New discussion on trajectory controllability of time invariant impulsive neutral stochastic functional integrodifferential equations via noncompact semigroup

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Abstract

The purpose of this paper is to determine a new discussion on trajectory controllability (T-controllability) of time invariant impulsive neutral stochastic functional integrodifferential equations (INSFIDEs) driven by a fractional Brownian motion (fBm) via noncompact semigroup in a Hilbert space. Initially, with the help of the Hausdorff measure of noncompactness, the Mönch fixed point theorem and some inequality technique, some new criteria to guarantee the mild solution for INSFIDEs are obtained. Next, the systems T-controllability is then examined using Gronwalls inequality. An example is given to validate the results at the end. Our work extends the work of [5, 6, 10, 11].

Keywords: T-controllability, Time invariant delay, Impulsive neutral stochastic integrodifferential system, Noncompact semigroup, Resolvent operator.

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1 Introduction

As we know, there are a variety of real-world situations where instantaneous perturbations and sudden changes occur at certain moments, including those involving mechanics, electronics, telecommunications, the financial market, and other domains. We commonly refer to the phenomena as impulsive effects, in which states are often subject to abrupt and short changes in discrete moments of time, can be neglected throughout the duration of the intended process, and are defined by impulsive differential equations. The study of impulsive ordinary differential equations, impulsive partial differential equations, and impulsive fractional differential equations has increased significantly over the past few decades. The theory and applications of above differential equations seem to have been matured. However, stochastic differential equations (SDEs) have received a lot of attention in recent years from scholars. SDEs are seen as a good tool for modelling real-life processes where noises are non-negligible since they naturally arise in a wide range of applications, including economics, finance, engineering, and social sciences. Meanwhile, it is becoming extremely difficult to ignore the existence of impulsive effects. In other words, noise or impulsive disturbance cannot be avoided in either natural or artificial systems [1], and that is why we need to study the corresponding property of SDEs disturbed by impulses. Fractional Brownian motion (fBm) is a family of centred Gaussian processes with continuous sample paths indexed by the Hurst parameter $\mathscr{H} \in (0, 1)$ which possesses many outstanding features such as continuous sample paths, self-similarity and stationary increments. As we know

(i) if $\mathscr{H} \in (0, \frac{1}{2})$, it reduces as a short-memory process,

(ii) if $\mathscr{H} = \frac{1}{2}$, it is a standard Brownian motion,

(iii) if $\mathscr{H} \in (\frac{1}{2}, 1)$, it is a long-memory process.

It is easy to see that fBm is a generalization of Brownian motion, but it behaves different significantly from the standard Brownian motion. This means that it is neither a semimartingale nor a Markov process when $\mathscr{H} \neq \frac{1}{2}$. This process is beneficial as driving noise in models that have been developed for biological systems, financial markets, and telecommunications networks, among other domains (see [2, 3]). Correspondingly, fBm with the Hurst index $\mathscr{H} \in (\frac{1}{2}, 1)$, has been efficiently presented as a replacement of the standard Brownian motion in studying stochastic differential systems.

One of the envisioned characteristics of stochasticdynamical systems is controllability, which confirms that an stochasticdynamical systemsmay be directed from an arbitrary initial state to a desired arbitrary final state using a variety of specific admissible methods of control. The idea of controllability was first proposed by Kalman [4] in 1963. Numerous ideas of controllability were discovered based on the literature that was available, including

• approximate controllability - it is possible to steer any state vector arbitrarily close to another state vector.

• exact controllability - Any two state vectors could be connected by a trajectory.

• the null controllability - State vectors could be arbitrarily pointed in the direction of zero.

• T-controllability - We are looking for a control that leads the system along a specified path rather

than one that moves it from a given initial state to a desired final state.
It has been demonstrated that T-controllability is a more powerful idea than other controllability concepts. For example: For cost-effectiveness, it may be preferable to launch a rocket in orbit using a specific course and target destination and so on, which is based on T-controllability notation. For more details on T-controllability one can see the papers [12, 14, 15, 16, 17] and reference their in.

Based on the above analysis, study on the existence, uniqueness, stability and controllability of the solutions of the differential systems which disturbed by stochastic effects or impulsive effects are driven by a fBm have been heated research topics. More and more researchers have paid their attention to such problems and some interesting results have brought to our view, see [5, 6, 7, 8, 9]. More precisely, Caraballo et al. [10] considered the asymptotic behaviour of mild solutions of stochastic delay evolution equations perturbed by a fBm as follow:

$$\begin{aligned} d[\mathsf{x}(\mathsf{t})] &= [\mathbb{A}\mathsf{x}(\mathsf{t}) + \mathfrak{f}(\mathsf{t}, \mathsf{x}(\mathsf{t} - \rho(\mathsf{t})))]d\mathsf{t} + \sigma(\mathsf{t})d\omega_Q^{\mathscr{H}}(\mathsf{t}), \\ \mathsf{x}(\mathsf{t}) &= \varphi(\mathsf{t}), \ \mathsf{t} \in [-r, 0]. \end{aligned}$$

In paper [5], Boufoussi and Hajji investigated the asymptotic behaviours of mild solutions for neutral SDEs driven by a fBm with finite delay:

$$\begin{aligned} d[\mathsf{x}(\mathsf{t})] &= [\mathbb{A}\mathsf{x}(\mathsf{t}) + \mathfrak{f}(\mathsf{t}, \mathsf{x}(\mathsf{t} - \delta(\mathsf{t})))]d\mathsf{t} + \sigma(\mathsf{t}, \mathsf{x}(\mathsf{t} - \rho(\mathsf{t})))d\omega_Q^{\mathscr{H}}(\mathsf{t}), \ t \geq 0, \ t \neq t_k, \\ \mathsf{x}(\mathsf{t}) &= \varphi(\mathsf{t}), \ \mathsf{t} \in [-r, 0]. \end{aligned}$$

Next, when the considered systems encountered impulsive disturbance, Boudaoui et al. [11] considered the existence of mild solutions to stochastic impulsive evolution equations with time delays driven by fBm as follow:

$$\begin{aligned} d[\mathsf{x}(\mathsf{t})] &= [\mathbb{A}\mathsf{x}(\mathsf{t}) + \mathfrak{f}(\mathsf{t}, \mathsf{x}_\mathsf{t})]d\mathsf{t} + \sigma(\mathsf{t})d\omega_Q^{\mathscr{H}}(\mathsf{t}), \\ \Delta\mathsf{x}(\mathsf{t}_k) &= \mathsf{x}(\mathsf{t}_k^+) - \mathsf{x}(\mathsf{t}_k^-) = I_k(\mathsf{x}(\mathsf{t}_k)), \ k = 1, 2, ..., m, \\ \mathsf{x}(\mathsf{t}) &= \varphi(\mathsf{t}), \ \mathsf{t} \in [-r, 0]. \end{aligned}$$

In paper [6], Chen considered the following impulsive stochastic partial differential equations perturbed by a standard Wiener process with delays:

$$\begin{aligned} d[\mathsf{x}(\mathsf{t})] &= [\mathbb{A}\mathsf{x}(\mathsf{t}) + \mathfrak{f}(\mathsf{t}, \mathsf{x}(\mathsf{t} - \delta(\mathsf{t})))]d\mathsf{t} + \sigma(\mathsf{t}, \mathsf{x}(\mathsf{t} - \rho(\mathsf{t})))d\omega_Q^{\mathscr{H}}(\mathsf{t}), \ t \geq 0, \ t \neq t_k, \\ \Delta\mathsf{x}(\mathsf{t}_k) &= I_k(\mathsf{x}(\mathsf{t}_k)), \ k = 1, 2, ... \\ \mathsf{x}(\mathsf{t}) &= \varphi(\mathsf{t}), \ \mathsf{t} \in [-r, 0]. \end{aligned}$$

However, after carefully examining the previously mentioned numerous literatures, it is demonstrated that the semigroups present in the above stochastic differential systems are compact, which is convenient to obtain the corresponding compact resolvent operators. To best of our knowledge, when the semigroups appeared in above stochastic differential systems are noncompact, it is not easy to obtain the corresponding compact resolvent operators. Also, there is no published paper has considered the T-controllability of INSFIDEs driven by a fractional Brownian motion with invarying-time delays. Inspired by these analysis, to fill this gap, in this paper, we consider time invariant INSFIDEs driven by fBm of the form:

$$\begin{aligned}
d\left[\mathsf{x}(\mathfrak{t})+\mathfrak{g}(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))\right] &= \mathbb{A}\left[\mathsf{x}(\mathfrak{t})+\mathfrak{g}\left(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right]d\mathfrak{t}+\int_0^{\mathfrak{t}}\Theta(\mathfrak{t}-s)\left[\mathsf{x}(s)+\mathfrak{g}(s,\mathsf{x}(s-\mathfrak{r}(s)))\right]dsd\mathfrak{t} \\
&+ \mathfrak{f}\left(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\rho(\mathfrak{t}))\right)d\mathfrak{t}+\mathfrak{h}\left(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\eta(\mathfrak{t}))\right)d\omega(\mathfrak{t})+\sigma(\mathfrak{t})d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(\mathfrak{t}), \\
\Delta\mathsf{x}(\mathfrak{t}_k) &:= \mathsf{x}(\mathfrak{t}_k^+)-\mathsf{x}(\mathfrak{t}_k)=\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k)),\quad k\in\mathbb{N},\ \mathfrak{t}\in\mathcal{J}=[0,\mathbb{T}],\ \mathfrak{t}\neq\mathfrak{t}_k, \\
\mathsf{x}(\mathfrak{t}) &= \varphi(\mathfrak{t}),\quad \mathfrak{t}\in(-\tau,0]\ (0<\tau\leq\infty),
\end{aligned} \tag{1.1}$$

where $\mathbb{A}$ is a generators a strongly continuous semigroup $\{\mathbb{R}(\mathfrak{t},\mathfrak{s}),\mathfrak{t}\geq 0\}$ on a Hilbert space $\mathbb{X}$. $\Theta(\mathfrak{t})$ is a closed linear operator on $\mathbb{X}$ with domain $\mathfrak{D}(\Theta)\supset\mathfrak{D}(\mathbb{A})$ which is independent of $\mathfrak{t}$. $\{\omega(\mathfrak{t});\ \mathfrak{t}\in\mathcal{J}\}$ is a standard Weiner process on a real and separable Hilbert space $\mathbb{X}$; $\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(\mathfrak{t})$ is a fractional Brownian motion with Hurst parameter $\mathscr{H}\in(\frac{1}{2},1)$. $\mathfrak{r},\rho,\eta:\mathcal{J}\to[0,\tau)$ are continuous; $\mathfrak{f},\mathfrak{g},\mathfrak{h}:\mathcal{J}\times\mathbb{X}\to\mathbb{X}$, $\sigma:\mathcal{J}\to\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$ are suitable functions. $\mathbb{I}_k:\mathbb{X}\to\mathbb{X}$ are continuous and $\varphi\in\mathcal{C}((-\tau,0],\mathbb{X})$, where $\mathcal{C}((-\tau,0],\mathbb{X})$ is the space of all continuous functions from $(-\tau,0]$ to $\mathbb{X}$.

Significance of this manuscript is presented below: (i) In this paper, firstly consider INSFIDEs driven by a fBm with invarying-time delays. Next, the system's T-controllability is then examined using Gronwalls inequality. (ii) The novelty of this article is that we consider the noncompact semigroup which the above papers [5, 6, 10, 11] we have referred are only considered compact semigroup case. There is no work for T-controllability of INSFIDEs driven by a fBm. In order to bridge this gap, we have looked into the T-controllability of 4.1. So our result is different from them.

Remarks: In our paper, when $\mathfrak{g}=0$, $\Theta=0$ and $\mathfrak{h}=0$ is reduced to the system in [10]. $\Theta=0$, $\mathfrak{h}=0$ and $\mathbb{I}_k=0$ is reduced to the system in [5]. $\Theta=0$ and $\mathfrak{h}=0$ is reduced to the system in [6]. So, our results not only include the systems in [5, 6, 10, 13], but also extends them to a much wider case.

The arrangement of the rest paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. Section 3 is devoted to study the existence of a mild solution to 1.1. Section 4 the system's T-controllability is then examined using Gronwalls inequality. An example will be given to illustrate the effectiveness and feasibility of the obtained results in Section 5.

2 Preliminaries

In this section, we recall basic knowledge of fBm and Hausdorff measure of noncompactness.

Let $(\Omega, \Im, \mathcal{P})$ be a complete probability space and $\mathbb{T} > 0$ be an arbitrary fixed horizon. A one-dimensional fractional Brownian motion with Hurst parameter $\mathscr{H} \in \left(\frac{1}{2}, 1\right)$ is a centred Gaussian process $\beta^{\mathscr{H}} = \{\beta^{\mathscr{H}}(\mathfrak{t}),\ 0 \le \mathfrak{t} \le \mathbb{T}\}$ with covariance function,

$$R_{\mathbb{H}}(s, \mathfrak{t}) = \mathbb{E}\left(\beta^{\mathscr{H}}(\mathfrak{t})\beta^{\mathscr{H}}(s)\right) = \frac{1}{2}\left(\mathfrak{t}^{2\mathscr{H}} + s^{2\mathscr{H}} - |\mathfrak{t} - s|^{2\mathscr{H}}\right),$$

here $\beta^{\mathscr{H}}$ has the following Weiner integral representation:

$$\beta^{\mathscr{H}}(\mathfrak{t}) = \int_0^{\mathfrak{t}} K_{\mathscr{H}}(\mathfrak{t}, s)d\beta(s),$$

where β is a standard Brownian motion and kernel $K_{\mathscr{H}}(\mathfrak{t} - s)$ is defined by

$$K_{\mathscr{H}}(\mathfrak{t}, s) = c_{\mathscr{H}} s^{\frac{1}{2}-\mathscr{H}} \int_s^{\mathfrak{t}} (\mathfrak{u} - s)^{\mathscr{H}-\frac{3}{2}} \mathfrak{u}^{\mathscr{H}-\frac{1}{2}} d\mathfrak{u}, \quad \mathfrak{t} > s,$$

where $c_{\mathscr{H}} = \sqrt{\frac{\mathscr{H}(2\mathscr{H}-1)}{\beta(2-2\mathscr{H}, \mathscr{H}-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ is the beta function. We refer to [?] for more details on the stochastic integral with respect fBm.

The fractional Weiner integral of the function $\psi : [0, \mathbb{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ with respect to $\mathcal{Q}-$Hilbert fBm is defined by

$$\int_0^{\mathfrak{t}} \psi(s)d\mathcal{B}^{\mathscr{H}}(s) = \sum_{n=1}^{\infty} \int_0^{\mathfrak{t}} \sqrt{\lambda_n}\psi(s)e_n d\beta_n^{\mathscr{H}}(s) = \sum_{n=1}^{\infty} \int_0^{\mathfrak{t}} \sqrt{\lambda_n} K_{\mathscr{H}}^*(\psi e_n)(s)d\beta_n(s), \tag{2.1}$$

where β_n is the standard Brownian motion.

Lemma 2.1. *[15] If $\psi : [0, \mathbb{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ satisfies $\int_0^{\mathfrak{t}} \|\psi(s)\|^2_{\mathcal{L}_2^0} < \infty$, then (2.1) is well defined as a $\mathbb{X}-$valued random variable and we have*

$$\mathbb{E}\left\|\int_0^{\mathfrak{t}} \psi(s)d\mathcal{B}^{\mathscr{H}}(s)\right\|^2 \le c_{\mathscr{H}} \mathfrak{t}^{2\mathscr{H}-1} \int_0^{\mathfrak{t}} \|\phi(s)\|^2_{\mathcal{L}_2^0} ds.$$

The Hausdorff measure of noncompactness $\alpha(\cdot)$ defined on each bounded subset $\mathscr{E}$ of Banach space $\mathbb{X}$ by

$$\alpha(\mathscr{E}) = \inf\{\epsilon > 0;\ \mathscr{E} \text{ has a finite } \epsilon- \text{ net in } \mathbb{X}\}.$$

Lemma 2.2. *[19] Let $\mathbb{X}$ be a real Banach space and $\mathscr{E}, \mathscr{F} \subset \mathbb{X}$ be bounded, the following properties hold:*

(1) $\mathscr{E}$ is precompact if and only if $\alpha(\mathscr{E}) = 0$;

(2) $\alpha(\mathscr{E}) = \alpha(\overline{\mathscr{E}}) = \alpha(conv\mathscr{E})$, where $\overline{\mathscr{E}}$ and conv $\mathscr{E}$ are the closure and convex hull of $\mathscr{E}$;

(3) $\alpha(\mathscr{E}) \le \alpha(\mathscr{F})$ when $\mathscr{E} \subset \mathscr{F}$;

(4) $\alpha(\mathscr{E}+\mathscr{F}) \leq \alpha(\mathscr{E})+\alpha(\mathscr{F})$, *where* $\mathscr{E}+\mathscr{F}=\{\mathsf{x}+\mathsf{y};\ \mathsf{x}\in\alpha(\mathscr{E}),\ \mathsf{y}\in\alpha(\mathscr{F})\}$;

(5) $\alpha(\mathscr{E})\cup\mathscr{F} \leq \max\{\alpha(\mathscr{E}),\alpha(\mathscr{F})\}$;

(6) $\alpha(\lambda\mathscr{E}) \leq |\lambda|\alpha(\mathscr{F})$ *for any* $\lambda\in\mathbb{R}$;

(7) *if* $K\subset\mathcal{C}([0,\mathbb{T}])$ *is bounded, then*

$$\alpha(K(\mathfrak{t})) \leq \alpha(K) \quad \textit{for all} \quad \mathfrak{t}\in[0,\mathbb{T}],$$

where $K(\mathfrak{t})=\{\mathfrak{u}(\mathfrak{t}):\mathfrak{u}\in K\subset\mathbb{X}\}$. *Further, if* K *is equicontinuous on* $[0,\mathbb{T}]$, *then* $\mathfrak{t}\to K(\mathfrak{t})$ *is continuous on* $[0,\mathbb{T}]$, *and*

$$\alpha(K)=\sup\{K(\mathfrak{t}):\quad \mathfrak{t}\in[0,\mathbb{T}]\};$$

(8) *if* $K\subset\mathcal{C}([0,\mathbb{T}];\mathbb{X})$ *is bounded and equicontinuous, then* $\mathfrak{t}\to\alpha(K(\mathfrak{t}))$ *is continuous on* $[0,\mathbb{T}]$ *and*

$$\alpha\left(\int_0^{\mathfrak{t}}K(s)ds\right) \leq \int_0^{\mathfrak{t}}\alpha(K(s))ds \quad \textit{for all}\ \mathfrak{t}\in[0,\mathbb{T}],$$

where

$$\int_0^{\mathfrak{t}}K(s)ds=\left\{\int_0^{\mathfrak{t}}\mathfrak{u}(s)ds:\quad \mathfrak{u}\in K\right\};$$

(9) *let* $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ *be a sequence of Bochner integrable functions from* $\mathcal{J}$ *to* $\mathbb{X}$ *with* $\|\mathfrak{u}_n(\mathfrak{t})\|\leq\hat{\mathfrak{m}}(\mathfrak{t})$ *for almost all* $\mathfrak{t}\in\mathcal{J}$ *and every* $n\geq 1$, *where* $\hat{\mathfrak{m}}(\mathfrak{t})\in\mathcal{L}(\mathcal{J};\mathbb{R}^+)$, *then the function* $\phi(\mathfrak{t})=\alpha(\{\mathfrak{u}_n\}_{n=1})\in\mathcal{L}(\mathcal{J};\mathbb{R}^+)$ *satisfies*

$$\alpha\left(\left\{\int_0^{\mathfrak{t}}\mathfrak{u}_n(s)ds:\ n\geq 1\right\}\right)\leq 2\int_0^{\mathfrak{t}}\psi(s)ds.$$

Lemma 2.3. *[20] If* $K\subset\mathcal{C}([0,\mathbb{T}];\mathcal{L}_2^0(\mathbb{Y},\mathbb{X}))$, ω *is a standard Weiner process, then*

$$\alpha\left(\int_0^{\mathfrak{t}}K(s)d\omega(s)\right)\leq\sqrt{T}\alpha(K(\mathfrak{t})),$$

where

$$\int_0^{\mathfrak{t}}K(s)d\omega(s)=\left\{\int_0^{\mathfrak{t}}\mathfrak{u}(s)d\omega(s);\quad \textit{for all}\quad \mathfrak{u}\in K,\quad \mathfrak{t}\in[0,\mathbb{T}]\right\}.$$

Lemma 2.4. *[19] Suppose that* $\mathbb{D}$ *is a closed convex subset of* $\mathbb{X}$, $0\in\mathbb{D}$. *If the map* $\Phi:\mathbb{D}\to\mathbb{X}$ *is continuous and of Mönch type, (i.e.)* Φ *satisfies the property,*

$$\mathscr{M}\subset\mathbb{D},\quad \mathscr{M}\ \textit{is countable},\quad \mathscr{M}\subset\overline{co}\left(\{0\}\cup\Phi(\mathscr{M})\right),$$

this implies $\overline{\mathscr{M}}$ *is compact, then* Φ *has a fixed point in* $\mathbb{D}$.

Before proceeding to the main result, we shall make the following assumptions:

(i) $\mathbb{A}(\mathfrak{t})$ is the infinitesimal generator of a C_0-semigroup $\mathbb{R}(\mathfrak{t})_{t>0}$ on $\mathbb{X}$.

(ii) Let $\mathbb{Y}$ the Banach space $\mathfrak{D}(\mathbb{A})$ equipped with the graph norm is $|\mathsf{y}|_{\mathbb{Y}} := |\mathbb{A}\mathsf{y}| + |\mathsf{y}|$ for $\mathsf{y} \in \mathbb{Y}$. $\mathbb{A}(\mathfrak{t})$ and $\Theta(\mathfrak{t}, s)$ are in the set of bounded linear operators from $\mathbb{Y} \to \mathbb{X}$. $\mathbb{A}(\mathfrak{t}), \Theta(\mathfrak{t}, s)$ are continuous on $0 \leq \mathfrak{t} \leq \mathrm{T}$ and $0 \leq s \leq \mathfrak{t} \leq \mathrm{T}$ into $\mathcal{L}(\mathbb{Y}, \mathbb{X})$.

Definition 2.1. *[18] A resolvent operator for* (1.1) *is a bounded linear operator valued function* $\mathbb{R}(\mathfrak{t}, s) \in \mathcal{L}(\mathbb{X})$ *for* $0 \leq s \leq \mathfrak{t} \leq \mathrm{T}$ *holds, the following properties:*

(i) $\mathbb{R}(\mathfrak{t}, \mathfrak{t}) = I$ *and* $\|\mathbb{R}(\mathfrak{t}, s)\| \leq N e^{\lambda(\mathfrak{t}-s)}$, $\mathfrak{t}, s \in \mathcal{J}$ *for* $N, \lambda \geq 0$.

(ii) $\mathbb{R}(\mathfrak{t}, s)$ *is strongly continuous in* s *and* $\mathfrak{t}$.

(iii) For $\mathsf{x} \in \mathbb{Y}$, $\mathbb{R}(\cdot)\mathsf{x} \in \mathfrak{C}^1([0, +\infty]; \mathbb{X}) \bigcap \mathfrak{C}([0, +\infty]; \mathbb{Y})$ *and*

$$\begin{aligned} d\mathbb{R}(\mathfrak{t})\mathsf{x} &= \left(\mathbb{A}\mathbb{R}(\mathfrak{t})\mathsf{x} + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t} - s)\mathbb{R}(s)\mathsf{x} ds\right) d\mathfrak{t} \\ &= \left(\mathbb{R}(\mathfrak{t})\mathbb{A}\mathsf{x} + \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\Theta(s)\mathsf{x} ds\right) d\mathfrak{t}. \end{aligned}$$

For more details, readers may refer to [18].

3 Existence of Mild Solution

In this section, the existence of mild solution for (1.1). Initially, let us introduce the following concept of mild solution for (1.1).

Definition 3.1. *An* $\mathbb{X}-$*valued stochastic process* $\{\mathsf{x}(\mathfrak{t}),\ \mathfrak{t} \in (-\tau, \mathbb{T}]\}$ *is called a mild solution of (1.1) if* $\mathsf{x}(\mathfrak{t}) = \varphi(\mathfrak{t})$ *on* $(-\tau, 0]$, *and the following conditions gets satisfied:*

(i) $\mathsf{x}(\cdot)$ *is continuous on* $(0, \mathfrak{t}_1]$ *and each interval* $(\mathfrak{t}_k, \mathfrak{t}_{k+1}]$, $k \in \mathbb{N}$;

(ii) for $\mathfrak{t}_k$, $\mathsf{x}(\mathfrak{t}_k^+) = \lim_{\mathfrak{t} \to \mathfrak{t}_k^+} \mathsf{x}(\mathfrak{t})$ *exists;*

(iii) for $\mathfrak{t} \geq 0$, *we have*

$$\begin{aligned} \mathsf{x}(\mathfrak{t}) &= \mathbb{R}\left[\varphi(0) + \mathfrak{g}\left(0, \varphi(0 - \mathfrak{r}(0))\right)\right] - \mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right) + \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\mathfrak{f}\left(s, \mathsf{x}(s - \rho(s))\right) ds \\ &+ \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\mathfrak{h}\left(s, \mathsf{x}(s - \eta(s))\right) d\omega(s) + \sum_{0 < \mathfrak{t}_k < \mathfrak{t}} \mathbb{R}(\mathfrak{t} - \mathfrak{t}_k)\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k)) \\ &+ \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\sigma(s) d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(s). \end{aligned} \tag{3.1}$$

In order to prove our results, the following assumptions are imposed

(H1) The function $\mathfrak{g} : \mathcal{J} \times \mathbb{X} \to \mathbb{X}$ fulfils

(i) $\mathfrak{g}$ is continuous and there exist constants $\mathscr{K}_1 > 0$ such that $\forall\ \mathfrak{t} \in \mathcal{J}$, $\mathsf{x}, \mathsf{y} \in \mathbb{X}$,

$$\begin{aligned} \|\mathfrak{g}(\mathfrak{t}, \mathsf{x}) - \mathfrak{g}(\mathfrak{t}, \mathsf{y})\|^2 &\leq \mathscr{K}_1 \|\mathsf{x} - \mathsf{y}\|^2, \\ \|\mathfrak{g}(\mathfrak{t}, \mathsf{x})\|^2 &\leq \mathscr{K}_1(1 + \|\mathsf{x}\|^2). \end{aligned}$$

(ii) There exist a positive function $\mathscr{C}_{\mathfrak{g}} \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}^+)$, then for any bounded subsets $\Theta_1 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{g}(\mathsf{t}, \Theta_1)) \leq \mathscr{C}_{\mathfrak{g}}(\mathsf{t}), \quad \sup_{\theta \in (-\tau, 0]} \alpha(\Theta_1(\theta)), \quad \overline{\Theta} = \sup_{\mathsf{t} \in \mathcal{J}} \mathscr{C}_{\mathfrak{g}}(\mathsf{t}).$$

(H2) The function $\mathfrak{f} : \mathcal{J} \times \mathbb{X} \to \mathbb{X}$ satisfies

(i) $\mathfrak{f}(\cdot, \mathsf{x}) : \mathcal{J} \times \mathbb{X}$ is measurable for each $\mathsf{x} \in \mathbb{X}$ and $\mathfrak{f}(\mathsf{t}, \cdot) : \mathbb{X} \to \mathbb{X}$ is continuous for each $\mathsf{t} \in \mathcal{J}$.

(ii) There exists a continuous function $\varsigma_{\mathfrak{f}}(\mathsf{t}) : \mathcal{J} \to \mathbb{R}^+$ and a continuous non-decreasing function $\Psi_{\mathfrak{f}} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathfrak{f}(\mathsf{t}, \mathsf{x})\|^2 \leq \varsigma_{\mathfrak{f}}(\mathsf{t}) \Psi_{\mathfrak{f}}(\|\mathsf{x}\|^2).$$

(iii) There exist a positive function $\mathscr{C}_{\mathfrak{f}} \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}^+)$, then for any bounded subsets $\Theta_2 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{f}(\mathsf{t}, \mathsf{x})) \leq \mathscr{C}_{\mathfrak{f}}(\mathsf{t}) \sup_{\theta \in (-\tau, 0]} \alpha(\Theta_2(\theta)).$$

(H3) The function $\mathfrak{h} : \mathcal{J} \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{X}, \mathbb{Y})$ satisfies

(i) $\mathfrak{h}(\cdot, \mathsf{x}) : \mathcal{J} \times \mathcal{L}_2^0(\mathbb{X}, \mathbb{Y})$ is measurable for each $\mathsf{x} \in \mathbb{X}$ and $\mathfrak{h}(\mathsf{t}, \cdot) : \mathbb{X} \to \mathcal{L}_2^0(\mathbb{X}, \mathbb{Y})$ is continuous for each $\mathsf{t} \in \mathcal{J}$.

(ii) There exists a continuous function $\varsigma_{\mathfrak{h}}(\mathsf{t}) : \mathcal{J} \to \mathbb{R}^+$ and a continuous non-decreasing function $\Psi_{\mathfrak{h}} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathfrak{h}(\mathsf{t}, \mathsf{x})\|^2 \leq \varsigma_{\mathfrak{h}}(\mathsf{t}) \Psi_{\mathfrak{h}}(\|\mathsf{x}\|^2).$$

(iii) There exist a positive function $\mathscr{C}_{\mathfrak{h}} \in \mathcal{L}^2(\mathcal{J}, \mathbb{R}^+)$, then for any bounded subsets $\Theta_3 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{h}(\mathsf{t}, \mathsf{x})) \leq \mathscr{C}_{\mathfrak{h}}(\mathsf{t}) \sup_{\theta \in (-\tau, 0]} \alpha(\Theta_3(\theta)).$$

(H4) The function $\mathbb{I}_k : \mathbb{X} \to \mathbb{X}$ is continuous that satisfies

(i) There exist positive constant d_k, $k = 1, 2, \cdots, m \ni$

$$\|\mathbb{I}_k(\mathsf{x}) - \mathbb{I}_k(\mathsf{y})\|^2 \leq \mathsf{d}_k \|\mathsf{x} - \mathsf{y}\|^2, \quad \|\mathbb{I}_k(0)\| = 0.$$

(ii) There exist a positive function $\mathscr{C}_k > 0$, $k = 1, 2, \cdots, m$, then for bounded subsets $\Theta_4 \subset \mathbb{X}$, we have

$$\alpha(\mathbb{I}_k(\Theta_4)) \leq \mathscr{C}_k \sup_{\theta \in (-\tau, 0]} \alpha(\Theta_4(\theta)).$$

.

(H5) The function $\sigma : \mathcal{J} \to \mathcal{L}_{\mathcal{Q}}^0(\mathbb{X}, \mathbb{Y})$ satisfies

(i) $\int_0^{\mathsf{t}} \|\sigma(s)\|^2_{\mathcal{L}_{\mathcal{Q}}^0} ds < \infty, \quad \mathsf{t} \in \mathcal{J}$,

(ii) $\sum_{n=1}^{\infty} \|\sigma \mathcal{Q}^{\frac{1}{2}} e_n\|_{\mathcal{L}^2([0, \mathbb{T}]; \mathbb{X})} < \infty$,

(iii) For $\mathsf{t} \in [0, \mathbb{T}]$, $\sum_{n=1}^{\infty} \|\sigma \mathcal{Q}^{\frac{1}{2}} e_n\|_{\mathbb{X}}$ is uniformly convergent for $\mathsf{t} \in \mathcal{J}$.

(H6)

$$6\left[\mathscr{K}_1 + m\mathbb{M}^2\sum_{k=1}^{m}\mathsf{d}_k + \mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{f}}(s)\frac{\Phi(\mathsf{r})}{\mathsf{r}}ds + \mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{h}}(s)\frac{\Phi(\mathsf{r})}{\mathsf{r}}ds\right] \leq 1.$$

Theorem 3.1. *Suppose that the condition (H1)-(H6) holds, the system (1.1) has a unique mild solution provided,*

$$\left[\overline{\Theta} + \mathbb{M}\mathbb{T}\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1(\mathcal{J},\mathbb{R}^+)} + \mathbb{M}\sqrt{\mathbb{T}}\|\mathscr{C}_{\mathfrak{h}}\|_{\mathcal{L}^2(\mathcal{J},\mathbb{R}^+)} + \mathbb{M}\sum_{k=1}^{m}\mathsf{d}_k\right] < 1. \tag{3.2}$$

Proof. Initially, let us introduce the set $\sigma_{\mathbb{T}} : \mathcal{PC}([-\tau,\mathbb{T}],\mathcal{L}^2(\Omega,\mathbb{X}))$ being Banach space of all continuous functions from $[-\tau,\mathbb{T}]$ to $\mathcal{L}^2(\Omega,\mathbb{X})$ is equipped with the norm $\|\chi\|^2 = \sup_{s\in[-\tau,\mathbb{T}]}(\mathbb{E}\|\chi\|^2)$. Let us consider the closed subset of $\lambda_{\mathbb{T}}$, defined by $\overline{\lambda}_{\mathbb{T}} = \{\mathsf{x} \in \lambda_{\mathbb{T}} : \mathsf{x}(\xi) = \varphi(\xi) \text{ for } \xi \in [-\tau, 0]\}$ provided with the norm $\|\cdot\|$. We may transform (1.1) into fixed point problem. Define an operator $\Phi : \overline{\lambda}_{\mathbb{T}} \to \overline{\lambda}_{\mathbb{T}}$ by

$$\begin{aligned}(\Phi\mathsf{x})(\mathsf{t}) &= \mathbb{R}\left[\varphi(0) + \mathfrak{g}\left(0,\varphi(0-\mathfrak{r}(0))\right)\right] - \mathfrak{g}\left(\mathsf{t},\mathsf{x}(\mathsf{t}-\mathfrak{r}(\mathsf{t}))\right) + \int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{f}\left(s,\mathsf{x}(s-\rho(s))\right)ds \\ &+ \int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{h}\left(s,\mathsf{x}(s-\eta(s))\right)d\omega(s) + \sum_{0<\mathsf{t}_k<\mathsf{t}}\mathbb{R}(\mathsf{t}-\mathsf{t}_k)\mathbb{I}_k(\mathsf{x}(\mathsf{t}_k)) \\ &+ \int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\sigma(s)d\mathcal{B}^{\mathscr{H}}_{\mathscr{Q}}(s), \qquad \mathsf{t} \in [0,\mathbb{T}].\end{aligned}$$

and $(\Phi\mathsf{x})(\mathsf{t}) = \varphi(\mathsf{t}),\ \ \mathsf{t} \in [-\tau, 0]$. We divide the proof into several steps:

Step 1: Consider a bounded, closed, convex set $\mathbb{B}_{\mathsf{r}} = \{\mathsf{x} \in \lambda_{\mathbb{T}} : \|\mathsf{x}\|^2 \leq \mathsf{r}\}$, then we verify $N(\mathbb{B}_{\mathsf{r}}) \subset \mathbb{B}_{\mathsf{r}}$. If it is inconsistent, then there exist a function $\mathsf{x} \in \mathbb{B}_{\mathsf{r}}$, such that $N(\mathbb{B}_{\mathsf{r}}) \not\subseteq \mathbb{B}_{\mathsf{r}}$, which implies there exist some $\mathsf{t} \in \mathcal{J}$ such that $\mathbb{E}\|(\Phi\mathsf{x})(\mathsf{t})\|^2 > \mathsf{r}$, we have

$$\begin{aligned}\mathsf{r} &< \mathbb{E}\left\|(\Phi\mathsf{x})(\mathsf{t})\right\|^2 \leq 6\mathbb{E}\left\|\mathbb{R}\left[\varphi(0) + \mathfrak{g}\left(0,\varphi(0-\mathfrak{r}(0))\right)\right]\right\|^2 + 6\mathbb{E}\left\|\mathfrak{g}\left(\mathsf{t},\mathsf{x}(\mathsf{t}-\mathfrak{r}(\mathsf{t}))\right)\right\|^2 \\ &+ 6\mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{f}\left(s,\mathsf{x}(s-\rho(s))\right)ds\right\|^2 + 6\mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{h}\left(s,\mathsf{x}(s-\eta(s))\right)d\omega(s)\right\|^2 \\ &+ 6\mathbb{E}\left\|\sum_{0<\mathsf{t}_k<\mathsf{t}}\mathbb{R}(\mathsf{t}-\mathsf{t}_k)\mathbb{I}_k(\mathsf{x}(\mathsf{t}_k))\right\|^2 + 6\mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\sigma(s)d\mathcal{B}^{\mathscr{H}}_{\mathscr{Q}}(s)\right\|^2 \\ &:= \sum_{i=1}^{6}\mathscr{N}_i.\end{aligned}$$

Employing assumptions (H1)-(H5), we have

$$\begin{aligned}
\mathscr{N}_1 &= \mathbb{E}\left\|\mathbb{R}\left[\varphi(0)+\mathfrak{g}\left(0,\varphi(0-\mathfrak{r}(0))\right)\right]\right\|^2 \\
&\leq 2\mathbb{E}\left\|\mathbb{R}(\mathsf{t})\varphi(0)\right\|^2+2\mathbb{E}\left\|\mathbb{R}(\mathsf{t})\mathfrak{g}\left(0,\varphi(0-\mathfrak{r}(0))\right)\right\|^2 \\
&\leq 2\mathbb{M}^2\mathbb{E}\|\varphi(0)\|^2+2\mathbb{M}^2\mathscr{K}_1\left(1+\|\varphi\|^2\right) \\
\mathscr{N}_2 &= \mathbb{E}\left\|\mathfrak{g}\left(\mathsf{t},\mathsf{x}(\mathsf{t}-\mathfrak{r}(\mathsf{t}))\right)\right\|^2 \leq \mathscr{K}_1\left(1+\|\mathsf{x}\|^2\right) \leq \mathscr{K}_1(1+\mathfrak{r}) \\
\mathscr{N}_3 &= \mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{f}\left(s,\mathsf{x}(s-\rho(s))\right)ds\right\|^2 \\
&\leq \mathbb{M}^2\mathbb{T}\int_0^{\mathsf{t}}\varsigma_{\mathfrak{f}}(s)\Phi_{\mathfrak{f}}(\|\mathsf{x}\|^2) \leq \mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{f}}(s)\Phi_{\mathfrak{f}}(\mathsf{r})ds \\
\mathscr{N}_4 &= \mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\mathfrak{h}\left(s,\mathsf{x}(s-\eta(s))\right)d\omega(s)\right\|^2 \\
&\leq \mathbb{M}^2\mathbb{T}\int_0^{\mathsf{t}}\varsigma_{\mathfrak{h}}(s)\Phi_{\mathfrak{h}}(\|\mathsf{x}\|^2)ds \leq \mathbb{M}^2\mathbb{T}\int_0^{\mathsf{t}}\varsigma_{\mathfrak{h}}(s)\Phi_{\mathfrak{h}}(\mathsf{r})ds \\
\mathscr{N}_5 &= \mathbb{E}\left\|\sum_{0<\mathsf{t}_k<\mathsf{t}}\mathbb{R}(\mathsf{t}-\mathsf{t}_k)\mathbb{I}_k(\mathsf{x}(\mathsf{t}_k))\right\|^2 \leq m\mathbb{M}^2\sum_{k=1}^{m}\mathsf{d}_k\mathsf{r} \\
\mathscr{N}_6 &= \mathbb{E}\left\|\int_0^{\mathsf{t}}\mathbb{R}(\mathsf{t}-s)\sigma(s)d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(s)\right\|^2 \\
&\leq \mathbb{T}\mathbb{M}^2c_{\mathscr{H}}(2\mathscr{H}-1)\mathbb{T}^{2\mathscr{H}-1}\sup_{0\leq\mathsf{t}\leq\mathbb{T}}\|\sigma(\mathsf{t})\|^2_{\mathcal{L}_2^0} \\
&\leq \mathbb{M}^2c_{\mathscr{H}}(2\mathscr{H}-1)\mathbb{T}^{2\mathscr{H}}\sup_{0\leq\mathsf{t}\leq\mathbb{T}}\|\sigma(\mathsf{t})\|^2_{\mathcal{L}_{\mathcal{Q}}^0}.
\end{aligned}$$

From $\mathscr{N}_1-\mathscr{N}_6$,

$$\begin{aligned}
\mathsf{r} &\leq 12\mathbb{M}^2\mathbb{E}\|\varphi(0)\|^2+12\mathbb{M}^2\mathscr{K}_1\left(1+\|\varphi\|^2\right)+6\mathscr{K}_1+6\mathbb{M}^2c_{\mathscr{H}}(2\mathscr{H}-1)\mathbb{T}^{2\mathscr{H}}\sup_{0\leq\mathsf{t}\leq\mathbb{T}}\|\sigma(\mathsf{t})\|^2_{\mathcal{L}_{\mathcal{Q}}^0} \\
&+ \left[6\mathscr{K}_1+6m\mathbb{M}^2\sum_{k=1}^{m}\mathsf{d}_k\right]\mathsf{r}+6\left[\mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{f}}(s)\Phi(\mathsf{r})ds+\mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{h}}(s)\Phi(\mathsf{r})ds\right]
\end{aligned}$$

Dividing both sides by r and let $\mathsf{r}\to\infty$, we obtain

$$\left[6\mathscr{K}_1+6m\mathbb{M}^2\sum_{k=1}^{m}\mathsf{d}_k\right]\mathsf{r}+6\left[\mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{f}}(s)\Phi(\mathsf{r})ds+\mathbb{T}\mathbb{M}^2\int_0^{\mathsf{t}}\varsigma_{\mathfrak{h}}(s)\Phi(\mathsf{r})ds\right]>1,$$

which contradicts (H6). Thus, there exist some function $\mathsf{x}\in\mathbb{B}_{\mathsf{r}}$, such that $N(\mathbb{B}_{\mathsf{r}})\subset\mathbb{B}_{\mathsf{r}}$.

Step 2: The operator Φ is continuous in $\mathbb{B}_{\mathsf{r}}$.

Let $\mathsf{x},\mathsf{x}_n\in\mathbb{B}_{\mathsf{r}}$, $\mathsf{t}\in[0,\mathbb{T}]$ and $\mathsf{x}_n\to\mathsf{x}$. Thus we have,

$$\begin{aligned}
\mathfrak{g}(\mathsf{t},\mathsf{x}_n) &\to \mathfrak{g}(\mathsf{t},\mathsf{x}),\quad n\to+\infty, \\
\mathfrak{f}(\mathsf{t},\mathsf{x}_n) &\to \mathfrak{f}(\mathsf{t},\mathsf{x}),\quad n\to+\infty, \\
\mathfrak{h}(\mathsf{t},\mathsf{x}_n) &\to \mathfrak{h}(\mathsf{t},\mathsf{x}),\quad n\to+\infty, \\
\mathbb{I}_k(\mathsf{x}_n(\mathsf{t}_k)) &\to \mathbb{I}_k(\mathsf{x}(\mathsf{t}_k)),\quad n\to+\infty.
\end{aligned}$$

Also,

$$\begin{aligned}\|\mathfrak{f}(\mathfrak{t},\mathsf{x}_n)-\mathfrak{f}(\mathfrak{t},\mathsf{x})\|^2 &\leq 2\varsigma_{\mathfrak{f}}(\mathfrak{t})\Phi_{\mathfrak{f}}(\mathsf{r})\\ \|\mathfrak{h}(\mathfrak{t},\mathsf{x}_n)-\mathfrak{h}(\mathfrak{t},\mathsf{x})\|^2 &\leq 2\varsigma_{\mathfrak{h}}(\mathfrak{t})\Phi_{\mathfrak{h}}(\mathsf{r}).\end{aligned}$$

Then by dominated convergence theorem, for $\mathfrak{t}\in[0,\mathbb{T}]$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi\mathsf{x}_n)(\mathfrak{t})-(\Phi\mathsf{x})(\mathfrak{t})\|^2 &\leq 4\mathbb{E}\|\mathfrak{g}(\mathfrak{t},\mathsf{x}_n(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))-\mathfrak{g}(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))\|^2\\ &+ 4\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{f}(s,\mathsf{x}_n(s-\rho(s)))-\mathfrak{f}(s,\mathsf{x}(s-\rho(s)))\right]ds\right\|^2\\ &+ 4\mathbb{E}\left\|\int_0^{\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{h}(s,\mathsf{x}_n(s-\eta(s)))-\mathfrak{h}(s,\mathsf{x}(s-\eta(s)))\right]d\omega(s)\right\|^2\\ &+ 4\mathbb{E}\left\|\sum_{0<\mathfrak{t}_k<\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s_k)\left[\mathbb{I}_k(\mathsf{x}_n(\mathfrak{t}_k))-\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k))\right]\right\|^2\\ &\leq 4\mathbb{E}\|\mathfrak{g}(\mathfrak{t},\mathsf{x}_n(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))-\mathfrak{g}(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))\|^2\\ &+ 4\mathbb{T}\mathbb{M}^2\int_0^{\mathfrak{t}}\mathbb{E}\|\mathfrak{f}(s,\mathsf{x}_n(s-\rho(s)))-\mathfrak{f}(s,\mathsf{x}(s-\rho(s)))\|^2ds\\ &+ 4\mathbb{T}\mathbb{M}^2\int_0^{\mathfrak{t}}\mathbb{E}\|\mathfrak{h}(s,\mathsf{x}_n(s-\eta(s)))-\mathfrak{h}(s,\mathsf{x}(s-\eta(s)))\|^2ds\\ &+ 4m\mathbb{M}^2\sum_{k=1}^{m}\mathbb{E}\|\mathbb{I}_k(\mathsf{x}_n(\mathfrak{t}_k))-\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k))\|^2\\ &\to 0 \;\text{ as }\; n\to\infty.\end{aligned}$$

Thus, Φ is continuous in $\mathbb{B}_{\mathsf{r}}$.

Step 3: The operator Φ is equicontinuous on $\mathcal{J}$.

Let $0<\mathfrak{t}_1<\mathfrak{t}_2<\mathbb{T}$ and $\mathsf{x}\in\mathbb{B}_{\mathsf{r}}$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi\mathsf{x})(\mathfrak{t}_2)-(\Phi\mathsf{x})(\mathfrak{t}_1)\|^2 &\leq 6\mathbb{E}\|[\mathbb{R}(\mathfrak{t}_2)-\mathbb{R}(\mathfrak{t}_1)]\left[\varphi(0)+\mathfrak{g}(0,\varphi(0-\mathfrak{r}(0)))\right]\|^2\\ &+ 6\mathbb{E}\|\mathfrak{g}(\mathfrak{t}_2,\mathsf{x}(\mathfrak{t}_2-\mathfrak{r}(\mathfrak{t}_2)))-\mathfrak{g}(\mathfrak{t}_1,\mathsf{x}(\mathfrak{t}_1-\mathfrak{r}(\mathfrak{t}_1)))\|^2\\ &+ 12\mathbb{E}\left\|\int_0^{\mathfrak{t}_1}[\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)]\,\mathfrak{f}(s,\mathsf{x}(s-\rho(s)))\,ds\right\|^2\\ &+ 12\mathbb{E}\left\|\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\mathbb{R}(\mathfrak{t}_2-s)\mathfrak{f}(s,\mathsf{x}(s-\rho(s)))\,ds\right\|^2\\ &+ 12\mathbb{E}\left\|\int_0^{\mathfrak{t}_1}[\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)]\,\mathfrak{h}(s,\mathsf{x}(s-\eta(s)))\,d\omega(s)\right\|^2\\ &+ 12\mathbb{E}\left\|\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\mathbb{R}(\mathfrak{t}_2-s)\mathfrak{h}(s,\mathsf{x}(s-\eta(s)))\,d\omega(s)\right\|^2\\ &+ 6\mathbb{E}\left\|\sum_{0<\mathfrak{t}_k<\mathfrak{t}}[\mathbb{R}(\mathfrak{t}_2-\mathfrak{t}_k)-\mathbb{R}(\mathfrak{t}_1-\mathfrak{t}_k)]\,\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k))\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+ \quad 12\mathbb{E}\left\|\int_0^{\mathfrak{t}_1}\left[\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)\right]\sigma(s)d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(s)\right\|^2+12\mathbb{E}\left\|\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\mathbb{R}(\mathfrak{t}_2-s)\sigma(s)d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(s)\right\|^2\\
&\leq \quad \left[12\mathbb{E}\|\varphi(0)\|^2+12\mathscr{K}_1(1+\|\varphi\|^2)\right]\mathbb{E}\left\|\mathbb{R}(\mathfrak{t}_2)-\mathbb{R}(\mathfrak{t}_1)\right\|^2+6\mathscr{K}_1\mathbb{E}\left\|\mathsf{x}(\mathfrak{t}_2)-\mathsf{x}(\mathfrak{t}_1)\right\|^2\\
&+ \quad 12\int_0^{\mathfrak{t}_1}\mathbb{E}\left\|\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)\right\|^2\varsigma_{\mathfrak{f}}(s)\Phi_{\mathfrak{f}}\|\mathsf{x}\|^2ds+12\mathbb{M}^2\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\varsigma_{\mathfrak{f}}(s)\Phi_{\mathfrak{f}}(\|\mathsf{x}\|^2)ds\\
&+ \quad 12\int_0^{\mathfrak{t}_1}\mathbb{E}\left\|\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)\right\|^2\varsigma_{\mathfrak{h}}(s)\Phi_{\mathfrak{h}}\|\mathsf{x}\|^2ds+12\mathbb{M}^2\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\varsigma_{\mathfrak{h}}(s)\Phi_{\mathfrak{h}}(\|\mathsf{x}\|^2)ds\\
&+ \quad 6\sum_{i=1}^{k}\mathsf{d}_k\mathbb{E}\left\|\mathbb{R}(\mathfrak{t}_2-\mathfrak{t}_k)-\mathbb{R}(\mathfrak{t}_1-\mathfrak{t}_k)\right\|^2+12c_{\mathscr{H}}(2\mathscr{H}-1)\mathbb{T}^{2\mathscr{H}-1}\int_0^{\mathfrak{t}_1}\mathbb{E}\left\|\mathbb{R}(\mathfrak{t}_2-s)-\mathbb{R}(\mathfrak{t}_1-s)\right\|^2\\
&\times \quad \|\sigma(s)\|^2_{\mathcal{L}_{\mathcal{Q}}^0}ds+12\mathbb{M}^2c_{\mathscr{H}}(2\mathscr{H}-1)\mathbb{T}^{2\mathscr{H}-1}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\mathbb{E}\|\sigma(s)\|^2_{\mathcal{L}_{\mathcal{Q}}^0}ds\\
&\to \quad 0 \ \text{ as } \ \mathfrak{t}_2\to\mathfrak{t}_1.
\end{aligned}
$$

This proves the equicontinuity property.

Step 4: Let us verify the Mönch condition gets satisfied.

We may define an

non-empty set $\Omega\subset\lambda_{\mathbb{T}}$, let $\mathsf{x}_1,\mathsf{x}_2\in\Omega$ then it is evident that $d\left(\Phi\mathsf{x}_1(\mathfrak{t}),\Phi\mathsf{x}_2(\mathfrak{t})\right)=d\left(\overline{\Phi}\mathsf{x}_1(\mathfrak{t}),\overline{\Phi}\mathsf{x}_2(\mathfrak{t})\right)$. By the similar proof of Lemma 2.3,

$$
\alpha(\Phi\mathsf{x}(\mathfrak{t}))=\alpha(\overline{\Phi}\mathsf{x}(\mathfrak{t})),
$$

where

$$
\begin{aligned}
(\phi\mathsf{x})(\mathfrak{t}) \quad &= \quad \mathbb{R}\left[\varphi(0)+\mathfrak{g}\left(0,\varphi(0-\mathfrak{r}(0))\right)\right]-\mathfrak{g}\left(\mathfrak{t},\mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)+\int_0^{\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s)\mathfrak{f}\left(s,\mathsf{x}(s-\rho(s))\right)ds\\
&+ \quad \int_0^{\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s)\mathfrak{h}\left(s,\mathsf{x}(s-\eta(s))\right)d\omega(s)+\sum_{0<\mathfrak{t}_k<\mathfrak{t}}\mathbb{R}(\mathfrak{t}-\mathfrak{t}_k)\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k))+\int_0^{\mathfrak{t}}\mathbb{R}(\mathfrak{t}-s)\sigma(s)d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(s)\\
&= \quad \overline{\Phi_1}+\overline{\Phi_2}+\overline{\Phi_3}.
\end{aligned}
$$

Let $\Delta\subset\mathbb{B}_{\mathsf{r}}$ be countable and $\Delta\subset\overline{co}(\{0\}\cup\Phi(\Delta))$. Then, we verify $\alpha(\Delta)=0$. Define $\Delta=\{\mathsf{x}^n\}_{n=1}^{\infty}$, we know $\Delta\subset\overline{co}(\{0\}\cup\Phi(\Delta))$ is equicontinuous on $\mathcal{J}$ by step 3.

Next, from (H1),(H2), we have

$$
\begin{aligned}
\alpha\left(\left\{\overline{\Phi}_1\mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) \quad &\leq \quad \overline{\Theta}_1\sup_{\theta\in(-\tau,0]}\alpha\left(\{\mathsf{x}^n(\theta-\mathfrak{r}(\theta))\}_{n=1}^{\infty}\right)\\
&\leq \quad \overline{\Theta}\sup_{\mathfrak{t}\in\mathcal{J}}\alpha\left(\{\mathsf{x}^n(\mathfrak{t})\}_{n=1}^{\infty}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\alpha\left(\left\{\overline{\Phi}_2\mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) \quad &\leq \quad \mathbb{M}\mathbb{T}\int_0^{\mathfrak{t}}\mathscr{C}_{\mathfrak{f}}(\mathfrak{t})\sup_{\theta\in(-\tau,0]}\alpha\left(\{\mathsf{x}^n(\theta-\rho(\theta))\}_{n=1}^{\infty}\right)ds\\
&+ \quad \mathbb{M}\sqrt{\mathbb{T}}\left\|\mathscr{C}_{\mathfrak{h}}\right\|_{\mathcal{L}^2(\mathcal{J},\mathbb{R}^+)}\sup_{\theta\in(-\tau,0]}\alpha\left(\{\mathsf{x}^n(\theta-\eta(\theta))\}_{n=1}^{\infty}\right)\\
&\leq \quad \left[\mathbb{M}\mathbb{T}\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1(\mathcal{J},\mathbb{R}^+)}+\mathbb{M}\sqrt{\mathbb{T}}\left\|\mathscr{C}_{\mathfrak{h}}\right\|_{\mathcal{L}^2(\mathcal{J},\mathbb{R}^+)}\right]\sup_{\mathfrak{t}\in\mathcal{J}}\alpha\left(\{\mathsf{x}^n(\mathfrak{t})\}_{n=1}^{\infty}\right)\\
\alpha\left(\left\{\overline{\Phi}_3\mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) \quad &\leq \quad \mathbb{M}\sum_{k=1}^{m}\mathsf{d}_k\sup_{\theta\in(-\tau,0]}\alpha\left(\left\{\mathsf{x}^n_{\mathfrak{t}_k}(\theta)\right\}_{n=1}^{\infty}\right)<\mathbb{M}\sum_{k=1}^{m}\mathsf{d}_k\sup_{\mathfrak{t}\in\mathcal{J}}\alpha\left(\{\mathsf{x}^n(\mathfrak{t})\}_{n=1}^{\infty}\right).
\end{aligned}
$$

Together with Φ_1, Φ_2 and Φ_3,

$$\begin{aligned}\alpha\left(\{\Phi \mathsf{x}^n(\mathfrak{t})\}_{n=1}^{\infty}\right) &\leq \alpha\left(\left\{\overline{\Phi}_1 \mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) + \alpha\left(\left\{\overline{\Phi}_2 \mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) + \alpha\left(\left\{\overline{\Phi}_3 \mathsf{x}^n(\mathfrak{t})\right\}_{n=1}^{\infty}\right) \\ &\leq \left[\overline{\Theta} + \mathbb{M}\mathbb{T}\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1(\mathcal{J},\mathbb{R}^+)} + \mathbb{M}\sqrt{\mathbb{T}}\,\|\mathscr{C}_{\mathfrak{h}}\|_{\mathcal{L}^2(\mathcal{J},\mathbb{R}^+)} + \mathbb{M}\sum_{k=1}^{m} \mathsf{d}_k\right] \sup_{\mathfrak{t}\in\mathcal{J}} \alpha\left(\{\mathsf{x}^n(\mathfrak{t})\}_{n=1}^{\infty}\right)\end{aligned}$$

by lemma 2.2,

$$\alpha(\Delta) \leq \alpha\left(\overline{co}(\{0\} \cup \Phi(\Delta))\right) = \alpha(\Phi(\Delta)) \leq \alpha(\Delta),$$

which implies $\alpha(\Delta) = 0$, Δ is a relatively compact set. Thus we deduce that Φ has a fixed point in Δ, which is a mild solution of system (1.1).

□

4 Trajectory Controllability

Definition 4.1. *The control system (4.1) is said to be trajectory controllable on $[0, \mathbb{T}]$, if for every $\mu \in \mathcal{V}$, such that the mild solution $\mathsf{x}(.)$ of (4.1) satisfies $\mu(t) = \mathsf{x}(t)$ almost everywhere.*

Lemma 4.1. *(Generalized Gronwall's inequality): If $\beta > 0$, $\tilde{a}(\mathfrak{t})$ is a non-negative function locally integrable on $0 \leq \mathfrak{t} \leq \mathbb{T}$ and $\mathfrak{q}(\mathfrak{t})$ is a non-negative, non-decreasing continuous function on $0 \leq \mathfrak{t} \leq \mathbb{T}$, $\mathfrak{q}(\mathfrak{t}) \leq \mathfrak{c}$ and suppose $\tilde{\mathfrak{u}}(\mathfrak{t}) \leq \tilde{a}(\mathfrak{t}) + \mathfrak{q}(\mathfrak{t})\int_0^{\mathfrak{t}}(\mathfrak{t}-s)^{\beta-1}\tilde{\mathfrak{u}}(s)ds$, on this interval. Then*

$$\tilde{\mathfrak{u}(\mathfrak{t})} \leq \tilde{a}(\mathfrak{t}) + \int_0^{\mathfrak{t}} \sum_{n=1}^{\infty} \frac{(\mathfrak{q}(\mathfrak{t})\Gamma(\beta))^n}{\Gamma(n\beta)}(\mathfrak{t}-s)^{\beta-1}\tilde{a}(s)ds, \quad 0 \leq \mathfrak{t} \leq \mathbb{T}.$$

In particular, when $\tilde{a}(\mathfrak{t}) = 0$, then $\tilde{\mathfrak{u}}(\mathfrak{t}) = 0 \;\; \forall \;\; 0 \leq \mathfrak{t} < \mathbb{T}$.

In order to prove the trajectory controllability of the system (4.1), let us impose the following hypotheses

(H7) The function $\mathfrak{f}$ and $\mathfrak{h}$, satisfies the Lipschitz condition with $\mathscr{K}_2, \mathscr{K}_3 > 0$ being constant

$$\begin{aligned}\mathbb{E}\,\|\mathfrak{f}(\mathfrak{t},\mathsf{x}) - \mathfrak{f}(\mathfrak{t},\mathsf{y})\|^2 &\leq \mathscr{K}_2\,\|\mathsf{x}-\mathsf{y}\|^2 \\ \mathbb{E}\,\|\mathfrak{h}(\mathfrak{t},\mathsf{x}) - \mathfrak{h}(\mathfrak{t},\mathsf{y})\|^2 &\leq \mathscr{K}_3\,\|\mathsf{x}-\mathsf{y}\|^2.\end{aligned}$$

Let us consider the control system of the form:

$$\begin{aligned}d\left[\mathsf{x}(\mathfrak{t}) + \mathfrak{g}(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t})))\right] &= \mathbb{A}\left[\mathsf{x}(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right]d\mathfrak{t} + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}-s)\left[\mathsf{x}(s) + \mathfrak{g}(s, \mathsf{x}(s-\mathfrak{r}(s)))\right]ds d\mathfrak{t} \\ &+ \mathbb{G}(\mathfrak{t})\mathfrak{u}(\mathfrak{t}) + \mathfrak{f}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\rho(\mathfrak{t}))\right)d\mathfrak{t} + \mathfrak{h}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\eta(\mathfrak{t}))\right)d\omega(\mathfrak{t}) + \sigma(\mathfrak{t})d\mathcal{B}_{\mathcal{Q}}^{\mathscr{H}}(\mathfrak{t}), \\ \Delta\mathsf{x}(\mathfrak{t}_k) &:= \mathsf{x}(\mathfrak{t}_k^+) - \mathsf{x}(\mathfrak{t}_k) = \mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k)), \quad k \in \mathbb{N}, \; \mathfrak{t} \in \mathcal{J} = [0, \mathbb{T}], \; \mathfrak{t} \neq \mathfrak{t}_k, \\ \mathsf{x}(\mathfrak{t}) &= \varphi(\mathfrak{t}), \quad \mathfrak{t} \in (-\tau, 0] \; (0 < \tau \leq \infty). \qquad (4.1)\end{aligned}$$

Here, the control function $\mathfrak{u}(\cdot) \in \mathcal{L}^2_{\Im}(\mathcal{J}, \mathbb{Z})$, where $\mathcal{L}^2_{\Im}(\mathcal{J}, \mathbb{Z})$ is the space of all admissible control functions, which is square integrable and $\Im_{\mathfrak{t}}$-adopted. $\mathbb{G}$ is a bounded linear operator from Hilbert space $\mathbb{Z}$ into $\mathbb{X}$.

Theorem 4.1. *If (H1)-(H7) are satisfied, for every $\mathfrak{u}(\cdot) \in \mathcal{L}^2_{\Im}(\mathcal{J}, \mathbb{Z})$ then $\exists$ a unique mild solution of 4.1.*

$$\begin{aligned} \mathsf{x}(\mathfrak{t}) &= \mathbb{R}\left[\varphi(0) + \mathfrak{g}\left(0, \varphi(0 - \mathfrak{r}(0))\right)\right] - \mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right) + \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\mathfrak{f}\left(s, \mathsf{x}(s - \rho(s))\right) ds \\ &+ \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\mathfrak{h}\left(s, \mathsf{x}(s - \eta(s))\right) d\omega(s) + \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\mathbb{G}(s)\mathfrak{u}(s) ds \\ &+ \sum_{0 < \mathfrak{t}_k < \mathfrak{t}} \mathbb{R}(\mathfrak{t} - \mathfrak{t}_k)\mathbb{I}_k(\mathsf{x}(\mathfrak{t}_k)) + \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t} - s)\sigma(s) d\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(s). \end{aligned} \tag{4.2}$$

Proof. This theorem's proof is similar to Theorem 3.1, and on can easily prove that solution of system 4.1 by using the Mönch fixed point theorem with the help of the Hausdorff measure of noncompactness, so it is omitted.

□

Theorem 4.2. *If the hypotheses (H5)(i), (H7) holds, the stochastic differential system (1.1) is trajectory controllable.*

Proof. Let $\mu(\mathfrak{t})$ be the given trajectory on $\mathfrak{T}$. Let us consider the feedback control $\mathfrak{u}(\mathfrak{t})$ as

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) &= \mathbb{G}^{-1}\Big[d\left[\mu(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}\mathfrak{r}(\mathfrak{t}))\right)\right] - \mathbb{A}\Big[\mu(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}\mathfrak{r}(\mathfrak{t}))\right) d\mathfrak{t} \\ &- \int_0^{\mathfrak{t}} \Theta(\mathfrak{t} - s)\left[\mu(s) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right)\right] ds\Big] d\mathfrak{t} - \mathfrak{f}\left(\mathfrak{t}, \mu(\mathfrak{t} - \rho(\mathfrak{t}))\right) d\mathfrak{t} \\ &- \mathfrak{h}\left(\mathfrak{t}, \mu(\mathfrak{t} - \eta(\mathfrak{t}))\right) d\omega(\mathfrak{t}) - \sigma(\mathfrak{t}) d\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(\mathfrak{t})\Big]. \end{aligned}$$

Thus (4.1) implies

$$\begin{aligned} d\left[\mathsf{x}(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right)\right] &= \mathbb{A}\left[\mathsf{x}(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right)\right] d\mathfrak{t} + \Big(d\left[\mu(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}\mathfrak{r}(\mathfrak{t}))\right)\right] \\ &- \mathbb{A}\Big[\mu(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}\mathfrak{r}(\mathfrak{t}))\right) d\mathfrak{t} - \int_0^{\mathfrak{t}} \Theta(\mathfrak{t} - s)\left[\mu(s) + \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t} - \mathfrak{r}(\mathfrak{t}))\right)\right] ds\Big] d\mathfrak{t} \\ &- \mathfrak{f}\left(\mathfrak{t}, \mu(\mathfrak{t} - \rho(\mathfrak{t}))\right) d\mathfrak{t} - \mathfrak{h}\left(\mathfrak{t}, \mu(\mathfrak{t} - \eta(\mathfrak{t}))\right) d\omega(\mathfrak{t}) - \sigma(\mathfrak{t}) d\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(\mathfrak{t})\Big) \\ &+ \int_0^{\mathfrak{t}} \Theta(\mathfrak{t} - s)\left[\mathsf{x}(s) + \mathfrak{g}\left(s, \mathsf{x}(s - \mathfrak{r}(s))\right)\right] ds.d\mathfrak{t} + \mathfrak{f}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \rho(\mathfrak{t}))\right) d\mathfrak{t} \\ &+ \mathfrak{h}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t} - \eta(\mathfrak{t}))\right) d\omega(\mathfrak{t}) + \sigma(\mathfrak{t}) d\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(\mathfrak{t}). \end{aligned}$$

Put $\Lambda(\mathfrak{t}) = \mathsf{x}(\mathfrak{t}) - \mu(\mathfrak{t})$,

$$\begin{aligned} d\left[\Lambda(\mathfrak{t}) + \left[\mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right) - \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right]\right] &= \mathbb{A}\Big[\Lambda(\mathfrak{t}) + \left[\mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right) - \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right]\Big] d\mathfrak{t} \\ &+ \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}-s)\left[\mathfrak{g}\left(s, \mathsf{x}(s-\mathfrak{r}(s))\right) - \mathfrak{g}\left(s, \mu(s-\mathfrak{r}(s))\right)\right] ds.d\mathfrak{t} \\ &+ \left[\mathfrak{f}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\rho(\mathfrak{t}))\right) - \mathfrak{f}\left(\mathfrak{t}, \mu(\mathfrak{t}-\rho(\mathfrak{t}))\right)\right] d\mathfrak{t} \\ &+ \left[\mathfrak{h}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\eta(\mathfrak{t}))\right) - \mathfrak{h}\left(\mathfrak{t}, \mu(\mathfrak{t}-\eta(\mathfrak{t}))\right)\right] d\omega(\mathfrak{t}), \\ \Delta\Lambda(\mathfrak{t}_k) = \left[\mathsf{x}(\mathfrak{t}_k^-) - \mu(\mathfrak{t}_k^+)\right] - \left[\mathsf{x}(\mathfrak{t}_k) - \mu(\mathfrak{t}_k)\right] &= \mathbb{I}_k\left(\mathsf{x}(\mathfrak{t}_k) - \mu(\mathfrak{t}_k)\right) \\ \Lambda(\mathfrak{t}) &= 0. \end{aligned}$$

The solution is

$$\begin{aligned} \Lambda(\mathfrak{t}) &= -\left[\mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right) - \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right] \\ &+ \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{f}\left(s, \mathsf{x}(s-\rho(s))\right) - \mathfrak{f}\left(s, \mu(s-\rho(s))\right)\right] ds \\ &+ \int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{h}\left(s, \mathsf{x}(s-\eta(s))\right) - \mathfrak{h}\left(s, \mu(s-\eta(s))\right)\right] d\omega(s) \\ &+ \sum_{0<\mathfrak{t}_k<\mathfrak{t}} \mathbb{R}(\mathfrak{t}-\mathfrak{t}_k)\left[\mathbb{I}_k(\mathfrak{x}(\mathfrak{t}_k)) - \mathbb{I}_k(\mu(\mathfrak{t}_k))\right] \end{aligned}$$

$$\begin{aligned} \mathbb{E}\left\|\Lambda(\mathfrak{t})\right\|^2 &\leq 4\mathbb{E}\left\|\mathfrak{g}\left(\mathfrak{t}, \mathsf{x}(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right) - \mathfrak{g}\left(\mathfrak{t}, \mu(\mathfrak{t}-\mathfrak{r}(\mathfrak{t}))\right)\right\|^2 \\ &+ 4\mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{f}\left(s, \mathsf{x}(s-\rho(s))\right) - \mathfrak{f}\left(s, \mu(s-\rho(s))\right)\right] ds\right\|^2 \\ &+ 4\mathbb{E}\left\|\int_0^{\mathfrak{t}} \mathbb{R}(\mathfrak{t}-s)\left[\mathfrak{h}\left(s, \mathsf{x}(s-\eta(s))\right) - \mathfrak{h}\left(s, \mu(s-\eta(s))\right)\right] d\omega(s)\right\|^2 \\ &+ 4\mathbb{E}\left\|\sum_{0<\mathfrak{t}_k<\mathfrak{t}} \mathbb{R}(\mathfrak{t}-\mathfrak{t}_k)\left[\mathbb{I}_k(\mathfrak{x}(\mathfrak{t}_k)) - \mathbb{I}_k(\mu(\mathfrak{t}_k))\right]\right\|^2 \\ &\leq 4\mathscr{K}_1\mathbb{E}\left\|\mathsf{x}(\mathfrak{t}) - \mu(\mathfrak{t})\right\|^2 + 4\mathbb{M}^2\mathbb{T}\int_0^{\mathfrak{t}} \mathscr{K}_2\mathbb{E}\left\|\mathsf{x}(s) - \mu(s)\right\|^2 ds + 4\mathbb{M}^2\mathbb{T}\int_0^{\mathfrak{t}} \mathscr{K}_3\mathbb{E}\left\|\mathsf{x}(s) - \mu(s)\right\|^2 ds \\ &+ 4\mathbb{M}^2 m\sum_{k=1}^{m} \mathsf{d}_k\left\|\mathsf{x}(\mathfrak{t}) - \mu(\mathfrak{t})\right\|^2 \\ &\leq \mathbb{V}^*\int_0^{\mathfrak{t}} \mathbb{E}\left\|\Lambda(s)\right\|^2 ds, \end{aligned}$$

where,

$$\mathbb{V}^* = \frac{4\mathbb{M}^2\mathbb{T}\left[\mathscr{K}_1 + \mathscr{K}_2\right]}{1 - 4\mathscr{K}_1 - 4\mathbb{M}^2 m\sum_{k=1}^{m} \mathsf{d}_k}.$$

By generalized Gronwall's inequality, $\mathbb{E}\|\Lambda(\mathfrak{t})\|^2 = 0$, $\mathsf{x}(\mathfrak{t}) = \mu(\mathfrak{t})$.

Thus the control system (4.1) is trajectory controllable on $\mathscr{J}$.

□

5 Illustration

Let us consider the system of the form:

$$
\begin{aligned}
d\left[u(\mathsf{t},\mathsf{x})-\frac{e^{-\mathsf{t}}}{10}\sin\left(u\left(\mathsf{t}-\frac{1}{2}\cos\mathsf{t}\right)\right)\right] &= \left[\frac{\partial^2}{\partial\mathsf{x}^2}u(\mathsf{t},\mathsf{x})-\frac{e^{-\mathsf{t}}}{10}\sin\left(u\left(\mathsf{t}-\frac{1}{2}\cos\mathsf{t}\right)\right)\right] \\
&+ \frac{e^{-2\mathsf{t}}}{70}\frac{u(\mathsf{t}-\frac{1}{3}(1+\cos\mathsf{t}))}{1+\left[u(\mathsf{t}-\frac{1}{3}(1+\cos\mathsf{t}))\right]^2}dt \\
&+ \int_0^{\mathsf{t}}\mathscr{B}(s)\frac{\partial^2}{\partial\mathsf{x}^2}\left[u(\mathsf{t},\mathsf{x})-\frac{e^{-\mathsf{t}}}{10}\sin\left(u\left(\mathsf{t}-\frac{1}{2}\cos\mathsf{t}\right)\right)\right]d\mathsf{t} \\
&+ \frac{e^{-2\mathsf{t}}}{70}\frac{u(\mathsf{t}-\frac{1}{4}(1+\cos\mathsf{t}))}{1+\left[u(\mathsf{t}-\frac{1}{4}(1+\cos\mathsf{t}))\right]^2}d\omega(\mathsf{t})+e^{-\pi^2\mathsf{t}}d\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(\mathsf{t}),\quad \mathsf{t}\neq\mathsf{t}_k,\mathsf{x}\in[0,\pi], \\
u(\mathsf{t},0) &= u(\mathsf{t},\pi)=0,\quad \mathsf{t}\in\mathcal{J}, \\
\Delta u(\mathsf{t}_k,\mathsf{x}) &= \frac{1}{100k^2}u(\mathsf{t}_k^-),\quad k=1,2, \\
u(\mathsf{t},\mathsf{x}) &= \varphi(\mathsf{t},\mathsf{x}),\quad \mathsf{t}\in(-\tau,0],\quad \mathsf{x}\in[0,\pi], \qquad (5.1)
\end{aligned}
$$

where $\omega(\mathsf{t})$ is a standard cylindrical Weiner process in $\mathbb{X}$, $\mathbb{A}:\mathbb{D}(\mathbb{A})\subset\mathbb{X}\to\mathbb{X}$, defined by $\mathbb{A}\mathsf{y}=z''$ with the domain $\mathbb{D}(\mathbb{A})=\{\mathsf{y}\in\mathbb{X},\mathsf{y},\mathsf{y}'$ are absolutely continuous $\mathsf{y}''\in\mathbb{X},\ \mathsf{y}(0)=\mathsf{y}(\pi)=0\}$ then $\mathbb{A}$ generates an analytic semigroup $\mathcal{T}(\mathsf{t})\in\mathbb{X}$. Moreover $\mathbb{A}$ has a discrete spectrum with eigenvalues $-n^2, n\in\mathbb{N}$ with the corresponding normalized Eigen functions $\mathsf{z}_n(\mathsf{x})=\sqrt{\frac{2}{\pi}}\sin(n\mathsf{x})$. Then the operator $(\mathbb{A})^{1/2}$ is given by

$$(\mathbb{A})^{1/2}\mathsf{x}=\sum_{n=1}^{\infty}n(\mathsf{x},\mathsf{z}_n)\mathsf{z}_n.$$

on the space $\mathbb{D}((\mathbb{A})^{1/2})=\{\mathsf{x}(.)\in\mathbb{X},\ \sum_{n=1}^{\infty}n(\mathsf{x},\mathsf{z}_n)\mathsf{z}_n\in\mathbb{X}\}$. Moreover $\mathcal{T}(\mathsf{t})$ is given by

$$\mathcal{T}(\mathsf{t})\mathsf{x}=\sum_{n=1}^{\infty}e^{n^2\mathsf{t}}(\mathsf{x},\mathsf{z}_n)\mathsf{z}_n.$$

Let $\mathcal{Q}:\mathbb{X}\to\mathbb{X}$ be an operator with sequence $\{\lambda_n\}_{n\geq1}\subset\mathbb{R}^+$ such that $\mathcal{Q}\mathsf{y}_n=\lambda_n\mathsf{y}_n$ and let $Tr\mathcal{Q}=\sum_{n=1}^{\infty}\sqrt{\lambda_n}<\infty$. Also, we may define the fractional Brownian motion:

$$\mathcal{B}^{\mathscr{H}}_{\mathcal{Q}}(\mathsf{t})=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\mathcal{B}^{\mathscr{H}}_n(\mathsf{t})\mathsf{y}_n,$$

where $\mathcal{B}^{\mathscr{H}}_n$ is a sequence of mutually independent two-sided one dimensional fBm. We know that

$$\mathfrak{g}(\mathsf{t},u)=\frac{e^{-\mathsf{t}}}{10}\sin u;\ \mathfrak{f}(\mathsf{t},u)=\frac{e^{-2\mathsf{t}}}{70}\frac{u}{1+u^2};\ \mathfrak{h}(\mathsf{t},u)=\frac{e^{-\mathsf{t}}}{70}\frac{u}{1+u^2}$$
$$\sigma(\mathsf{t}) = e^{-\pi^2\mathsf{t}},\quad \mathbb{I}_k(\mathsf{x}(\mathsf{t}_k))=\frac{1}{100k^2}u(\mathsf{t}_k^-)$$

The delay terms $\mathfrak{r}(\mathsf{t})=\frac{1}{2}\cos\mathsf{t}$, $\rho(\mathsf{t})=\frac{1}{3}(1+\cos\mathsf{t})$, $\eta(\mathsf{t})=\frac{1}{4}(1+\sin\mathsf{t})$. It is evident that the conditions (H1)-(H6) holds. This implies that the conditions of the Theorem 3.1 and Theorem 4.1 gets satisfied. Thus the system (5.1) exist and is T-controllable on $[0,1]$.

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