s}_j)) \\ -d_2\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_2+\eta_n;\psi}\rho(\xi_n)+d_2I_{a^+}^{\tau_2;\psi}\rho(\mathfrak{s}_j)\Big)+\Big(\Lambda_{2_j}\mathfrak{S}(\varepsilon,\bar{\gamma}_2-1) \\ -\Lambda_{4_j}\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)\Big)\Big(\mathcal{N}_j(\varepsilon_j,\rho(\varepsilon_j))-I_{a^+}^{\tau_1+\tau_2;\psi}g(\varepsilon_j,\varphi(\varepsilon_j),\rho(\varepsilon_j)) \\ +d_2I_{a^+}^{\tau_2;\psi}\rho(\varepsilon_j)\Big), & \varepsilon\in(\varepsilon_j,\mathfrak{s}_j], \\ \mathcal{N}_j(\varepsilon_j,\rho(\varepsilon_j)), & \varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j],j=1,2,\dots,r, \end{cases} \tag{7}$$

*where*

$$\Lambda_{1_j}=\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)-\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\gamma_1+\delta_2+\theta_m-1),$$

$$\Lambda_{2_j}=\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma}_1+\tau_2-1)-\sum_{n=1}^{q}\mu_n\mathfrak{S}(\xi_n,\bar{\gamma}_1+\tau_2+\eta_n-1),$$

$$\Lambda_{3_j}=\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)-\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\gamma_2+\theta_m-1),$$

$$\Lambda_{4_j}=\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma}_2-1)-\sum_{n=1}^{q}\mu_n\mathfrak{S}(\xi_n,\bar{\gamma}_2+\eta_n-1),\ j=0,1,\dots,r.$$

*Proof* Let $\varphi$ and $\rho$ be the solution of (1).

*Case 1:* For $\varepsilon\in[a,\mathfrak{s}_0]$, consider

$$^{H}D_{a^+}^{\delta_1,\alpha_1:\psi}\Big(^{H}D_{a^+}^{\delta_2,\alpha_2:\psi}+d_1\Big)\varphi(\varepsilon)=f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)),$$

$$\varphi(a)=0,\ \varphi(\mathfrak{s}_0)=\sum_{m=1}^{p}\lambda_m I_{a^+}^{\theta_m:\psi}\varphi(\zeta_m).$$

Using Lemma 2 and applying operators $I_{a^+}^{\delta_1;\psi}$ and $I_{a^+}^{\delta_2;\psi}$ on both sides of the above sequential differential equation, we have

$$\varphi(\varepsilon)=I_{a^+}^{\delta_1+\delta_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_1I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon)+c_1\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)+c_2\mathfrak{S}(\varepsilon,\gamma_2-1). \tag{8}$$

When $\varphi(a)=0$, we get $c_2=0$. Then the above equation is reduced to

$$\varphi(\varepsilon)=I_{a^+}^{\delta_1+\delta_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_1I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon)+c_1\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1). \tag{9}$$

Applying the boundary condition and simplifying, we get

$$c_1 = \frac{1}{\Lambda_{1_0}}\Big[\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_0,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0))$$
$$+ \sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_0)\Big].$$

Thus,

$$\varphi(\varepsilon) = I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon) + \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m$$
$$\times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_0,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0))$$
$$- d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m) + d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_0)\Big), \qquad \varepsilon\in[a,\mathfrak{s}_0].$$

Similarly, for

$${}^{H}D_{a^+}^{\tau_1,\beta_1:\psi}\Big({}^{H}D_{a^+}^{\tau_2,\beta_2:\psi} + d_2\Big)\varphi(\varepsilon) = g(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)),$$

$$\rho(a) = 0,\ \rho(\mathfrak{s}_0) = \sum_{n=1}^{q}\mu_n I_{a^+}^{\eta_n:\psi}\rho(\xi_n),$$

$$\rho(\varepsilon) = I_{a^+}^{\tau_1+\tau_2;\psi} f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)) - d_2 I_{a^+}^{\tau_2;\psi}\rho(\varepsilon) + \frac{\mathfrak{S}(\varepsilon,\bar{\gamma_1}+\tau_2-1)}{\Lambda_{2_0}}\Big(\sum_{n=1}^{q}\mu_n$$
$$\times I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi} g(\xi_n,\varphi(\xi_n),\rho(\xi_n)) - I_{a^+}^{\tau_1+\tau_2;\psi} g(\mathfrak{s}_0,\varphi(\mathfrak{s}_0),\rho(\mathfrak{s}_0))$$
$$- d_2\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_2+\eta_n;\psi}\rho(\xi_n) + d_2 I_{a^+}^{\tau_2;\psi}\rho(\mathfrak{s}_0)\Big), \qquad \varepsilon\in[a,\mathfrak{s}_0].$$

For $\varepsilon\in(\mathfrak{s}_0,\varepsilon_1]$, $\varphi(\varepsilon_1) = \mathcal{M}_1(\varepsilon,\varphi(\varepsilon))$, $\rho(\varepsilon_1) = \mathcal{N}_1(\varepsilon,\rho(\varepsilon))$.

*Case 2:* For $\varepsilon\in(\varepsilon_1,\mathfrak{s}_1]$, consider

$${}^{H}D_{a^+}^{\delta_1,\alpha_1:\psi}\Big({}^{H}D_{a^+}^{\delta_2,\alpha_2:\psi} + d_1\Big)\varphi(\varepsilon) = f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)),$$

$$\varphi(\varepsilon_1) = \mathcal{M}_1(\varepsilon_1,\varphi(\varepsilon_1)),\ \varphi(\mathfrak{s}_1) = \sum_{m=1}^{p}\lambda_m I_{a^+}^{\theta_m:\psi}\varphi(\zeta_m).$$

Repeating the same process as above, we obtain

$$\mathcal{M}_1(\varepsilon_1,\varphi(\varepsilon_1)) = I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon_1,\rho(\varepsilon_1),\varphi(\varepsilon_1)) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon_1)$$
$$+ c_1\mathfrak{S}(\varepsilon_1,\gamma_1+\delta_2-1) + c_2\mathfrak{S}(\varepsilon_1,\gamma_2-1),$$

and

$$I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_1,\rho(\mathfrak{s}_1),\varphi(\mathfrak{s}_1)) - d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_1) + c_1\mathfrak{S}(\mathfrak{s}_1,\gamma_1+\delta_2-1) + c_2\mathfrak{S}(\mathfrak{s}_1,\gamma_2-1)$$

$$= \sum_{m=1}^{p} \lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m, \rho(\zeta_m), \varphi(\zeta_m)) - d_1 \sum_{m=1}^{p} \lambda_m I_{a^+}^{\delta_2+\theta_m;\psi} \varphi(\zeta_m)$$
$$+ c_1 \sum_{m=1}^{p} \lambda_m \mathfrak{S}(\zeta_m, \gamma_1 + \delta_2 + \theta_m - 1) + \sum_{m=1}^{p} \lambda_m c_2 \mathfrak{S}(\zeta_m, \gamma_2 + \theta_m - 1).$$

Solving for $c_1$ and $c_2$ and substituting, we obtain

$$\varphi(\varepsilon) = \begin{cases} I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - d_1 I_{a^+}^{\delta_2;\psi} \varphi(\varepsilon) \\ +\Big( \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_1,\gamma_2-1)-\mathfrak{S}(\varepsilon,\gamma_2-1)\mathfrak{S}(\varepsilon_1,\gamma_1+\delta_2-1)}{\Lambda_{1_1}\mathfrak{S}(\varepsilon_1,\gamma_2-1)-\Lambda_{3_1}\mathfrak{S}(\varepsilon_1,\gamma_1+\delta_2-1)} \Big) \Big( \sum_{m=1}^{p} \lambda_m \\ \times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi} f(\zeta_m, \rho(\zeta_m), \varphi(\zeta_m)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\mathfrak{s}_1, \rho(\mathfrak{s}_1), \varphi(\mathfrak{s}_1)) \\ -d_1 \sum_{m=1}^{p} \lambda_m I_{a^+}^{\delta_2+\theta_m;\psi} \varphi(\zeta_m) + d_1 I_{a^+}^{\delta_2;\psi} \varphi(\mathfrak{s}_1) \Big) + \Big( \Lambda_{1_1} \mathfrak{S}(\varepsilon, \gamma_2 - 1) \\ -\Lambda_{3_1} \mathfrak{S}(\varepsilon, \gamma_1 + \delta_2 - 1) \Big) \Big( \mathcal{M}_j(\varepsilon_1, \varphi(\varepsilon_1)) - I_{a^+}^{\delta_1+\delta_2;\psi} f(\varepsilon_1, \rho(\varepsilon_1), \varphi(\varepsilon_1)) \\ +d_1 I_{a^+}^{\delta_2;\psi} \varphi(\varepsilon_1) \Big), \qquad \varepsilon \in (\varepsilon_1, \mathfrak{s}_1]. \end{cases}$$

Similarly, for

$${}^{H}D_{a^+}^{\tau_1,\beta_1:\psi} \Big( {}^{H}D_{a^+}^{\tau_2,\beta_2:\psi} + d_2 \Big) \varphi(\varepsilon) = g(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)),$$
$$\rho(\varepsilon_1) = \mathcal{N}_1(\varepsilon_1, \rho(\varepsilon)), \ \rho(\mathfrak{s}_1) = \sum_{n=1}^{q} \mu_n I_{a^+}^{\eta_n:\psi} \rho(\xi_n),$$

$$\rho(\varepsilon) = \begin{cases} I_{a^+}^{\tau_1+\tau_2;\psi} f(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - d_2 I_{a^+}^{\tau_2;\psi} \rho(\varepsilon) \\ +\Big( \frac{\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)\mathfrak{S}(\varepsilon_1,\bar{\gamma}_2-1)-\mathfrak{S}(\varepsilon,\bar{\gamma}_2-1)\mathfrak{S}(\varepsilon_1,\bar{\gamma}_1+\tau_2-1)}{\Lambda_{2_1}\mathfrak{S}(\varepsilon_1,\bar{\gamma}_2-1)-\Lambda_{4_1}\mathfrak{S}(\varepsilon_1,\bar{\gamma}_1+\tau_2-1)} \Big) \Big( \sum_{n=1}^{q} \mu_n \\ \times I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi} g(\xi_n, \varphi(\xi_n), \rho(\xi_n)) - I_{a^+}^{\tau_1+\tau_2;\psi} g(\mathfrak{s}_1, \varphi(\mathfrak{s}_1), \rho(\mathfrak{s}_1)) \\ -d_2 \sum_{n=1}^{q} \mu_n I_{a^+}^{\tau_2+\eta_n;\psi} \rho(\xi_n) + d_2 I_{a^+}^{\tau_2;\psi} \rho(\mathfrak{s}_1) \Big) + \Big( \Lambda_{2_1} \mathfrak{S}(\varepsilon, \bar{\gamma}_2 - 1) \\ -\Lambda_{4_1} \mathfrak{S}(\varepsilon, \bar{\gamma}_1 + \tau_2 - 1) \Big) \Big( \mathcal{N}_j(\varepsilon_1, \rho(\varepsilon_1)) - I_{a^+}^{\tau_1+\tau_2;\psi} g(\varepsilon_1, \varphi(\varepsilon_1), \rho(\varepsilon_1)) \\ +d_2 I_{a^+}^{\tau_2;\psi} \rho(\varepsilon_1) \Big), \qquad \varepsilon \in (\varepsilon_1, \mathfrak{s}_1]. \end{cases}$$

Generally, for $\varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]$, $\varphi(\varepsilon_j) = \mathcal{M}_j(\varepsilon_j, \varphi(\varepsilon_j))$ and $\rho(\varepsilon_j) = \mathcal{N}_j(\varepsilon_j, \rho(\varepsilon_j))$.

*Case 3:* For $\varepsilon \in (\varepsilon_j, \mathfrak{s}_j]$, consider

$${}^{H}D_{a^+}^{\delta_1,\alpha_1:\psi} \Big( {}^{H}D_{a^+}^{\delta_2,\alpha_2:\psi} + d_1 \Big) \varphi(\varepsilon) = f(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)),$$
$$\varphi(\varepsilon_j) = \mathcal{M}_j(\varepsilon_j, \varphi(\varepsilon_j)), \ \varphi(\mathfrak{s}_j) = \sum_{m=1}^{p} \lambda_m I_{a^+}^{\theta_m:\psi} \varphi(\zeta_m),$$

and

$${}^{H}D_{a^+}^{\tau_1,\beta_1:\psi} \Big( {}^{H}D_{a^+}^{\tau_2,\beta_2:\psi} + d_2 \Big) \varphi(\varepsilon) = g(\varepsilon, \varphi(\varepsilon), \rho(\varepsilon)),$$
$$\rho(\varepsilon_j) = \mathcal{N}_j(\varepsilon_j, \rho(\varepsilon_j)), \ \rho(\mathfrak{s}_j) = \sum_{n=1}^{q} \mu_n I_{a^+}^{\eta_n:\psi} \rho(\xi_n).$$

Proceeding in the same way, we obtain (6) and (7), which are the solutions of the impulsive fractional BVPs (1).

Conversely, by using standard steps, we verify that (6) and (7) satisfy (1). □

## 4 Main results

The existence of a solution to (1) are determined in this section.

We transform our system into a fixed-point problem.

Define the operators $\mathcal{F},\mathcal{G},\mathcal{J}:\mathcal{Z}\times\mathcal{Z}\longrightarrow\mathcal{Z}\times\mathcal{Z}$ by

$$\begin{aligned}&\mathcal{F}(\varphi,\rho)(\varepsilon)=(\mathcal{F}_1(\varphi,\rho)(\varepsilon),\mathcal{F}_2(\varphi,\rho)(\varepsilon)),\qquad \mathcal{G}(\varphi,\rho)(\varepsilon)=(\mathcal{G}_1(\varphi,\rho)(\varepsilon),\mathcal{G}_2(\varphi,\rho)(\varepsilon)),\\ &\mathcal{J}(\varphi,\rho)(\varepsilon)=\mathcal{F}(\varphi,\rho)(\varepsilon)+\mathcal{G}(\varphi,\rho)(\varepsilon),\end{aligned}$$

where $\mathcal{F}_1,\mathcal{F}_2,\mathcal{G}_1,\mathcal{G}_2:\mathcal{Z}\longrightarrow\mathcal{Z}$ are given by

$$\mathcal{F}_1\varphi(\varepsilon)=\begin{cases}\frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))\\ -I_{a^+}^{\delta_1+\delta_2;\psi}f(\mathfrak{s}_0,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0))-d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m)\\ +d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_0)\Big), & \varepsilon\in[a,\mathfrak{s}_0],\\ \Big(\frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\varepsilon,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\Big(\sum_{m=1}^{p}\lambda_m\\ \times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))-I_{a^+}^{\delta_1+\delta_2;\psi}f(\mathfrak{s}_j,\rho(\mathfrak{s}_j),\varphi(\mathfrak{s}_j))\\ -d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}\varphi(\zeta_m)+d_1 I_{a^+}^{\delta_2;\psi}\varphi(\mathfrak{s}_j)\Big)+\Big(\Lambda_{1_j}\mathfrak{S}(\varepsilon,\gamma_2-1)\\ -\Lambda_{3_j}\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\Big)\Big(\mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j))-I_{a^+}^{\delta_1+\delta_2;\psi}f(\varepsilon_j,\rho(\varepsilon_j),\varphi(\varepsilon_j))\\ +d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon_j)\Big), & \varepsilon\in(\varepsilon_j,\mathfrak{s}_j],\\ \mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j)), & \varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j],\ j=1,2,\dots,r,\end{cases}$$

$$\mathcal{F}_2\rho(\varepsilon)=\begin{cases}\frac{\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)}{\Lambda_{2_0}}\Big(\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi}g(\xi_n,\varphi(\xi_n),\rho(\xi_n))\\ -I_{a^+}^{\tau_1+\tau_2;\psi}g(\mathfrak{s}_0,\varphi(\mathfrak{s}_0),\rho(\mathfrak{s}_0))-d_2\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_2+\eta_n;\psi}\rho(\xi_n)\\ +d_2 I_{a^+}^{\tau_2;\psi}\rho(\mathfrak{s}_0)\Big), & \varepsilon\in[a,\mathfrak{s}_0],\\ \Big(\frac{\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma}_2-1)-\mathfrak{S}(\varepsilon,\bar{\gamma}_2-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma}_1+\tau_2-1)}{\Lambda_{2_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma}_2-1)-\Lambda_{4_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma}_1+\tau_2-1)}\Big)\Big(\sum_{n=1}^{q}\mu_n\\ \times I_{a^+}^{\tau_1+\tau_2+\eta_n;\psi}g(\xi_n,\varphi(\xi_n),\rho(\xi_n))-I_{a^+}^{\tau_1+\tau_2;\psi}g(\mathfrak{s}_j,\varphi(\mathfrak{s}_j),\rho(\mathfrak{s}_j))\\ -d_2\sum_{n=1}^{q}\mu_n I_{a^+}^{\tau_2+\eta_n;\psi}\rho(\xi_n)+d_2 I_{a^+}^{\tau_2;\psi}\rho(\mathfrak{s}_j)\Big)+\Big(\Lambda_{2_j}\mathfrak{S}(\varepsilon,\bar{\gamma}_2-1)\\ -\Lambda_{4_j}\mathfrak{S}(\varepsilon,\bar{\gamma}_1+\tau_2-1)\Big)\Big(\mathcal{N}_j(\varepsilon_j,\rho(\varepsilon_j))-I_{a^+}^{\tau_1+\tau_2;\psi}g(\varepsilon_j,\varphi(\varepsilon_j),\rho(\varepsilon_j))\\ +d_2 I_{a^+}^{\tau_2;\psi}\rho(\varepsilon_j)\Big), & \varepsilon\in(\varepsilon_j,\mathfrak{s}_j],\\ \mathcal{N}_j(\varepsilon_j,\rho(\varepsilon_j)), & \varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j],\ j=1,2,\dots,r,\end{cases}$$

$$\mathcal{G}_1\varphi(\varepsilon)=\begin{cases}I_{a^+}^{\delta_1+\delta_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon), & \varepsilon\in[a,\mathfrak{s}_0],\\ I_{a^+}^{\delta_1+\delta_2;\psi}f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))-d_1 I_{a^+}^{\delta_2;\psi}\varphi(\varepsilon), & \varepsilon\in(\varepsilon_j,\mathfrak{s}_j],\\ 0, & \varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j],\end{cases}$$

$$\mathcal{G}_2\varphi(\varepsilon) = \begin{cases} I_{a^+}^{\tau_1+\tau_2;\psi} f(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - d_2 I_{a^+}^{\tau_2;\psi} \rho(\varepsilon), & \varepsilon \in [a, \mathfrak{s}_0], \\ I_{a^+}^{\tau_1+\tau_2;\psi} f(\varepsilon, \rho(\varepsilon), \varphi(\varepsilon)) - d_2 I_{a^+}^{\tau_2;\psi} \rho(\varepsilon), & \varepsilon \in (\varepsilon_j, \mathfrak{s}_j], \\ 0, & \varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]. \end{cases}$$

The fixed points of the operator equation $\mathcal{J}(\varphi, \rho) = (\varphi, \rho)$ are the solution of equation (1).

In order to obtain the existence results, we need the following hypotheses:

$(H_1)$ Let $f, g \in C(\mathcal{T} \times \mathcal{R} \times \mathcal{R}, \mathcal{R})$. For each $(\varepsilon, \varphi, \rho) \in \mathcal{T} \times \mathcal{R} \times \mathcal{R}$, there exist constants $l_{f_1}, l_{f_2}, M_f, l_{g_1}, l_{g_2}, M_g \in [0, 1)$ such that

$$|f(\varepsilon, \rho, \varphi)| \leq l_{f_1}|\rho(\varepsilon)| + l_{f_2}|\varphi(\varepsilon)| + M_f,$$
$$|g(\varepsilon, \varphi, \rho)| \leq l_{g_1}|\varphi(\varepsilon)| + l_{g_2}|\rho(\varepsilon)| + M_g.$$

$(H_2)$ Let $f, g \in C(\mathcal{T} \times \mathcal{R} \times \mathcal{R}, \mathcal{R})$. For each $(\varepsilon, \varphi, \rho), (\varepsilon, \bar{\varphi}, \bar{\rho}) \in \mathcal{T} \times \mathcal{R} \times \mathcal{R}$, there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that

$$|f(\varepsilon, \rho, \varphi) - f(\varepsilon, \bar{\rho}, \bar{\varphi})| \leq \lambda_1\big(|\rho(\varepsilon) - \bar{\rho}(\varepsilon)| + |\varphi(\varepsilon) - \bar{\varphi}(\varepsilon)|\big),$$
$$|g(\varepsilon, \varphi, \rho) - g(\varepsilon, \bar{\varphi}, \bar{\rho})| \leq \lambda_2\big(|\varphi(\varepsilon) - \bar{\varphi}(\varepsilon)| + |\rho(\varepsilon) - \bar{\rho}(\varepsilon)|\big).$$

$(H_3)$ Let $\mathcal{M}_j, \mathcal{N}_j \in C((\mathfrak{s}_{j-1}, \varepsilon_j] \times \mathcal{R}, \mathcal{R})$. For each $(\varepsilon, \varphi, \rho) \in \mathcal{T} \times \mathcal{R} \times \mathcal{R}$, there exist constants $L_1, M_1, L_2, M_2 \in [0, 1)$ such that

$$|\mathcal{M}_j(\varepsilon, \varphi)| \leq L_1|\varphi(\varepsilon)| + M_1,$$
$$|\mathcal{N}_j(\varepsilon, \rho)| \leq L_2|\rho(\varepsilon)| + M_2.$$

$(H_4)$ Let $\mathcal{M}_j, \mathcal{N}_j \in C((\mathfrak{s}_{j-1}, \varepsilon_j] \times \mathcal{R}, \mathcal{R})$. For each $\varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]$, and $\varphi$, $\bar{\varphi}$, $\rho, \bar{\rho} \in \mathcal{R}$, there exist constants $P_{\mathcal{M}_j}, P_{\mathcal{N}_j} \in [0, \frac{1}{r}]$ such that

$$|\mathcal{M}_j(\varepsilon, \varphi) - \mathcal{M}_j(\varepsilon, \bar{\varphi})| \leq P_{\mathcal{M}_j}|\varphi(\varepsilon) - \bar{\varphi}(\varepsilon)|,$$
$$|\mathcal{N}_j(\varepsilon, \rho) - \mathcal{N}_j(\varepsilon, \bar{\rho})| \leq P_{\mathcal{N}_j}|\rho(\varepsilon) - \bar{\rho}(\varepsilon)|.$$

For easy understandability, we use the following notations:

$$\mathcal{A}_{1_j} = \sum_{m=1}^{p} \lambda_m \mathfrak{S}(\zeta_m, \delta_1 + \delta_2 + \theta_m) + \mathfrak{S}(\mathfrak{s}_j, \delta_1 + \delta_2),\ j = 0, 1, \ldots, r,$$
$$\mathcal{A}_{2_j} = d_1 \sum_{m=1}^{p} \lambda_m \mathfrak{S}(\zeta_m, \delta_2 + \theta_m) + \mathfrak{S}(\mathfrak{s}_j, \delta_2),\ j = 0, 1, \ldots, r,$$
$$\mathcal{A}_{3_j} = \sum_{n=1}^{q} \mu_n \mathfrak{S}(\xi_n, \tau_1 + \tau_2 + \eta_n) + \mathfrak{S}(\mathfrak{s}_j, \tau_1 + \tau_2),\ j = 0, 1, \ldots, r,$$
$$\mathcal{A}_{4_j} = d_2 \sum_{n=1}^{q} \mu_n \mathfrak{S}(\xi_n, \tau_2 + \eta_n) + \mathfrak{S}(\mathfrak{s}_j, \tau_2),\ j = 0, 1, \ldots, r,$$

$$\Theta_1 = \max\Big\{\mathfrak{S}(\varepsilon_j, \delta_1+\delta_2),\ j=1,2,\dots,r\Big\}, \quad \Theta_2 = \max\Big\{d_1\mathfrak{S}(\varepsilon_j, \delta_2),\ j=1,2,\dots,r\Big\},$$

$$\Theta_3 = \max\Big\{\mathfrak{S}(\varepsilon_j, \tau_1+\tau_2),\ j=1,2,\dots,r\Big\}, \quad \Theta_4 = \max\Big\{d_2\mathfrak{S}(\varepsilon_j, \tau_2),\ j=1,2,\dots,r\Big\}$$

$$\mathcal{P}_1 = \max\Big\{P_{\mathcal{M}_j},\ j=1,2,\dots,r\Big\}, \quad \mathcal{P}_2 = \max\Big\{P_{\mathcal{N}_j},\ j=1,2,\dots,r\Big\},$$

$$\mathcal{E}_1 = \frac{\mathfrak{S}(\mathfrak{s}_0, \gamma_1+\delta_2-1)}{\Lambda_{1_0}}, \quad \mathcal{E}_2 = \frac{\mathfrak{S}(\mathfrak{s}_0, \bar{\gamma_1}+\tau_2-1)}{\Lambda_{2_0}},$$

$$\mathcal{E}_3 = \max\Bigg\{\frac{\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)},\ j=1,2,\dots,r\Bigg\},$$

$$\mathcal{E}_4 = \max\Bigg\{\Lambda_{1_j}\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1),\ j=1,2,\dots,r\Bigg\},$$

$$\mathcal{E}_5 = \max\Bigg\{\frac{\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma_1}+\tau_2-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma_2}-1)-\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma_2}-1)\mathfrak{S}(\varepsilon_j,\bar{\gamma_1}+\tau_2-1)}{\Lambda_{2_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma_2}-1)-\Lambda_{4_j}\mathfrak{S}(\varepsilon_j,\bar{\gamma_1}+\tau_2-1)},\ j=1,2,\dots,r\Bigg\},$$

$$\mathcal{E}_6 = \max\Bigg\{\Lambda_{2_j}\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma_2}-1)-\Lambda_{4_j}\mathfrak{S}(\mathfrak{s}_j,\bar{\gamma_1}+\tau_2-1),\ j=1,2,\dots,r\Bigg\}.$$

**Theorem 6** *The operator $\mathcal{F}$ is Lipschitz with constant $\varpi$. Consequently, $\mathcal{F}$ is $\sigma$-Lipschitz with the same constant $\varpi$ and satisfies the growth condition*

$$\begin{aligned}
&\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}} \le \mathcal{L}_{\mathcal{FG}}\|(\varphi,\rho)\|_{\mathcal{Z}} + \mathcal{M}_{\mathcal{FG}},\ \textit{where}\\
&\mathcal{L}_{\mathcal{FG}} = \max\big\{\mathcal{E}_1\mathcal{A}_{1_0}(l_{f_1}+l_{f_2}) + \mathcal{E}_2\mathcal{A}_{3_0}(l_{g_1}+l_{g_2}) + \mathcal{E}_1\mathcal{A}_{2_j} + \mathcal{E}_2\mathcal{A}_{4_0},\\
&\qquad (\mathcal{E}_3\mathcal{A}_{1_0} + \mathcal{E}_4\Theta_1)(l_{f_1}+l_{f_2})(\mathcal{E}_5\mathcal{A}_{3_j} + \mathcal{E}_6\Theta_3)(l_{g_1}+l_{g_2}) + \mathcal{E}_3\mathcal{A}_{2_j}\\
&\qquad + \mathcal{E}_5\mathcal{A}_{4_j} + \mathcal{E}_4L_1 + \mathcal{E}_6L_2 + \mathcal{E}_4\Theta_2 + \mathcal{E}_6\Theta_4,\ L_1+L_2\big\}\ \textit{and}\\
&\mathcal{M}_{\mathcal{FG}} = \max\big\{\mathcal{E}_1\mathcal{A}_{1_0}M_f + \mathcal{E}_2\mathcal{A}_{3_0}M_g, (\mathcal{E}_3\mathcal{A}_{1_j} + \mathcal{E}_4\Theta_1)M_f + (\mathcal{E}_5\mathcal{A}_{3_j} + \mathcal{E}_6\Theta_3)M_g\\
&\qquad + \mathcal{E}_4M_1 + \mathcal{E}_6M_2, M_1+M_2\big\}.
\end{aligned}$$

*Proof* Let $\varphi, \bar{\varphi}, \rho, \bar{\rho} \in \mathcal{Z}$. Using $(H_1)-(H_4)$, we proceed as follows:

*Case 1:* For $\varepsilon \in [a, \mathfrak{s}_0]$,

$$\begin{aligned}
&|\mathcal{F}_1(\varphi,\rho)(\varepsilon) - \mathcal{F}_1(\bar{\varphi},\bar{\rho})(\varepsilon)|\\
&\quad\le \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}|f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))\\
&\quad -f(\zeta_m,\bar{\rho}(\zeta_m),\bar{\varphi}(\zeta_m))| + I_{a^+}^{\delta_1+\delta_2;\psi}|f(b,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0)) - f(b,\bar{\rho}(\mathfrak{s}_0),\bar{\varphi}(\mathfrak{s}_0))|\\
&\quad + d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}|\varphi(\zeta_m)-\bar{\varphi}(\zeta_m)| + d_1 I_{a^+}^{\delta_2;\psi}|\varphi(\mathfrak{s}_0)-\bar{\varphi}(\mathfrak{s}_0)|\Big).
\end{aligned}$$

$$\begin{aligned}
&\|\mathcal{F}_1(\varphi,\rho)-\mathcal{F}_1(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}}\\
&\quad\le \frac{\mathfrak{S}(\mathfrak{s}_0,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_1+\delta_2+\theta_m)\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)\\
&\quad+\mathfrak{S}(\mathfrak{s}_0,\delta_1+\delta_2)\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)+d_1\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_2+\theta_m)\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\\
&\quad+d_1\mathfrak{S}(\mathfrak{s}_0,\delta_2)\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\Big)\\
&\le \mathcal{E}_1\mathcal{A}_{1_0}\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)+\mathcal{E}_1\mathcal{A}_{2_0}\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}.
\end{aligned}$$

Similarly,

$$\|\mathcal{F}_2(\varphi,\rho)-\mathcal{F}_1(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}}\le \mathcal{E}_2\mathcal{A}_{3_0}\lambda_2\big(\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}+\|\rho-\bar{\rho}\|_{\mathcal{Z}}\big)+\mathcal{E}_2\mathcal{A}_{4_0}\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}.$$

Thus,

$$\|\mathcal{F}(\varphi,\rho)-\mathcal{F}(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}}\le(\mathcal{E}_1\mathcal{A}_{1_0}\lambda_1+\mathcal{E}_2\mathcal{A}_{3_0}\lambda_2+\mathcal{E}_1\mathcal{A}_{2_0}+\mathcal{E}_2\mathcal{A}_{4_0})\big(\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}+\|\rho-\bar{\rho}\|_{\mathcal{Z}}\big).$$

*Case 2:* For $\varepsilon\in(\varepsilon_j,\mathfrak{s}_j]$,

$$\begin{aligned}
&|\mathcal{F}_1(\varphi,\rho)(\varepsilon)-\mathcal{F}_1(\bar{\varphi},\bar{\rho})(\varepsilon)|\\
&\quad\le\Big(\frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\varepsilon,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\\
&\quad\times\Big(\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}|f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))-f(\zeta_m,\bar{\rho}(\zeta_m),\bar{\varphi}(\zeta_m))|+I_{a^+}^{\delta_1+\delta_2;\psi}\\
&\quad\times|f(\mathfrak{s}_j,\rho(\mathfrak{s}_j),\varphi(\mathfrak{s}_j))-f(\mathfrak{s}_j,\bar{\rho}(\mathfrak{s}_j),\bar{\varphi}(\mathfrak{s}_j))|+d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}|\varphi(\zeta_m)-\bar{\varphi}(\zeta_m)|\\
&\quad+d_1 I_{a^+}^{\delta_2;\psi}|\varphi(\mathfrak{s}_j)-\bar{\varphi}(\mathfrak{s}_j)|\Big)+\Big(\Lambda_{1_j}\mathfrak{S}(\varepsilon,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\Big)\\
&\quad\times\Big(|\mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j))-\mathcal{M}_j(\varepsilon_j,\bar{\varphi}(\varepsilon_j))|+I_{a^+}^{\delta_1+\delta_2;\psi}|f(\varepsilon_j,\rho(\varepsilon_j),\varphi(\varepsilon_j))\\
&\quad-f(\varepsilon_j,\bar{\rho}(\varepsilon_j),\varphi(\varepsilon_j))|+d_1 I_{a^+}^{\delta_2;\psi}|\varphi(\varepsilon_j)-\bar{\varphi}(\varepsilon_j)\Big).
\end{aligned}$$

$$\begin{aligned}
&\|\mathcal{F}_1(\varphi,\rho)-\mathcal{F}_1(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}}\\
&\quad\le\Big(\frac{\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\\
&\quad\times\Big(\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_1+\delta_2+\theta_m)\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)+\mathfrak{S}(\mathfrak{s}_j,\delta_1+\delta_2)\\
&\quad\times\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)+d_1\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_2+\theta_m)\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}+d_1\mathfrak{S}(\mathfrak{s}_j,\delta_2)\\
&\quad\times\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\Big)+\Big(\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)\Lambda_{1_j}-\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)\Lambda_{3_j}\Big)\Big(P_{\mathcal{M}_j}\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\\
&\quad+\mathfrak{S}(\varepsilon_j,\delta_1+\delta_2)\lambda_1\big(\|\rho-\bar{\rho}\|_{\mathcal{Z}}+\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\big)+d_1\mathfrak{S}(\varepsilon_j,\delta_2)\|\varphi-\bar{\varphi}\|_{\mathcal{Z}}\Big)
\end{aligned}$$

$$\leq (\mathcal{E}_3\mathcal{A}_{1_j} + \mathcal{E}_4\Theta_1)\lambda_1\big(\|\rho - \bar{\rho}\|_{\mathcal{Z}} + \|\varphi - \bar{\varphi}\|_{\mathcal{Z}}\big) + (\mathcal{E}_3\mathcal{A}_{2_j} + \mathcal{E}_4\mathcal{P}_1 + \mathcal{E}_4\Theta_2)\|\varphi - \bar{\varphi}\|_{\mathcal{Z}}.$$

Similarly,

$$\begin{aligned}\|\mathcal{F}_2(\varphi,\rho) - \mathcal{F}_2(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} &\leq (\mathcal{E}_5\mathcal{A}_{3_j} + \mathcal{E}_6\Theta_3)\lambda_2\big(\|\varphi - \bar{\varphi}\|_{\mathcal{Z}} + \|\rho - \bar{\rho}\|_{\mathcal{Z}}\big) + (\mathcal{E}_5\mathcal{A}_{4_j} \\ &\quad + \mathcal{E}_6\mathcal{P}_2 + \mathcal{E}_6\Theta_4)\|\rho - \bar{\rho}\|_{\mathcal{Z}}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{F}(\varphi,\rho) - \mathcal{F}(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} &\leq (\mathcal{E}_3\mathcal{A}_{1_j}\lambda_1 + \mathcal{E}_5\mathcal{A}_{3_j}\lambda_2 + \mathcal{E}_4\Theta_1\lambda_1 + \mathcal{E}_6\Theta_3\lambda_2 + \mathcal{E}_3\mathcal{A}_{2_j} + \mathcal{E}_5\mathcal{A}_{4_j} \\ &\quad + \mathcal{E}_4\mathcal{P}_1 + \mathcal{E}_6\mathcal{P}_2 + \mathcal{E}_4\Theta_2 + \mathcal{E}_6\Theta_4)\big(\|\varphi - \bar{\varphi}\|_{\mathcal{Z}} + \|\rho - \bar{\rho}\|_{\mathcal{Z}}\big).\end{aligned}$$

*Case 3:* For $\varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]$,

$$\|\mathcal{F}_1(\varphi,\rho) - \mathcal{F}_1(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} \leq P_{\mathcal{M}_j}\|\varphi - \bar{\varphi}\|_{\mathcal{Z}}, \ \ \|\mathcal{F}_2(\varphi,\rho) - \mathcal{F}_2(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} \leq P_{\mathcal{N}_j}\|\rho - \bar{\rho}\|_{\mathcal{Z}}.$$

Thus,

$$\|\mathcal{F}(\varphi,\rho) - \mathcal{F}(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} \leq \big(\mathcal{P}_1 + \mathcal{P}_2\big)\big(\|\varphi - \bar{\varphi}\|_{\mathcal{Z}} + \|\rho - \bar{\rho}\|_{\mathcal{Z}}\big).$$

Consequently,

$$\begin{aligned}&\|\mathcal{F}(\varphi,\rho) - \mathcal{F}(\bar{\varphi},\bar{\rho})\|_{\mathcal{Z}} \leq \varpi\big(\|\varphi - \bar{\varphi}\|_{\mathcal{Z}} + \|\rho - \bar{\rho}\|_{\mathcal{Z}}\big), \text{ where} \\ &\varpi = \max\Big\{\mathcal{E}_1\mathcal{A}_{1_0}\lambda_1 + \mathcal{E}_2\mathcal{A}_{3_0}\lambda_2 + \mathcal{E}_1\mathcal{A}_{2_0} + \mathcal{E}_2\mathcal{A}_{4_0},\ \mathcal{E}_3\mathcal{A}_{1_j}\lambda_1 + \mathcal{E}_5\mathcal{A}_{3_j}\lambda_2 + \mathcal{E}_4\Theta_1\lambda_1 \\ &\quad + \mathcal{E}_6\Theta_3\lambda_2 + \mathcal{E}_3\mathcal{A}_{2_j} + \mathcal{E}_5\mathcal{A}_{4_j} + \mathcal{E}_4\mathcal{P}_1 + \mathcal{E}_6\mathcal{P}_2 + \mathcal{E}_4\Theta_2 + \mathcal{E}_6\Theta_4,\ \mathcal{P}_1 + \mathcal{P}_2\Big\}.\end{aligned}$$

Thus, $\mathcal{F}$ is Lipschitz with constant $\varpi$.

According to Proposition 2, $\mathcal{F}$ is $\sigma$-Lipschitz with constant $\varpi$.

Next, we proceed to derive the growth condition.

*Case 1:* For $\varepsilon \in [a, \mathfrak{s}_0]$,

$$\begin{aligned}&|\mathcal{F}_1(\varphi,\rho)(\varepsilon)| \\ &\quad\leq \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}|f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))| \\ &\qquad + I_{a^+}^{\delta_1+\delta_2;\psi}|f(\mathfrak{s}_0,\rho(\mathfrak{s}_0),\varphi(\mathfrak{s}_0))| + d_1\sum_{m=1}^{p}\lambda_m I_{a^+}^{\delta_2+\theta_m;\psi}|\varphi(\zeta_m)| + d_1 I_{a^+}^{\delta_2;\psi}|\varphi(\mathfrak{s}_0)|\Big).\end{aligned}$$

$$\begin{aligned}&\|\mathcal{F}_1(\varphi,\rho)\|_{\mathcal{Z}} \\ &\quad\leq \frac{\mathfrak{S}(\mathfrak{s}_0,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_1+\delta_2+\theta_m) + \mathfrak{S}(\mathfrak{s}_0,\delta_1+\delta_2)\Big)(l_{f_1}\|\rho\|_{\mathcal{Z}} \\ &\qquad + l_{f_2}\|\varphi\|_{\mathcal{Z}} + M_f) + \frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)}{\Lambda_{1_0}}\Big(d_1\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_2+\theta_m)\end{aligned}$$

$$
\begin{aligned}
&\quad + d_1\mathfrak{S}(\mathfrak{s}_0,\delta_2)\Big)\|\varphi\|_{\mathcal{Z}}\\
&\le \big(\mathcal{E}_1\mathcal{A}_{1_0}l_{f_1}+\mathcal{E}_1\mathcal{A}_{2_0}\big)\|\rho\|_{\mathcal{Z}}+\mathcal{E}_1\mathcal{A}_{1_0}l_{f_2}\|\varphi\|_{\mathcal{Z}}+\mathcal{E}_1\mathcal{A}_{1_0}M_f.
\end{aligned}
$$

Similarly,

$$
\|\mathcal{F}_2(\varphi,\rho)\|_{\mathcal{Z}}\le(\mathcal{E}_2\mathcal{A}_{3_0}l_{g_1}+\mathcal{E}_2\mathcal{A}_{4_0})\|\varphi\|_{\mathcal{Z}}+\mathcal{E}_2\mathcal{A}_{3_0}l_{g_2}\|\rho\|_{\mathcal{Z}}+\mathcal{E}_2\mathcal{A}_{3_0}M_g.
$$

Thus,

$$
\begin{aligned}
\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}}\le&(\mathcal{E}_1\mathcal{A}_{1_0}(l_{f_1}+l_{f_2})+\mathcal{E}_2\mathcal{A}_{3_0}(l_{g_1}+l_{g_2})+\mathcal{E}_1\mathcal{A}_{2_0}+\mathcal{E}_2\mathcal{A}_{4_0})\big(\|\varphi\|_{\mathcal{Z}}+\|\rho\|_{\mathcal{Z}}\big)\\
&+\mathcal{E}_1\mathcal{A}_{1_0}M_f+\mathcal{E}_2\mathcal{A}_{3_0}M_g.
\end{aligned}
$$

*Case 2:* For $\varepsilon\in(\varepsilon_j,\mathfrak{s}_j]$,

$$
\begin{aligned}
&|\mathcal{F}_1(\varphi,\rho)(\varepsilon)|\\
&\le\Big(\frac{\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\varepsilon,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\Big(\sum_{m=1}^{p}\lambda_m\\
&\quad\times I_{a^+}^{\delta_1+\delta_2+\theta_m;\psi}|f(\zeta_m,\rho(\zeta_m),\varphi(\zeta_m))|+I_{a^+}^{\delta_1+\delta_2;\psi}|f(\mathfrak{s}_j,\rho(\mathfrak{s}_j),\varphi(\mathfrak{s}_j))|+d_1\sum_{m=1}^{p}\lambda_m\\
&\quad\times I_{a^+}^{\delta_2+\theta_m;\psi}|\varphi(\zeta_m)|+d_1I_{a^+}^{\delta_2;\psi}|\varphi(\mathfrak{s}_j)|\Big)+\Big(\Lambda_{1_j}\mathfrak{S}(\varepsilon,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon,\gamma_1+\delta_2-1)\Big)\\
&\quad\times\Big(|\mathcal{M}_j(\varepsilon_j,\varphi(\varepsilon_j))|+I_{a^+}^{\delta_1+\delta_2;\psi}|f(\varepsilon_j,\rho(\varepsilon_j),\varphi(\varepsilon_j))|+d_1I_{a^+}^{\delta_2;\psi}|\varphi(\varepsilon_j)-\bar{\varphi}(\varepsilon_j)\Big).\\
&\|\mathcal{F}_1(\varphi,\rho)\|_{\mathcal{Z}}\\
&\le\Big(\frac{\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}{\Lambda_{1_j}\mathfrak{S}(\varepsilon_j,\gamma_2-1)-\Lambda_{3_j}\mathfrak{S}(\varepsilon_j,\gamma_1+\delta_2-1)}\Big)\\
&\quad\times\Big[\Big(\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_1+\delta_2+\theta_m)+\mathfrak{S}(\mathfrak{s}_j,\delta_1+\delta_2)\Big)(l_{f_1}\|\rho\|_{\mathcal{Z}}+l_{f_2}\|\varphi\|_{\mathcal{Z}}+M_f)\\
&\quad+\Big(d_1\sum_{m=1}^{p}\lambda_m\mathfrak{S}(\zeta_m,\delta_2+\theta_m)+d_1\mathfrak{S}(\mathfrak{s}_j,\delta_2)\Big)\|\varphi\|_{\mathcal{Z}}\Big]\\
&\quad+\Big(\mathfrak{S}(\mathfrak{s}_j,\gamma_2-1)\Lambda_{1_j}-\mathfrak{S}(\mathfrak{s}_j,\gamma_1+\delta_2-1)\Lambda_{3_j}\Big)\Big(L_1\|\varphi\|_{\mathcal{Z}}+M_1\\
&\quad+\mathfrak{S}(\varepsilon_j,\delta_1+\delta_2)(l_{f_1}\|\rho\|_{\mathcal{Z}}+l_{f_2}\|\varphi\|_{\mathcal{Z}}+M_f)+d_1\mathfrak{S}(\varepsilon_j,\delta_2)\|\varphi\|_{\mathcal{Z}}\Big)\\
&\le(\mathcal{E}_3\mathcal{A}_{1_j}+\mathcal{E}_4\Theta_1)(l_{f_1}+l_{f_2})\big(\|\rho\|_{\mathcal{Z}}+\|\varphi\|_{\mathcal{Z}}\big)+(\mathcal{E}_3\mathcal{A}_{2_j}+\mathcal{E}_4L_1\\
&\quad+\mathcal{E}_4\Theta_2)\|\varphi\|_{\mathcal{Z}}+(\mathcal{E}_3\mathcal{A}_{1_j}+\mathcal{E}_4\Theta_1)M_f+\mathcal{E}_4M_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|\mathcal{F}_2(\varphi,\rho)\|_{\mathcal{Z}}\le&(\mathcal{E}_5\mathcal{A}_{3_j}+\mathcal{E}_6\Theta_3)(l_{g_1}+l_{g_2})\big(\|\varphi\|_{\mathcal{Z}}+\|\rho\|_{\mathcal{Z}}\big)+(\mathcal{E}_5\mathcal{A}_{4_j}\\
&+\mathcal{E}_6L_2+\mathcal{E}_6\Theta_4)\|\rho\|_{\mathcal{Z}}+(\mathcal{E}_5\mathcal{A}_{3_j}+\mathcal{E}_6\Theta_3)M_g+\mathcal{E}_6M_2.
\end{aligned}
$$

Thus,

$$\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}} \leq \Big((\mathcal{E}_3\mathcal{A}_{1_j} + \mathcal{E}_4\Theta_1)(l_{f_1} + l_{f_2})(\mathcal{E}_5\mathcal{A}_{3_j} + \mathcal{E}_6\Theta_3)(l_{g_1} + l_{g_2}) + \mathcal{E}_3\mathcal{A}_{2_j} + \mathcal{E}_5\mathcal{A}_{4_j} + \mathcal{E}_4L_1 + \mathcal{E}_6L_2 + \mathcal{E}_4\Theta_2 + \mathcal{E}_6\Theta_4\Big)\big(\|\varphi\|_{\mathcal{Z}} + \|\rho\|_{\mathcal{Z}}\big) + (\mathcal{E}_3\mathcal{A}_{1_j} + \mathcal{E}_4\Theta_1)M_f + (\mathcal{E}_5\mathcal{A}_{3_j} + \mathcal{E}_6\Theta_3)M_g + \mathcal{E}_4M_1 + \mathcal{E}_6M_2.$$

*Case 3:* For $\varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]$,

$$\|\mathcal{F}_1(\varphi,\rho)\|_{\mathcal{Z}} \leq L_1\|\varphi\|_{\mathcal{Z}} + M_1, \qquad \|\mathcal{F}_2(\varphi,\rho)\|_{\mathcal{Z}} \leq L_2\|\rho\|_{\mathcal{Z}} + M_2.$$

Thus,

$$\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}} \leq \big(L_1 + L_2\big)\big(\|\varphi\|_{\mathcal{Z}} + \|\rho\|_{\mathcal{Z}}\big) + M_1 + M_2.$$

Consequently,

$$\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}} \leq \mathcal{L}_{\mathcal{F}\mathcal{G}}\|(\varphi,\rho)\|_{\mathcal{Z}} + \mathcal{M}_{\mathcal{F}\mathcal{G}}.$$

Hence the growth condition is satisfied. □

**Theorem 7** *The operator $\mathcal{G}$ is continuous and satisfies the following growth condition*:

$$\|\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}} \leq \mathcal{L}^*_{\mathcal{F}\mathcal{G}}\|(\varphi,\rho)\|_{\mathcal{Z}} + \mathcal{M}^*_{\mathcal{F}\mathcal{G}},\ \textit{where}$$

$$\mathcal{L}^*_{\mathcal{F}\mathcal{G}} = \max\{\mathfrak{S}(\varepsilon,\delta_1+\delta_2)(l_{f_1} + l_{f_2}) + \mathfrak{S}(\varepsilon,\tau_1+\tau_2)(l_{g_1} + l_{g_2}) + d_1\mathfrak{S}(\varepsilon,\delta_2) + d_2\mathfrak{S}(\varepsilon,\tau_2),\ 0\}\ \textit{and}$$

$$\mathcal{M}^*_{\mathcal{F}\mathcal{G}} = \max\{\mathfrak{S}(\varepsilon,\delta_1+\delta_2)M_f + \mathfrak{S}(\varepsilon,\tau_1+\tau_2)M_g,\ 0\}.$$

*Proof* Consider a bounded subset $\mathcal{B}_\varsigma = \{\|(\varphi,\rho)\|_{\mathcal{Z}} \leq \varsigma : (\varphi,\rho) \in \mathcal{Z} \times \mathcal{Z}\}$.

Let $\{(\varphi_n,\rho_n)\}$ be a sequence such that $(\varphi_n,\rho_n) \longrightarrow (\varphi,\rho)$ as $n \longrightarrow \infty$ within $\mathcal{B}_\varsigma$.

To show that $\mathcal{G}$ is continuous. Let $\varphi, \bar{\varphi}, \rho, \bar{\rho} \in \mathcal{B}_\varsigma$.

For $\varepsilon \in [a, \mathfrak{s}_0]$ and $\varepsilon \in (\varepsilon_j, \mathfrak{s}_j]$,

$$|\mathcal{G}_1(\varphi_n,\rho_n)(\varepsilon) - \mathcal{G}_1(\varphi,\rho)(\varepsilon)| \leq I_{a^+}^{\delta_1+\delta_2;\psi}|f(\varepsilon,\rho_n(\varepsilon),\varphi_n(\varepsilon)) - f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))| + d_1 I_{a^+}^{\delta_2;\psi}|\varphi_n(\varepsilon) - \varphi(\varepsilon)|.$$

Using $(H_1) - (H_4)$, we obtain that

$$\|\mathcal{G}_1(\varphi_n,\rho_n) - \mathcal{G}_1(\varphi,\rho)\|_{\mathcal{Z}} \leq \mathfrak{S}(\varepsilon,\delta_1+\delta_2)\lambda_1\big(\|\rho_n - \rho\|_{\mathcal{Z}} + \|\varphi_n - \varphi\|_{\mathcal{Z}}\big) + d_1\mathfrak{S}(\varepsilon,\delta_2)\|\varphi_n - \varphi\|_{\mathcal{Z}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Similarly,

$$\|\mathcal{G}_2(\varphi_n,\rho_n) - \mathcal{G}_2(\varphi,\rho)\|_{\mathcal{Z}}$$

$$
\begin{aligned}
&\le \mathfrak{S}(\varepsilon,\tau_1+\tau_2)\big(\|\varphi_n-\varphi\|_{\mathcal{Z}}+\|\varphi_n-\varphi\|_{\mathcal{Z}}\big)+d_2\mathfrak{S}(\varepsilon,\tau_2)\|\rho_n-\rho\|_{\mathcal{Z}}\\
&\quad\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\mathcal{G}(\varphi_n,\rho_n)-\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}} \le{}& \Big(\mathfrak{S}(\varepsilon,\delta_1+\delta_2)\lambda_1+\mathfrak{S}(\varepsilon,\tau_1+\tau_2)\lambda_2\\
&+d_1\mathfrak{S}(\varepsilon,\delta_2)+d_2\mathfrak{S}(\varepsilon,\tau_2)\Big)\|(\varphi_n,\rho_n)-(\varphi,\rho)\|_{\mathcal{Z}}\\
&\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}
$$

For $\varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j]$,

$$
\|\mathcal{G}(\varphi_n,\rho_n)(\varepsilon)-\mathcal{G}(\varphi,\rho)(\varepsilon)\|_{\mathcal{Z}}=0.
$$

Consequently,

$$
\begin{aligned}
\|\mathcal{G}(\varphi_n,\rho_n)-\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}} \le{}& \max\{\mathfrak{S}(\varepsilon,\delta_1+\delta_2)\lambda_1+\mathfrak{S}(\varepsilon,\tau_1+\tau_2)\lambda_2\\
&+d_1\mathfrak{S}(\varepsilon,\delta_2)+d_2\mathfrak{S}(\varepsilon,\tau_2),0\}\|(\varphi_n,\rho_n)-(\varphi,\rho)\|_{\mathcal{Z}}\\
&\longrightarrow 0 \text{ as } n\longrightarrow\infty.
\end{aligned}
$$

$\Longrightarrow \mathcal{G}$ is continuous.

For $\varepsilon\in[a,\mathfrak{s}_0]$ and $\varepsilon\in(\varepsilon_j,\mathfrak{s}_j]$,

$$
|\mathcal{G}_1(\varphi,\rho)|\le I_{a^+}^{\delta_1+\delta_2;\psi}|f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))|+d_1 I_{a^+}^{\delta_2;\psi}|\varphi(\varepsilon)|.
$$

Using $(H_1)-(H_4)$, we obtain

$$
\|\mathcal{G}_1(\varphi,\rho)\|_{\mathcal{Z}}\le\mathfrak{S}(\varepsilon,\delta_1+\delta_2)(l_{f_1}\|\rho\|_{\mathcal{Z}}+l_{f_2}\|\varphi\|_{\mathcal{Z}}+M_f)+d_1\mathfrak{S}(\varepsilon,\delta_2)\|\varphi\|_{\mathcal{Z}}.
$$

Similarly,

$$
\|\mathcal{G}_2(\varphi,\rho)\|_{\mathcal{Z}}\le\mathfrak{S}(\varepsilon,\tau_1+\tau_2)(l_{g_1}\|\varphi\|_{\mathcal{Z}}+l_{g_2}\|\rho\|_{\mathcal{Z}}+M_g)+d_2\mathfrak{S}(\varepsilon,\tau_2)\|\rho\|_{\mathcal{Z}}.
$$

Thus,

$$
\begin{aligned}
\|\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}}\le{}&\Big(\mathfrak{S}(\varepsilon,\delta_1+\delta_2)(l_{f_1}+l_{f_2})+\mathfrak{S}(\varepsilon,\tau_1+\tau_2)(l_{g_1}+l_{g_2})+d_1\mathfrak{S}(\varepsilon,\delta_2)\\
&+d_2\mathfrak{S}(\varepsilon,\tau_2)\Big)\|(\varphi,\rho)\|_{\mathcal{Z}}+\mathfrak{S}(\varepsilon,\delta_1+\delta_2)M_f+\mathfrak{S}(\varepsilon,\tau_1+\tau_2)M_g.
\end{aligned}
$$

For $\varepsilon\in(\mathfrak{s}_{j-1},\varepsilon_j]$,

$$
\|\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}}=0.
$$

Consequently,

$$
\|\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}}\le\mathcal{L}^*_{\mathcal{F}\mathcal{G}}\|(\varphi,\rho)\|_{\mathcal{Z}}+\mathcal{M}^*_{\mathcal{F}\mathcal{G}}.
$$

Hence the growth condition is satisfied. □

**Theorem 8** *The operator $\mathcal{G}$ is compact.*

*Proof* Let $\chi \subset \mathcal{B}_\varsigma$ be bounded and $\{(\varphi_n, \rho_n)\}$ be a sequence in $\chi$.

From the growth condition of $\mathcal{G}$, it is clear that $\mathcal{G}\chi$ is uniformly bounded in $\mathcal{Z} \times \mathcal{Z}$.

To show that $\mathcal{G}$ is equicontinuous.

Let $a \le \varepsilon_1 \le \varepsilon_2 \le b$.

For $\varepsilon \in [a, \mathfrak{s}_0]$ and $\varepsilon \in (\varepsilon_j, \mathfrak{s}_j]$, we obtain

$$\begin{aligned}
&|\mathcal{G}_1(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}_1(\varphi_n,\rho_n)(\varepsilon_1)| \\
&\quad \le \frac{1}{\Gamma(\delta_1+\delta_2+1)}\Big[2(\psi(\varepsilon_2)-\psi(\varepsilon_1))^{\delta_1+\delta_2} + (\psi(\varepsilon_2)-\psi(a))^{\delta_1+\delta_2} \\
&\qquad - (\psi(\varepsilon_1)-\psi(a))^{\delta_1+\delta_2}\Big](l_{f_1}\|\rho\|_{\mathcal{Z}} + l_{f_2}\|\varphi\|_{\mathcal{Z}} + M_f) + \frac{1}{\Gamma(\delta_2+1)}\Big[2(\psi(\varepsilon_2) \\
&\qquad - \psi(\varepsilon_1))^{\delta_2} + (\psi(\varepsilon_2)-\psi(a))^{\delta_2} - (\psi(\varepsilon_1)-\psi(a))^{\delta_2}\Big]\varsigma.
\end{aligned}$$

Using $(H_1) - (H_4)$, we get

$$\|\mathcal{G}_1(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}_1(\varphi_n,\rho_n)(\varepsilon_1)\|_{\mathcal{Z}} \longrightarrow 0 \text{ as } \varepsilon_2 \longrightarrow \varepsilon_1.$$

Similarly,

$$\|\mathcal{G}_2(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}_2(\varphi_n,\rho_n)(\varepsilon_1)\|_{\mathcal{Z}} \longrightarrow 0 \text{ as } \varepsilon_2 \longrightarrow \varepsilon_1.$$

Thus,

$$\|\mathcal{G}(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}(\varphi_n,\rho_n)(\varepsilon_1)\|_{\mathcal{Z}} \longrightarrow 0 \text{ as } \varepsilon_2 \longrightarrow \varepsilon_1.$$

For $\varepsilon \in (\mathfrak{s}_{j-1}, \varepsilon_j]$,

$$\|\mathcal{G}(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}(\varphi,\rho)(\varepsilon_1)\|_{\mathcal{Z}} = 0.$$

Consequently,

$$\|\mathcal{G}(\varphi_n,\rho_n)(\varepsilon_2) - \mathcal{G}(\varphi_n,\rho_n)(\varepsilon_1)\|_{\mathcal{Z}} \longrightarrow 0 \text{ as } \varepsilon_2 \longrightarrow \varepsilon_1.$$

This implies that $\mathcal{G}$ is equicontinuous. By the Arzela–Ascoli theorem [27], $\mathcal{G}$ is compact.

According to Proposition 3, $\mathcal{G}$ is $\sigma$-Lipschitz with constant 0. □

**Theorem 9** *If $(H_1) - (H_4)$ hold and $\varpi = \max\{\Omega, \Upsilon, \Delta\} \in [0,1)$, where*

$$\begin{aligned}
&\Omega = \mathcal{E}_1\mathcal{A}_{1_0}\lambda_1 + \mathcal{E}_2\mathcal{A}_{3_0}\lambda_2 + \mathcal{E}_1\mathcal{A}_{2_0} + \mathcal{E}_2\mathcal{A}_{4_0}, \\
&\Upsilon = \mathcal{E}_3\mathcal{A}_{1_j}\lambda_1 + \mathcal{E}_5\mathcal{A}_{3_j}\lambda_2 + \mathcal{E}_4\Theta_1\lambda_1 + \mathcal{E}_6\Theta_3\lambda_2 + \mathcal{E}_3\mathcal{A}_{2_j} + \mathcal{E}_5\mathcal{A}_{4_j} + \mathcal{E}_4\mathcal{P}_1 \\
&\qquad + \mathcal{E}_6\mathcal{P}_2 + \mathcal{E}_4\Theta_2 + \mathcal{E}_6\Theta_4 \text{ and} \\
&\Delta = \mathcal{P}_1 + \mathcal{P}_2,
\end{aligned}$$

*then the coupled system has at least one solution* $(\varphi,\rho)\in\mathcal{Z}\times\mathcal{Z}$ *and the solution set of* (1) *is bounded in* $\mathcal{Z}\times\mathcal{Z}$.

*Proof* We observe that $\mathcal{F}$ is $\sigma$-Lipschitz with constant $\varpi\in[0,1)$ from Theorem 6 and $\mathcal{G}$ is $\sigma$-Lipschitz with constant 0 from Theorem 7. By Proposition 1 and Definition 1, $\mathcal{J}$ is a strict $\sigma$-contraction with constant $\varpi$. Hence $\mathcal{J}$ is $\sigma$-condensing.

Now consider the set

$$\mathcal{S}=\Big\{(\varphi,\rho)\in\mathcal{Z}\times\mathcal{Z}:\text{there exists }\omega\in[0,1],\ (\varphi,\rho)=\omega\,\mathcal{J}(\varphi,\rho)\Big\}.$$

We need to show that $\mathcal{S}$ is bounded in $\mathcal{Z}\times\mathcal{Z}$.

Let $(\varphi,\rho)\in\mathcal{S}$. Then from the growth conditions of Theorem 6 and Theorem 7, we have

$$(\varphi,\rho)=\omega\,\mathcal{J}(\varphi,\rho)=\omega\big(\mathcal{F}(\varphi,\rho)+\mathcal{G}(\varphi,\rho)\big),$$

and

$$\begin{aligned}\|(\varphi,\rho)\|_{\mathcal{Z}}=&\omega\|\mathcal{J}(\varphi,\rho)\|_{\mathcal{Z}}\\ \leq&\omega\Big(\|\mathcal{F}(\varphi,\rho)\|_{\mathcal{Z}}+\|\mathcal{G}(\varphi,\rho)\|_{\mathcal{Z}}\Big)\\ \leq&\omega\Big(\mathcal{L}_{\mathcal{FG}}\|(\varphi,\rho)\|_{\mathcal{Z}}+\mathcal{M}_{\mathcal{FG}}+\mathcal{L}^{*}_{\mathcal{FG}}\|(\varphi,\rho)\|_{\mathcal{Z}}+\mathcal{M}^{*}_{\mathcal{FG}}\Big)\\ \leq&\omega\Big(\mathcal{L}_{\mathcal{FG}}\|(\varphi,\rho)\|_{\mathcal{Z}}+\mathcal{L}^{*}_{\mathcal{FG}}\|(\varphi,\rho)\|_{\mathcal{Z}}\Big)+\omega\Big(\mathcal{M}_{\mathcal{FG}}+\mathcal{M}^{*}_{\mathcal{FG}}\Big).\end{aligned}$$

Thus, $\mathcal{S}$ is bounded in $\mathcal{Z}\times\mathcal{Z}$. According to Theorem 4, there exists $\varsigma>0$ such that $\mathcal{S}\subset\mathcal{B}_{\varsigma}(0)$, hence

$$D(I-\omega\,\mathcal{J},\mathcal{B}_{\varsigma}(0),0)=1,\ \text{for all }\omega\in[0,1].$$

Therefore, $\mathcal{J}$ has at least one fixed point, and thus the coupled system (1) has at least one solution. □

## 5 Example

In this section, we provide an example to demonstrate our results. The following boundary value problem finds applications in various fields where complex dynamics, memory effects, and non-local interactions play a significant role. A few examples include population dynamics with delays, chemical reaction networks, epidemiological models, financial systems with delayed reactions, and many more.

In particular, we can explain the following boundary value problem in the context of a biological system. The state variables $\varphi(\varepsilon)$ and $\rho(\varepsilon)$ may represent the concentration of a hormone in the bloodstream and the concentration of a cytokine produced by immune cells in response to the hormone, respectively. The sequential derivatives corresponding to $\varphi(\varepsilon)$ help to capture the complex temporal behaviour and memory effects of hormone levels in the bloodstream, while the sequential derivatives corresponding to $\rho(\varepsilon)$ model the delayed and memory-dependent response of cytokine production. The functions $\mathcal{M}_j$ and $\mathcal{N}_j$ describe gradual changes in hormone and cytokine levels, respectively, over certain intervals. This model could be used to predict how the hormone and cytokine levels evolve over time, to design medical treatments by understanding the delayed responses and interactions between hormones and cytokines, and to analyse biological rhythms and cycles, where the past states significantly influence future behaviour.

Investigating the existence of a solution is essential for ensuring the mathematical validity, practical applicability, and predictive capability of the model.

*Example* 1 Let us consider the coupled system of $\psi$-Hilfer sequential fractional boundary value problem with non-instantaneous impulses

$$\begin{cases} {}^{H}D_{a^+}^{\frac{7}{9},\frac{1}{4}:e^{\frac{\varepsilon-12}{16}}}\left({}^{H}D_{a^+}^{\frac{4}{5},\frac{2}{3}:e^{\frac{\varepsilon-12}{16}}}+\frac{1}{7}\right)\varphi(\varepsilon)=f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon)),\ \varepsilon\in[\frac{1}{4},\frac{3}{4}]\bigcup(1,\frac{3}{2}],\\ {}^{H}D_{a^+}^{\frac{5}{8},\frac{1}{2}:e^{\frac{\varepsilon-12}{16}}}\left({}^{H}D_{a^+}^{\frac{9}{10},\frac{3}{4}:e^{\frac{\varepsilon-12}{16}}}+\frac{1}{9}\right)\rho(\varepsilon)=g(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon)),\ \varepsilon\in[\frac{1}{4},\frac{3}{4}]\bigcup(1,\frac{3}{2}],\\ \varphi(\varepsilon)=\mathcal{M}_j(\varepsilon,\varphi(\varepsilon)),\quad \rho(\varepsilon)=\mathcal{N}_j(\varepsilon,\rho(\varepsilon)),\quad \varepsilon\in(\frac{3}{4},1],\\ \varphi(\frac{1}{4})=0,\ \varphi(\mathfrak{s}_j)=\sum_{m=1}^{3}(\frac{3m}{m^2+9})I_{a^+}^{\frac{m+4}{7}:e^{\frac{\varepsilon-12}{16}}}\varphi(\frac{m}{5}),\\ \rho(\frac{1}{4})=0,\ \rho(\mathfrak{s}_j)=\sum_{n=1}^{2}(\frac{n}{n+5})^2 I_{a^+}^{\frac{n^2}{5}:e^{\frac{\varepsilon-12}{16}}}\rho(\frac{n}{3}),\ j=0,1. \end{cases} \tag{10}$$

where

$$\begin{aligned} f(\varepsilon,\rho(\varepsilon),\varphi(\varepsilon))&=\frac{1}{16(1+\varepsilon^2)}\sin|\rho(\varepsilon)|+\frac{\cos\varepsilon}{3+\varepsilon}\left(\frac{1+|\varphi(\varepsilon)|}{4+|\varphi(\varepsilon)|}\right)+\frac{1}{11},\\ g(\varepsilon,\varphi(\varepsilon),\rho(\varepsilon))&=\frac{\sin^2|\varphi(\varepsilon)|}{12\sqrt{5+\varepsilon^2}}+\frac{|\rho(\varepsilon)|}{(1+\varepsilon)(12+|\rho(\varepsilon)|)}+\frac{1}{14},\\ \mathcal{M}_j(\varepsilon,\varphi(\varepsilon))&=\sqrt{\varepsilon^2+3}\left(\frac{|\varphi(\varepsilon)|}{20+|\varphi(\varepsilon)|}\right)+\frac{1}{11},\quad \mathcal{N}_j(\varepsilon,\rho(\varepsilon))=\frac{2\varepsilon}{\varepsilon+1}\left(\frac{|\rho(\varepsilon)|}{16+|\rho(\varepsilon)|}\right)+\frac{1}{18}. \end{aligned}$$

Here,

$$\begin{aligned} &\delta_1=\frac{7}{9},\ \delta_2=\frac{4}{5},\ \alpha_1=\frac{1}{4},\ \alpha_2=\frac{2}{3}\ \tau_1=\frac{5}{8},\ \tau_2=\frac{9}{10},\ \beta_1=\frac{1}{2},\ \beta_2=\frac{3}{4},\ d_1=\frac{1}{7},\\ &d_2=\frac{1}{9},\ \lambda_m=\frac{3m}{m^2+9},\ \theta_m=\frac{m+4}{7},\ \zeta_m=\frac{m}{5},\ \mu_n=\left(\frac{n}{n+5}\right)^2,\ \eta_n=\frac{n^2}{5},\ \xi_n=\frac{n}{3}\\ &a=\varepsilon_0=\frac{1}{4}<\mathfrak{s}_0=\frac{3}{4}<\varepsilon_1=1<\mathfrak{s}_1=b=\frac{3}{2}. \end{aligned}$$

We calculate:

$$\begin{aligned} &\gamma_1\approx 0.8333,\ \bar{\gamma}_1\approx 0.9333,\ \gamma_2\approx 0.8125,\ \bar{\gamma}_2\approx 0.9750,\ \Lambda_{1_0}\approx 0.0781,\ \Lambda_{1_1}\approx 0.1421,\\ &\Lambda_{2_0}\approx 0.0555,\ \Lambda_{2_1}\approx 0.1086,\ \Lambda_{3_0}\approx 1.2463, \Lambda_{3_1}\approx 1.1693,\ \Lambda_{4_0}\approx 1.0803,\\ &\Lambda_{4_1}\approx 1.0549,\ \mathcal{A}_{1_0}\approx 0.0029,\ \mathcal{A}_{1_1}\approx 0.0128,\ \mathcal{A}_{2_0}\approx 0.1133,\ \mathcal{A}_{2_1}\approx 0.2403,\\ &\mathcal{A}_{3_0}\approx 0.0025,\ \mathcal{A}_{3_1}\approx 0.0105,\ \mathcal{A}_{4_0}\approx 0.0481,\ \mathcal{A}_{4_1}\approx 0.1122,\ \Theta_1\approx 0.0019,\\ &\Theta_2\approx 0.0075,\ \Theta_3\approx 0.0023,\ \Theta_4\approx 0.0039,\ \mathcal{P}_1\approx 0.1000,\ \mathcal{P}_2\approx 0.0714,\\ &\mathcal{E}_1\approx 1.0078,\ \mathcal{E}_2\approx 1.0037,\ \mathcal{E}_3\approx 0.9752,\ \mathcal{E}_4\approx 0.0022,\ \mathcal{E}_5\approx 0.9796,\ \mathcal{E}_6\approx 0.0013. \end{aligned}$$

Comparing with $(H_1)$ – $(H_4)$, we observe that

$$l_{f_1} = \frac{1}{17},\ l_{f_2} = \frac{1}{13},\ M_f = \frac{1}{11},\ l_{g_1} = \frac{1}{27},\ l_{g_2} = \frac{1}{15},\ M_g = \frac{1}{14},\ L_1 = \frac{1}{10},$$
$$M_1 = \frac{1}{11},\ L_2 = \frac{1}{14},\ M_2 = \frac{1}{18},\ \lambda_1 = \frac{1}{13},\ \lambda_2 = \frac{1}{15},\ P_{\mathcal{M}_1} = \frac{1}{10},\ P_{\mathcal{N}_1} = \frac{1}{14}.$$

We determine $\varpi = \max\{\Omega,\ \Upsilon,\ \Delta\} = 0.3462 < 1$.

This implies that $\mathcal{F}$ is $\sigma$-Lipschitz with constant 0.3462, and thus, $\mathcal{G}$ is $\sigma$-Lipschitz with constant zero. Consequently, $\mathcal{J}$ is $\sigma$-Lipschitz with constant 0.3462.

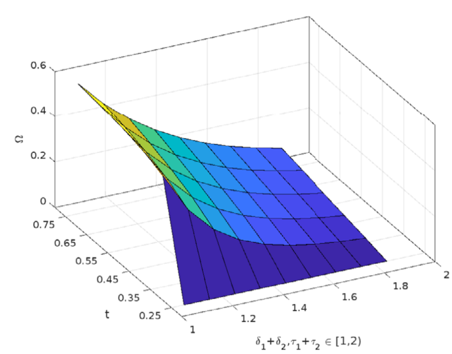
Since $\mathcal{S} = \{(\varphi,\rho) \in \mathcal{Z}\times\mathcal{Z} : \text{there exist } \omega \in [0,1],\ (\varphi,\rho) = \omega\,\mathcal{J}(\varphi,\rho)\}$, by calculation, we obtain $\|(\varphi,\rho)\| \approx 0.0586$.

Then $\mathcal{S}$ is bounded, and by Theorem 9, the BVP (10) has at least one solution.

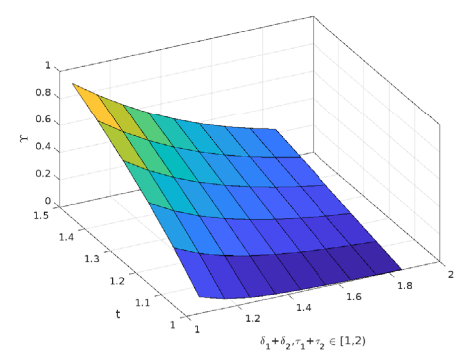
Moreover, the numerical results of $\Omega$ for all $\varepsilon \in [\frac{1}{4}, \frac{3}{4}]$ and $\Upsilon$ for all $\varepsilon \in [1, \frac{3}{2}]$ for various values of order $0 < \delta_1, \delta_2, \tau_1, \tau_2 < 1$ are obtained and are graphically presented in Fig. 1a and Fig. 1b, respectively.

We observe that $\Delta = 0.1714$ for all $\varepsilon \in (\frac{3}{4}, 1]$.

We observe that for an increase in time, $\varpi$ increases gradually and is clearly less than 1. Also, when the order increases, $\varpi$ decreases gradually. The results are shown in Table 1 and graphically presented in 2.




(a) $\Omega$ for all $\varepsilon \in (\frac{1}{4}, \frac{3}{4}]$ and order $\in [1, 2)$.




(b) $\Upsilon$ for all $\varepsilon \in (1, \frac{3}{2}]$ and order $\in [1, 2)$.

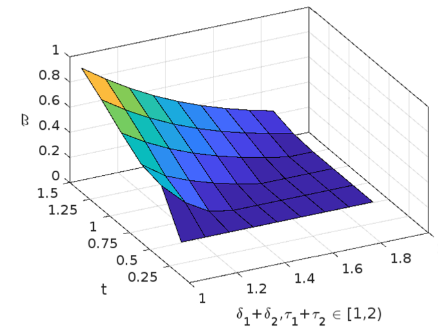
**Figure 1** The Lipschitz constant in different time intervals

**Table 1** $\varpi$ for various orders of the FDE

| $\varepsilon$ | $\varpi$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $O = \frac{21}{20}$ | $O = \frac{23}{20}$ | $O = \frac{25}{20}$ | $O = \frac{27}{20}$ | $O = \frac{29}{20}$ | $O = \frac{31}{20}$ | $O = \frac{33}{20}$ | $O = \frac{35}{20}$ | $O = \frac{37}{20}$ |
| 0.25 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 |
| 0.50 | 0.4886 | 0.3130 | 0.2035 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 | 0.1714 |
| 0.75 | 0.5238 | 0.3929 | 0.3161 | 0.2561 | 0.2085 | 0.1714 | 0.1714 | 0.1714 | 0.1714 |
| 1.00 | 0.6412 | 0.5289 | 0.4366 | 0.3608 | 0.2984 | 0.2468 | 0.2042 | 0.1714 | 0.1714 |
| 1.25 | 0.7730 | 0.6522 | 0.5482 | 0.4596 | 0.3844 | 0.3210 | 0.2676 | 0.2228 | 0.1853 |
| 1.5 | 0.8912 | 0.7653 | 0.6524 | 0.5532 | 0.4672 | 0.3932 | 0.3300 | 0.2763 | 0.2308 |

[1] $O$ denotes the order of the FDE.
[2] $\varepsilon$ denotes the time interval.
[3] $\varpi = \max\{\Omega,\ \Upsilon,\ \Delta\}$.

**Figure 2** $\varpi = \max\{\Omega, \Upsilon, \Delta\}$




## 6 Conclusion

In this paper, we investigated the coupled system of $\psi$-Hilfer sequential fractional BVPs with non-instantaneous impulses. In a piece-wise continuous space, we derived the solution of the system. On the basis of TDT, the existence results of the system were proved. An example was constructed to demonstrate the results. Additionally, a graphical analysis was carried out to verify the results.

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All authors equally contributed to the manuscript.

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**Data Availability**
No datasets were generated or analysed during the current study.

## Declarations

**Competing interests**
The authors declare no competing interests.

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