Optimal controls for neutral stochastic integrodifferential equations with infinite time delay and deviated argument driven by Rosenblatt Process

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Abstract

In this article, the authors set up an optimal control for a class of neutral Stochastic Integro-Differential Equations (SIDEs) with infinite delay and deviated arguments driven by Rosenblatt process in Hilbert space. Sufficient conditions for the existence of mild solution are formulated and proved by using fixed point theorem and stochastic analysis techniques. We have used the axiomatic definition of phase space for infinite time delay process. We have extended the problem in [5] to neutral SIDEs with infinite delay and have used modified techniques to make it compatible with integro-differential system. In addition, the existence of optimal control of the proposed problem is presented by using Balder's theorem. Our result extends the work of [3, 5]. Finally, an example illustrates the potential of the main results.

Keywords: Optimal Controllability, Rossenblatt Process, Neutral functional Stochastic integrodifferential equations, Resolvent operator

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1 Introduction

The concept of Differential Equations (DEs) with deviated argument is a momentous field of nonlinear analysis that seems often in considering systems in the fields of science, engineering, statistics, and so on [1]. In some real-life situations, the delay is not defined on time but also by an unknown variable.

DEs with deviated arguments are a type of delay DEs in which the unknown quantity and its derivative appear in different argument values. For real and fractional differential models, various controllability features involving DEs with deviated argument are examined by authors, see [2, 3]. Moreover, the study of SDEs is important in view of the fact that its applications can be found in biology, chemistry, mechanics, and other fields. In the literature, there is a lot of information about SDEs with Brownian motion, see [8, 16, 17, 23, 24] and the references therein.

Consider $(\xi_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with correlation function holds $\mathfrak{R}(n) := \mathbb{E}(\xi_0\xi_n) = n^{\frac{2\mathscr{H}-2}{\mathrm{k}}}\mathscr{L}(n)$, with $\mathscr{H} \in \left(\frac{1}{2}, 1\right)$ and $\mathscr{L} \to \infty$. Set $\mathscr{G}$ denote the Hermite function of rank $\mathscr{K}$. Also, if $\mathscr{G}$ admits the following,

$$\mathscr{G}(\mathfrak{r}) = \sum_{j\geq 0}\mathfrak{c}_j\mathscr{H}_j(\mathfrak{r}),\ \mathfrak{c}_j = \frac{1}{j!}\mathbb{E}(\mathscr{G}(\xi_0\mathscr{H}(\xi_0))),$$

then $\mathscr{K} = \min\{j|\mathfrak{c}_j \neq 0\} \geq 1$, where $\mathscr{H}_j(\mathfrak{r})$ is the Hermite polynomial of degree j is $\mathscr{H}_j(\mathfrak{r}) = (-1)^j e^{\frac{\mathfrak{r}^2}{2}}\frac{\mathfrak{d}_j}{\mathfrak{d}\mathfrak{r}_j}e^{-\frac{\mathfrak{r}^2}{2}}$. Then by the Non-Central Limit Theorem, $\frac{1}{n^{\mathscr{H}}}\sum_{[nt]}^{j=1}\mathscr{G}(\xi_j)$ converges as $n \to \infty$ in the sense of finite-dimensional distributions to the process

$$\mathbb{Z}_{\mathscr{H}}^{\mathscr{K}}(\iota) = \mathfrak{c}(\mathscr{H},\mathscr{K})\int_{\mathbb{R}^{\mathscr{K}}}\int_0^{\iota}\left(\prod_{j=1}^{\mathscr{K}}(\chi - y_j)_+^{-\left(\frac{1}{2}+\frac{1-\mathscr{H}}{\mathscr{K}}\right)}\right)\mathrm{d}\chi\mathrm{d}\mathbb{B}(y_1)...\mathrm{d}\mathbb{B}(y_{\mathscr{K}}), \tag{1.1}$$

The (1.1) is a Wiener-It$\hat{o}$ multiple integral of order $\mathscr{K}$ w.r.t the standard Bm $(\mathbb{B}(y))_{y\in\mathbb{R}}$ and $\mathfrak{c}(\mathscr{H},\mathscr{K})$ is normalizing constant depends on $\mathscr{H}$ and $\mathscr{K}$. The process $(\mathbb{Z}_{\mathscr{H}}^{\mathscr{K}}(\iota))_{\iota\geq 0}$ is known as the Hermite process.

- If $\mathscr{K} = 1$, the process (1.1) is the fBm with Hurst index $\mathscr{H} \in \left(\frac{1}{2}, 1\right)$ [13].
- If $\mathscr{K} = 2$, the process given by (1.1) is called the Rosenblatt process, and it's not Gausian process. (see [11, 12]).

Self-similar processes with long-range dependence are seen in a variety of fields, including econometrics, internet traffic, hydrology, turbulence, and finance [14, 15]. The Rosenblatt process is a self-similar process with stationary increments that occurs as the limit of long-range-dependent stationary series. Still, it is not, a Gaussian process. However, in real situations when the Gaussianity is not plausible for the model, one can use the Rosenblatt process. Comparatively, Rosenblatt process gains its interest due to its convolution of the dependence structures and the property of non-Gaussianity. Therefore, it seems stimulating to establish the SDEs with Rosenblatt process. Following are some practical uses of Rossenblatt process:

1. Rossenblatt process is useful to study the wavelet expansion.
2. Limiting distributions of the parabolically rescaled solutions of the heat equation with singular non-Gaussian data have similar behavior to the Rosenblatt distribution.

3. The asymptotic distributions in the model are demonstrated to be functionals of Hermite processes in the unit root testing problem with failures being nonlinear transforms of linear processes with long-range dependency.

Many researchers have been established SDEs via the Rosenblatt process, readers one can obtained see [8, 9, 17, 19]. The fractional stochastic optimal control problem refers to optimize the cost functional subject to dynamical constraints on the stochastic control parameter and state variables that having fractional models with noise. The essential purpose of optimal control is to find the ideal control values for the dynamic system under open-loop control that maximize or reduce a particular performance index. Because determining optimal control problems is more complicated than the nonlinear dynamic systems and open-ended endeavors. Optimal control is subsequently applied to biomedicine in cancer chemotherapy, and contemporarily applied to epidemiological models see [18, 21]. Recently, many literature have been demonstrated the optimal control SDEs [22, 23, 24, 25, 26] and references therein).

Motivated by the above fact, we consider the neutral SIDEs with deviated argument governed by Rosenblatt process of the form:

$$\begin{aligned} \mathrm{d}_\iota\left[\mathfrak{x}(\iota)+\mathfrak{h}(\iota,\mathfrak{x}_\iota)\right] &= \Bigg[\mathfrak{A}\left[\mathfrak{x}(\iota)+\mathfrak{h}(\iota,\mathfrak{x}_\iota)\right]+\int_0^\iota \Lambda(\iota-\chi)\left[\mathfrak{x}(\chi)+\mathfrak{h}(\chi,\mathfrak{x}_\chi)\right]\mathrm{d}\chi \\ &+ \mathfrak{f}(\iota,\mathfrak{x}_\iota,\mathfrak{x}(\varpi(\mathfrak{x}(\iota),\iota)))+\mathbf{B}(\iota)\mathbf{u}(\iota)\Bigg]\mathrm{d}\iota+\sigma(\iota,\mathfrak{x}_\iota)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\iota),\ l=\iota\in[0,\mathrm{T}], \\ \mathfrak{x}(\iota) &= \mathfrak{x}_0=\phi\in\mathscr{B}_h,\ \iota\in(-\infty,0]; \end{aligned} \tag{1.2}$$

where the deviating argument $\mathfrak{x}(\varpi(\mathfrak{x}(\iota)\in\mathfrak{C}(\mathbb{X},l)$ is one whose range $\varpi(\mathbb{H},l)$ is disjoint from l. $\mathfrak{A}:\mathfrak{D}(\mathfrak{A})\subset\mathbb{H}\to\mathbb{H}$ generators a strongly continuous semigroup $\{\mathfrak{S}(\iota)\}_{\iota\geq 0}$ in $\mathbb{H}$. Let $\mathbb{Z}_{\mathscr{H}}(\iota)$ be a $\mathbb{K}$-valued Rosenblatt process with parameter $\mathscr{H}\in\left(\frac{1}{2},1\right)$ where $\mathbb{K}$ is an another real separable Hilbert space. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a complete probability space and for $\{\Im_\iota\}_{\iota\geq 0}$ be the σ-field generated by $\{\mathbb{Z}_{\mathscr{H}}(\chi),\ \chi\in[0,\mathrm{T}]\}$ and the $\mathbb{P}$-null sets. Let $\mathbb{U}$ be the Hilbert space and $\mathbf{B}\in\mathcal{L}(\mathbb{U},\mathbb{H})$ be bounded. The control function is $\mathbf{u}(\cdot)\in\mathcal{L}^2_{\Im}(l,\mathbb{U})$. The time history $\mathfrak{x}_\iota:(-\infty,0]\to\mathbb{H}$ defined by $\mathfrak{x}_\iota(\theta)=\mathfrak{x}(\iota+\theta)$ is in the abstract phase space $\mathscr{B}_h$. The nonlinear functions $\mathfrak{h}:l\times\mathscr{B}_h\to\mathbb{H}$, $\mathfrak{f}:l\times\mathscr{B}_h\times\mathbb{H}\to\mathbb{H}$, and $\sigma:l\times\mathscr{B}_h\to\mathcal{L}^0_2(\mathbb{K},\mathbb{H})$ are appropriate mappings and specified in the next section. Let $\mathfrak{C}$ be the closed subspace of all continuous process $\mathfrak{x}$ that belonging to the space $\mathfrak{C}((-\infty,\mathrm{T}],\mathcal{L}^2(\Im,\mathbb{H}))$ consisting of $\Im_\iota$-adapted measurable processes $\{\mathfrak{x}(\iota):\iota\in l\}$ such that $\mathfrak{x}$ is continuous.

Novelty of this research work:

- This work have obliterated the growth conditions utilized in [3, 4] and also added the efficiency of deviating argument.
- The supplemental boundedness and range conditions used in [5] are removed, thereby modifying the limitations on the nonlinearity operator.

- In comparison to [3, 4, 5], we enhance the approach and ease the conditions.
- We have extended the problem in [5] to neutral SIDEs with infinite delay and have used modified techniques to make it compatible with introdifferential system.
- Optimal controllability of neutral SIDEs with deviating argument have received little attention in the literature. In order to bridge this gap, we have looked into the optimal controllability of (1.2).

2 Notations and Preliminary

In this manuscript, we suppose that the phase space is defined axiomatically [27]. Assume that $\mathscr{B}_h$ are developed for $\Im_0$-measurable functions from $(-\infty, 0]$ equipped with the norm $\|\cdot\|_{\mathscr{B}_h}$. Define the abstract phase space for an infinite time delay process by

$$\mathscr{B}_h = \Big\{\zeta : (-\infty, 0] \to \mathrm{X} \text{ for any } \tau > 0 \ (\mathbf{E}\,\|\zeta\|^2)^{1/2} \text{ is bounded and measurable function } [\tau, 0] \text{and } \int_{-\infty}^{0} h(\iota) \sup_{\iota \le \tau \le 0} (\mathbf{E}\,\|\zeta(s)\|^2)^{1/2} \mathrm{d}\iota < +\infty\Big\}$$

Clearly, $\mathscr{B}_h$ is a complete Banach space equipped with the norm $\|\zeta\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(\iota) \sup_{\iota\le\chi\le 0}(\mathbf{E}\,\|\zeta\|^2)^{1/2}\mathrm{d}\iota$

Lemma 2.1. *Presume $\mathfrak{x} \in \mathbb{H}$, then $\forall\ \iota \in [0, \mathrm{T}]$, $\mathfrak{x}_\iota \in \mathscr{B}_h$ and*

$$l(\mathbf{E}(\|\mathfrak{x}(\iota)\|^2))^{\frac{1}{2}} \le l_1 \sup_{0\le\chi\le\iota} (\mathbf{E}\,\|\mathfrak{x}(\chi)\|^2)^{\frac{1}{2}} + \|\mathfrak{x}_0\|_{\mathscr{B}_h},$$

where $l_1 = \int_{-\infty}^{0} h(\chi)\mathrm{d}\chi < \infty$.

Proof: For any $\iota \in [0, a]$, y_ι is bounded and measurable on $[-a, 0]$ for $a > 0$. Further,

$$\begin{aligned}
|||y_\iota|||_{\mathcal{C}_b} &= \int_{-\infty}^{0} b(\chi) \sup_{\theta\in[\chi,0]} \mathbf{E}||y_\iota(\theta)||\mathrm{d}\chi \\
&= \int_{-\infty}^{-\iota} b(\chi) \sup_{\theta\in[\chi,0]} \mathbf{E}||y(\iota+\theta)||\mathrm{d}\chi + \int_{-\iota}^{0} b(\chi) \sup_{\theta\in[\chi,0]} \mathbf{E}||y(\iota+\theta)||\mathrm{d}\chi \\
&= \int_{-\infty}^{-\iota} b(\chi) \sup_{\theta_1\in[\iota+\chi,\iota]} \mathbf{E}||y(\theta_1)||\mathrm{d}\chi + \int_{-\iota}^{0} b(\chi) \sup_{\theta_1\in[\iota+\chi,\iota]} \mathbf{E}||y(\theta_1)||\mathrm{d}\chi \\
&\le \int_{-\infty}^{-\iota} b(\chi)\left[\sup_{\theta_1\in[\iota+\chi,0]} \mathbf{E}||y(\theta_1)|| + \sup_{\theta_1\in[0,\iota]} (\mathbf{E}||y(\theta_1)||^2)^{1/2}\right]\mathrm{d}\chi + \int_{-\iota}^{0} b(\chi) \sup_{\theta_1\in[0,\iota]} (\mathbf{E}||y(\theta_1)||^2)^{1/2}\mathrm{d}\chi \\
&= \int_{-\infty}^{-\iota} b(\chi) \sup_{\theta_1\in[\iota+\chi,0]} \mathbf{E}||y(\theta_1)||\mathfrak{d}\chi + \int_{-\infty}^{0} b(\chi)\mathrm{d}\chi. \sup_{\chi\in[0,\iota]} (\mathbf{E}||y(\chi)||^2)^{1/2} \\
&\le \int_{-\infty}^{0} b(\chi) \sup_{\theta_1\in[\chi,0]} \mathbf{E}||y(\theta_1)||\mathrm{d}\chi + l_1 \sup_{\chi\in[0,\iota]} (\mathbf{E}||y(\chi)||^2)^{1/2} \\
&= \int_{-\infty}^{0} b(\chi) \sup_{\theta_1\in[\chi,0]} ||y_0(\theta_1)||\mathrm{d}\chi + l_1 \sup_{\chi\in[0,\iota]} (\mathbf{E}||y(\chi)||^2)^{1/2} \\
&= l_1 \sup_{\chi\in[0,\iota]} (\mathbf{E}||y(\chi)||^2)^{1/2} + |||y_0|||_{\mathcal{C}_b}
\end{aligned}$$

Since $\phi \in \mathcal{C}_b$, then $y_\iota \in \mathcal{C}_b$. Moreover,

$$\|y_\iota\|_{\mathcal{C}_b} = \int_{-\infty}^{0} b(\chi) \sup_{\theta \in [\chi, 0]} ||y_\iota(\theta)|| \mathrm{d}\chi \geq ||y_\iota(\theta)|| \int_{-\infty}^{0} b(\chi)\mathrm{d}\chi \ \mathbf{E}||y(\iota)||$$

The proof is complete. □

Now assume the following that, if $\mathfrak{x} : (-\infty, \mathrm{T}) \to \mathbb{H}$, $\mathrm{T} > 0$ is continuous on $[0, \mathrm{T})$ and $\mathfrak{x}_0 \in \mathscr{B}$, then for $\iota \in [0, \mathrm{T})$, the following holds:

1. $\mathfrak{x}_\iota \in \mathscr{B}_h$. and $\|\mathfrak{x}(\iota)\| \leq \mathrm{K} \|\mathfrak{x}_\iota\|_{\mathscr{B}_h}$.
2. $\|\mathfrak{x}(\iota)\|_{\mathscr{B}_h} \leq \mathrm{M}(\iota - \mathfrak{k}) \sup\{\|\mathfrak{x}(\chi)\| : 0 \leq \chi \leq \iota\} + \mathrm{N}(\iota - \mathfrak{k}) \|\mathfrak{x}_0\|_{\mathscr{B}_h}$, where $\mathrm{K} > 0$
3. $\mathrm{M}, \mathrm{N} : [0, +\infty) \to [0, +\infty)$ is locally bounded and continuous.
   K, M and N are independent of $\mathfrak{x}(\cdot)$.

Partial Integro-differential Equations: Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces such that

$$|\mathrm{y}|_{\mathbb{Y}} := |\mathfrak{A}\mathrm{y}| + |\mathrm{y}| \ \text{ for } \mathrm{y} \in \mathbb{Y}.$$

and $\Lambda(\iota)$ are closed linear operators on $\mathbb{X}$.

Consider the following system

$$\begin{aligned} \mathrm{du}(\iota) &= \left(\mathfrak{A}\mathrm{u}(\iota) + \int_0^\iota \Lambda(\iota - \chi)\mathrm{u}(\chi)\mathrm{d}\chi\right)\mathrm{d}\iota, \ \iota \geq 0, \\ \mathrm{u}(0) &= \mathrm{u}_0 \in \mathbb{X}. \end{aligned} \tag{2.1}$$

Definition 2.1. *A resolvent for Equation (2.1) is a bounded linear operator valued function $\mathscr{R}(\iota) \in \mathcal{L}(\mathbb{X})$, $\iota \geq 0$, having the following properties:*

(i) $\mathscr{R}(0) = I$ *and* $\|\mathscr{R}(\iota)\| \leq \mathrm{N}e^{\alpha\iota}$ *for all* $\iota \geq 0$ *for* $\mathrm{N} \geq 1$ *and* $\alpha > 0$

(ii) For $\mathfrak{x} \in \mathbb{X}$, $\mathscr{R}(\iota)\mathfrak{x}$ *is continuous for* $\iota \geq 0$

(iii) For $\mathfrak{x}$ *in* $\mathbb{Y}$, $\mathscr{R}(\cdot)\mathfrak{x} \in \mathfrak{C}^1([0, +\infty); \mathbb{X}) \bigcap \mathfrak{C}([0, +\infty); \mathbb{Y})$ *and*

$$\begin{aligned} \mathrm{d}\mathscr{R}(\iota)\mathfrak{x} &= \left(\mathfrak{A}\mathscr{R}(\iota)\mathfrak{x} + \int_0^\iota \Lambda(\iota - \chi)\mathscr{R}(\chi)\mathfrak{x}\mathrm{d}\chi\right)\mathrm{d}\iota \\ &= \left(\mathscr{R}(\iota)\mathfrak{A}\mathfrak{x} + \int_0^\iota \mathscr{R}(\iota - \chi)\Lambda(\chi)\mathfrak{x}\mathrm{d}\chi\right)\mathrm{d}\iota. \end{aligned}$$

The reader is recommended to [6] for more information on the resolvent operator. We use the following assumptions to deal with the existence of a resolvent operator:

(A1) For all $\iota \geq 0$, $\Lambda(\iota) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes a closed, continuous operator. For any $\mathrm{y} \in \mathbb{Y}$, the map $\iota \to \Lambda(\iota)\mathrm{y}$ is bounded, differentiable, and its derivative $\mathrm{d}\Lambda(\iota)y/\mathrm{d}\iota$ is bounded and uniformly continuous on $[0, \infty)$.

The following theorem provides sufficient criteria for the existence of the resolvent operator of (2.1).

Theorem 2.1. *Assume (A1) hold. Then, $\exists$ a unique resolvent operator for the Cauchy problem (2.1).*

Theorem 2.2. *Assume (A1) hold. Then, the corresponding resolvent operator $\Lambda(\iota)$ of (2.1) is continuous for $\iota \geq 0$ on the operator norn, namely for $\iota_0 \geq 0$, it holds that*

$$\lim_{\gamma \to 0} \|\Lambda(\iota_0 - \gamma) - \Lambda(\iota_0)\| = 0.$$

Theorem 2.3. *Assume (A1) be satisfied. Then $\exists\ \mathrm{C} > 0 \ \ni \|\Lambda(\iota + \epsilon) - \Lambda(\epsilon)\Lambda(\iota)\| \leq \mathrm{C}\epsilon$.*

Lemma 2.2. *If $\Xi : l \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ satisfies $\int_0^{\mathrm{T}} \|\Xi(\chi)\|^2_{\mathcal{L}_2^0} \mathrm{d}\chi < \infty$. Then, (1.1) is well defined and $\mathbb{H}$-valued random variable and*

$$\mathbb{E}\left\| \int_0^{\iota} \Xi(\chi) \mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) \right\|^2 < \mathrm{C}_{\mathscr{H}} \iota^{2\mathscr{H}-1} \int_0^{\iota} \|\Xi(\chi)\|^2_{\mathcal{L}_2^0} \mathrm{d}\chi.$$

Definition 2.2. *A $\mathbb{H}$-valued stochastic processes $\mathfrak{x} : (0, \mathrm{T}] \to \mathbb{H}$ is called a mild solution of equation (1.1), if $(\iota) = \mathfrak{x}_0 = \phi \in \mathscr{B}$, $\iota \in (-\infty, 0]$,*

(i) $\mathfrak{x}(\iota)$ is measurable and $\Im_\iota$-adapted, $\iota \geq 0$

(ii) For $\iota \in l$ a.s., $\mathfrak{x}(\iota) \in \mathbb{H}$ has a Cadlag path and for every $\iota \in l$, $\mathfrak{x}(\iota)$, satisfies

$$\begin{aligned} \mathfrak{x}(\iota) &= \mathscr{R}(\iota)\left[\phi(0) + \mathfrak{h}(0, \phi(0))\right] - \mathfrak{h}(\iota, \mathfrak{x}_\iota) + \int_0^{\iota} \mathscr{R}(\iota - \chi) \mathfrak{f}(\chi, \mathfrak{x}_\chi, \mathfrak{x}(\varpi(\mathfrak{x}(\chi), \chi))) \mathrm{d}\chi \\ &+ \int_0^{\iota} \mathscr{R}(\iota - \chi) \mathbf{B}(\iota) \mathbf{u}(\iota) \mathrm{d}\chi + \int_0^{\iota} \mathscr{R}(\iota - \chi) \sigma(\chi, \mathfrak{x}(\chi)) \mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi). \end{aligned} \tag{2.2}$$

3 Main Results

In this section, we prove the existence of the mild solution for system (1.2). In the whole of this work, we assume (A1) and (A2) are true. The following assumptions are necessary to prove the main results:

(H1) $\mathscr{R}(\iota)$ is compact, for $\iota > 0$.

(H2) The function $\mathfrak{f} : l \times \mathscr{B} \times \mathbb{H} \to \mathbb{H}$ satisfies, $\forall\ \mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{B}$ and $\acute{\mathfrak{x}}_1, \acute{\mathfrak{x}}_2 \in \mathbb{H}$

$$\mathbb{E}\|\mathfrak{f}(\iota, \mathfrak{x}_1, \acute{\mathfrak{x}}_1) - \mathfrak{f}(\iota, \mathfrak{x}_2, \acute{\mathfrak{x}}_2)\|^2 \leq \Theta_{\mathfrak{f}}\left[\mathbb{E}\|\mathfrak{x}_1 - \mathfrak{x}_2\|^2_{\mathscr{B}} + \mathbb{E}\|\acute{\mathfrak{x}}_1 - \acute{\mathfrak{x}}_2\|\right].$$

(H3) Let $\varpi : \mathbb{H} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies and with constant $\Theta_\varpi > 0$ such that

$$\mathbb{E}\|\varpi(\mathfrak{x}_1, \chi) - \varpi(\mathfrak{x}_2, \chi)\|^2_{\mathbb{R}^+} \leq \Theta_\varpi\left[\mathbb{E}\|\mathfrak{x}_1 - \mathfrak{x}_2\|^2_{\mathscr{B}}\right].$$

and $\varpi(\cdot, 0) = 0$.

(H4) The function $\mathfrak{h} : l \times \mathscr{B}_h \to \mathbb{H}$ is continuous. Moreover,

(i) $\mathfrak{h}(\cdot, \mathfrak{x})$ is strongly measurable, $\iota \in l$.

(ii) For $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{B}_h$, $\mathfrak{h}(\iota, \cdot)$ satisfies, $\Theta_{\mathfrak{h}} > 0$,

$$\begin{aligned} \mathbb{E} \left\| \mathfrak{h}(\iota, \mathfrak{x}_1) - \mathfrak{h}(\iota, \mathfrak{x}_2) \right\|^2 &\leq \Theta_{\mathfrak{h}} \mathbb{E} \left\| \mathfrak{x}_1 - \mathfrak{x}_2 \right\|^2_{\mathscr{B}_h}, \\ \mathbb{E} \left\| \mathfrak{h}(\iota, \mathfrak{x}) \right\|^2 &\leq \Theta_{\mathfrak{h}} \left(1 + \mathbb{E} \left\| \mathfrak{x} \right\|^2 \right). \end{aligned}$$

(H5) The function $\sigma : l \times \mathscr{B}_h \to \mathcal{L}_2^0$ satisfies, for every $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{B}_h$,

$$\mathbb{E} \left\| \sigma(\iota, \mathfrak{x}_1) - \sigma(\iota, \mathfrak{x}_2) \right\|^2 \leq \Theta_{\sigma} \mathbb{E} \left\| \mathfrak{x}_1 - \mathfrak{x}_2 \right\|^2_{\mathscr{B}_h}.$$

(H6) The linear operator W from $\mathcal{L}^2(l, \mathbb{U})$ into $\mathcal{L}^2(\mathrm{W}, \mathbb{H})$, defined by

$$\mathrm{W}\mathbf{u} = \int_l \mathscr{R}(\iota - \chi) \mathbf{B} \mathbf{u}(\chi) \mathrm{d}\chi$$

has an induced inverse W^{-1} which takes values in $\mathcal{L}^2(l, \mathbb{U}) / \ker \mathrm{W}$ and $\exists$ two +ve constant $\mathrm{M}_{\mathbf{B}}$ and M_{W} such that

$$\|\mathbf{B}\| \leq \mathrm{M}_{\mathbf{B}}, \qquad \left\| \mathrm{W}^{-1} \right\| \leq \mathrm{M}_{\mathrm{W}}.$$

Theorem 3.1. *Under (H1)-(H6) hold, then there exists a mild solution of (1.2) on l provided that*

$$\mathfrak{D} = 2\Theta_{\mathfrak{h}} \mathrm{M}_{\mathrm{T}} + 2\Theta_{\mathfrak{f}} \mathrm{T}^2 \Theta_{\mathscr{R}}^2 \left[2\Theta_{\varpi} \widetilde{r} + \mathrm{M}_{\mathrm{T}} \right].$$

Proof. Let $\mathscr{B}_{\mathrm{T}}$ be the set defined by $\mathscr{B}_{\mathrm{T}} = \left\{ \mathfrak{x} : (-\infty, \mathrm{T}] \to \mathbb{H} \ \ni \ \mathfrak{x}|(-\infty, 0] \in \mathscr{B}_h, \mathfrak{x}|l \in \mathfrak{C} \right\}$.

Define the operator $\Phi : \mathscr{B}_{\mathrm{T}} \to \mathscr{B}_{\mathrm{T}}$ by

$$(\Phi\mathfrak{x})(\iota) = \begin{cases} 0, & \iota \in (-\infty, 0], \\ \mathscr{R}(\iota) \left[\phi(0) + \mathfrak{h}(0, \phi(0)) \right] - \mathfrak{h}(\iota, \mathfrak{x}_\iota) + \int_0^\iota \mathscr{R}(\iota - \chi) \mathfrak{f}(\chi, \mathfrak{x}_\chi, \mathfrak{x}(\varpi(\mathfrak{x}(\chi), \chi))) \mathrm{d}\chi \\ + \int_0^\iota \mathscr{R}(\iota - \chi) \mathbf{B}(\iota) \mathbf{u}(\iota) \mathrm{d}\chi + \int_0^\iota \mathscr{R}(\iota - \chi) \sigma(\chi, \mathfrak{x}(\chi)) \mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi), & \mathrm{t} \in l. \end{cases}$$

Using (H6), we get

$$\mathbb{E} \left\| \int_\chi^\iota \mathbf{B}(\chi) \mathbf{u}(\chi) \mathrm{d}\chi \right\|^2 \leq \Theta_{\mathscr{R}}^2 \mathrm{T} \left\| \mathbf{B} \right\|^2 \left\| \mathbf{u} \right\|^2_{\mathcal{L}_{\mathfrak{F}}}.$$

Then, from the Bochner theorem, it follows that $\mathscr{R}(\iota - \chi) \mathbf{B}(\chi) \mathbf{u}(\chi)$ is integrable on l. The nonlinear functions $\mathfrak{h}, \mathfrak{f}$ and σ are continuous on l, and the set $\Phi(\mathfrak{x})$ is well defined on l. Next, to prove that Φ has a fixed point. For $\phi \in \mathscr{B}_h$,

$$\widetilde{\phi} = \begin{cases} \phi(\iota), & \iota \in (-\infty, 0], \\ \mathscr{R}(\iota) \phi(0), & \iota \in l. \end{cases} \tag{3.1}$$

Then, $\widetilde{\phi} \in \mathscr{B}_{\mathrm{T}}$. Let $\mathfrak{x}(\iota)=\widetilde{\phi}(\iota)+\zeta(\iota)$, $-\infty<\iota\leq \mathrm{T}$. It is clear that $\mathfrak{x}(\iota)$ satisfies Definition 3.1 iff $\zeta(\iota)$ satisfies that $\zeta(0)=0$ and

$$\begin{aligned}\zeta(\iota) &= \mathscr{R}(\iota)\mathfrak{h}(0,\phi(0))-\mathfrak{h}(\iota,\widetilde{\phi}(\iota)+\zeta(\iota))+\int_0^\iota \mathscr{R}(\iota-\chi)\mathfrak{f}(\chi,\phi_\chi+\zeta(\chi),\widetilde{\phi}(\chi)+\zeta(\chi)(\varpi(\widetilde{\phi}(\chi)+\zeta(\chi),\chi)))\mathrm{d}\chi \\ &+ \int_0^\iota \mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}(\chi)\mathrm{d}\chi+\int_0^\iota \mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi).\end{aligned}$$

Let $\mathscr{B}_{\mathrm{T}}^0=\{\zeta\in\mathscr{B}_{\mathrm{T}},\ \zeta(0)=0\in\mathscr{B}\}$. For every $\zeta\in\mathscr{B}_{\mathrm{T}}^0$,

$$\|\zeta\|_{\mathrm{T}}=\|\zeta_0\|_{\mathscr{B}}+\sup_{0\leq\iota\leq \mathrm{T}}\mathbb{E}\|\zeta(\iota)\|=\sup_{0\leq\iota\leq \mathrm{T}}\mathbb{E}\|\zeta(\iota)\|.$$

Thus, $\left(\mathscr{B}_{\mathrm{T}}^0,\|\cdot\|_{\mathrm{T}}\right)$ is a Banach space.

Define the set $\Lambda_r=\left\{\mathrm{v}\in\mathscr{B}_{\mathrm{T}}^0,\|\mathrm{v}\|^2\leq r\right\}$, $r\geq 0$. Then, obviously, $\Lambda_r\subset\mathscr{B}_{\mathrm{T}}^0$ is uniformly bounded. Furthermore, we have $\zeta\in\Lambda_r$,

$$\begin{aligned}\left\|\widetilde{\phi}(\iota)+\zeta(\iota)\right\|_{\mathscr{B}}^2 &= 2\left(\left\|\widetilde{\phi}(\iota)\right\|_{\mathscr{B}}^2+\|\zeta(\iota)\|_{\mathscr{B}}^2\right), \\ &\leq 2\left(\mathrm{M}_{\mathrm{T}}^2\sup_{\chi\in l}\mathbb{E}\|\zeta(\chi)\|^2+\mathrm{N}_{\mathrm{T}}^2\sup_{\chi\in l}\mathbb{E}\|\zeta(0)\|^2+\mathrm{M}_{\mathrm{T}}^2\sup_{\chi\in l}\mathbb{E}\left\|\widetilde{\phi}(\iota)\right\|^2+\mathrm{N}_{\mathrm{T}}^2\sup_{\chi\in l}\mathbb{E}\left\|\widetilde{\phi}(0)\right\|^2\right), \\ &\leq 2\mathrm{M}_{\mathrm{T}}^2\left(\zeta+\mathrm{M}^2\|\phi\|^2\right)+2\mathrm{M}_{\mathrm{T}}^2\|\phi\|_{\mathscr{B}}^2, \\ &= \widetilde{r}.\end{aligned}$$

Now, we define the operator $\widetilde{\Phi}:\mathscr{B}_{\mathrm{T}}^0\to\mathscr{B}_{\mathrm{T}}^0$ by

$$(\widetilde{\Phi}\zeta)(\iota)=\begin{cases}0, \quad \iota\in(-\infty,0], \\ \mathscr{R}(\iota)\mathfrak{h}(0,\phi(0))-\mathfrak{h}(\iota,\widetilde{\phi}(\iota)+\zeta(\iota)) \\ +\int_0^\iota\mathscr{R}(\iota-\chi)\mathfrak{f}(\chi,\phi_\chi+\zeta(\chi),\widetilde{\phi}(\chi)+\zeta(\chi)(\varpi(\widetilde{\phi}(\chi)+\zeta(\chi),\chi))) \\ +\int_0^\iota\mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}(\chi)\mathrm{d}\chi+\int_0^\iota\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi),\ \mathrm{t}\in l.\end{cases}$$

Notice that $\widetilde{\Phi}$ is well defined on Λ_r, for $r>0$. Note that Φ has a fixed point iff $\widetilde{\Phi}$ has a fixed point. First, we decompose $\widetilde{\Phi}$ as $\widetilde{\Phi}=\widetilde{\Phi}_1+\widetilde{\Phi}_2$. Define

$$\begin{aligned}(\widetilde{\Phi}_1\zeta)(\iota) &= \mathscr{R}(\iota)\mathfrak{h}(0,\phi(0))-\mathfrak{h}(\iota,\widetilde{\phi}(\iota)+\zeta(\iota)) \\ &+ \int_0^\iota\mathscr{R}(\iota-\chi)\mathfrak{f}(\chi,\phi_\chi+\zeta(\chi),\widetilde{\phi}(\chi)+\zeta(\chi)(\varpi(\widetilde{\phi}(\chi)+\zeta(\chi),\chi)))\mathrm{d}\chi \\ (\widetilde{\Phi}_2\zeta)(\iota) &= \int_0^\iota\mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}(\chi)\mathrm{d}\chi+\int_0^\iota\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi).\end{aligned}$$

Now, we shall show that the operators $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ satisfy all the conditions of Theorem 3.1.

Step: 1 $\widetilde{\Phi}_1$ is a contraction on $\mathscr{B}_{\mathrm{T}}^0$.

Let $t \in l$ and $r_1, r_2 \in \mathscr{B}_r$. Consider

$$\begin{aligned}
\mathbb{E}\left\|(\widetilde{\Phi}_1 r_1)(\iota) - (\widetilde{\Phi}_2 r_2)(\iota)\right\|^2 &\leq 2\mathbb{E}\left\|\mathfrak{h}(\iota, \widetilde{\phi}(\iota) + \zeta_{1\iota}) - \mathfrak{h}(\iota, \widetilde{\phi}(\iota) + \zeta_{2\iota})\right\|^2 \\
&+ 2\mathbb{E}\Big\|\int_0^\iota \mathscr{R}(\iota - \chi)\big[\mathfrak{f}(\chi, \phi_\chi + \zeta_{1\chi}, (\widetilde{\phi} + \zeta_1)(\varpi(\widetilde{\phi}(\chi) + \zeta_1(\chi), \chi))) \\
&- \mathfrak{f}(\chi, \phi_\chi + \zeta_{2\chi}, (\widetilde{\phi} + \zeta_2)(\varpi(\widetilde{\phi}(\chi) + \zeta_2(\chi), \chi)))\big]\mathrm{d}\chi\Big\|^2 \\
&\leq 2\Theta_{\mathfrak{h}}\left\|\zeta_{1\iota} - \zeta_{2\iota}\right\|^2_{\mathscr{B}_h} \\
&+ \mathrm{T}\Theta^2_{\mathscr{R}}\Theta_{\mathfrak{f}}\int_0^\iota \left[\widetilde{r}\Theta_\varpi \mathbb{E}\left\|\zeta_1 - \zeta_2\right\|^2_{\mathscr{B}_h} + \mathbb{E}\left\|\zeta_1 - \zeta_2\right\|^2 + \widetilde{r}\Theta_\varpi \mathbb{E}\left\|\zeta_1 - \zeta_2\right\|^2\right]\mathrm{d}\chi \\
&\leq 2\Theta_{\mathfrak{h}}\mathrm{M}_\mathrm{T}\sup_{\chi\in(0,\iota)}\mathbb{E}\left\|\zeta_1(\chi) - \zeta_2(\chi)\right\|^2_{\mathscr{B}_h} + 2\Theta_{\mathfrak{f}}\mathrm{T}^2\Theta^2_{\mathscr{R}}\big[2\Theta_\varpi\widetilde{r}\mathbb{E}\left\|\zeta_1 - \zeta_2\right\|^2_{\mathscr{B}_h} \\
&+ \mathrm{M}_\mathrm{T}\sup_{\chi\in(0,\iota)}\mathbb{E}\left\|\zeta_1(\chi) - \zeta_2(\chi)\right\|^2_{\mathscr{B}_h}\big] \\
&\leq 2\Theta_{\mathfrak{h}}\mathrm{M}_\mathrm{T}\sup_{\chi\in(0,\iota)}\mathbb{E}\left\|\zeta_1(\chi) - \zeta_2(\chi)\right\|^2_{\mathscr{B}_h} \\
&+ 2\Theta_{\mathfrak{f}}\mathrm{T}^2\Theta^2_{\mathscr{R}}\big[2\Theta_\varpi\widetilde{r} + \mathrm{M}_\mathrm{T}\big]\sup_{\chi\in(0,\iota)}\mathbb{E}\left\|\zeta_1(\chi) - \zeta_2(\chi)\right\|^2_{\mathscr{B}_h} \\
&\leq \big\{2\Theta_{\mathfrak{h}}\mathrm{M}_\mathrm{T} + 2\Theta_{\mathfrak{f}}\mathrm{T}^2\Theta^2_{\mathscr{R}}\big[2\Theta_\varpi\widetilde{r} + \mathrm{M}_\mathrm{T}\big]\big\}\sup_{\chi\in(0,\iota)}\mathbb{E}\left\|\zeta_1(\chi) - \zeta_2(\chi)\right\|^2_{\mathscr{B}_h} \\
&:= \mathfrak{D}\mathbb{E}\left\|\zeta_1 - \zeta_2\right\|^2_{\mathscr{B}_h},
\end{aligned}$$

where $\mathfrak{D} = 2\Theta_{\mathfrak{h}}\mathrm{M}_\mathrm{T} + 2\Theta_{\mathfrak{f}}\mathrm{T}^2\Theta^2_{\mathscr{R}}\big[2\Theta_\varpi\widetilde{r} + \mathrm{M}_\mathrm{T}\big]$.

We have $\mathfrak{D} < 1$, and thus, $\widetilde{\Phi}_1$ is a contraction mapping.

Step: 2 $\widetilde{\Phi}_1$ maps bounded sets into bounded set in Λ_r.

It is sufficient to show that there exists a +ve constant $\mathfrak{D}_1$ such that, we have for any $\zeta \in \mathscr{B}_r$, $\mathbb{E}\left\|(\widetilde{\Phi}_2\zeta)(\mathrm{t})\right\| \leq \mathfrak{D}_1$.

$$\begin{aligned}
\mathbb{E}\left\|(\widetilde{\Phi}_2\zeta)(\iota)\right\|^2 &\leq 2\mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota - \chi)\mathbf{B}(\chi)\mathbf{u}(\chi)\mathrm{d}\chi + \int_0^\iota \mathscr{R}(\iota - \chi)\sigma(\chi, \widetilde{\phi}(\chi) + \zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right\|^2 \\
&\leq 2\mathrm{T}\Theta^2_{\mathscr{R}}\left\|\mathbf{B}\right\|^2\left\|\mathbf{u}\right\|^2_{\mathcal{L}_\Im} + 2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\Theta^2_{\mathscr{R}}\mathbb{E}\left\|\sigma(\chi, \widetilde{\phi}(\chi) + \zeta(\chi))\right\|^2 \\
&\leq 2\mathrm{T}\Theta^2_{\mathscr{R}}\left\|\mathbf{B}\right\|^2\left\|\mathbf{u}\right\|^2_{\mathcal{L}_\Im} + 2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\Theta^2_{\mathscr{R}}\widetilde{r} \\
&\leq 2\Theta^2_{\mathscr{R}}\left[\mathrm{T} + \left\|\mathbf{B}\right\|^2\left\|\mathbf{u}\right\|^2_{\mathcal{L}_\Im} + 2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\widetilde{r}\right] \\
&:= \mathfrak{D}_1.
\end{aligned}$$

Step 3: The operator $\widetilde{\Phi}_2$ is completely continuous.

First, we show that $\widetilde{\Phi}_2$ maps $\mathscr{B}_r$ into an equicontinuous family. Suppose if $0 < \epsilon < \gamma < \gamma + \mathscr{T} < \mathrm{T}$

with $\epsilon > 0$ and $\mathscr{T} > 0$.

$$
\begin{aligned}
&\mathbb{E}\left\|(\widetilde{\Phi}_2\zeta)(\gamma+\mathscr{T}) - (\widetilde{\Phi}_2\zeta)(\gamma)\right\|^2 \\
&\leq \mathbb{E}\left\|\int_0^{\gamma+\mathscr{T}} \mathscr{R}(\gamma+\mathscr{T}-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) - \int_0^{\gamma} \mathscr{R}(\gamma-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right\|^2 \\
&\leq \mathbb{E}\left\|\int_0^{\gamma} [\mathscr{R}(\gamma+\mathscr{T}-\chi)-\mathscr{R}(\gamma-\chi)]\,\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) + \int_{\gamma}^{\gamma+\mathscr{T}} \mathscr{R}(\gamma+\mathscr{T}-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right\|^2 \\
&\leq 3\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\sup_{\chi\in[0,\gamma]}\|\mathscr{R}(\gamma+\mathscr{T}-\chi)-\mathscr{R}(\gamma-\chi)\|^2\int_0^{\gamma}\mathbb{E}\left\|\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\right\|^2_{\mathcal{L}^2_{\mathfrak{I}}}\mathrm{d}\chi \\
&\quad + 3\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta^2_{\mathscr{R}}\int_{\gamma}^{\gamma+\mathscr{T}}\mathbb{E}\left\|\sigma(\chi,\widetilde{\phi}(\chi)+\zeta(\chi))\right\|^2_{\mathcal{L}^2_{\mathfrak{I}}}\mathrm{d}\chi \\
&\leq 3\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\sup_{\chi\in[0,\gamma]}\|\mathscr{R}(\gamma+\mathscr{T}-\chi)-\mathscr{R}(\gamma-\chi)\|^2\,\widetilde{r}\gamma + 3\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta^2_{\mathscr{R}}\mathscr{T}\widetilde{r}.
\end{aligned}
$$

Applying Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above results tends to 0 as $\mathscr{T} \to 0$. Thus, $\widetilde{\Phi}_2$ is continuous from the right in $(0,\mathrm{T}]$. A similar argument prove that it is also continuous from the left in $(0,\mathrm{T}]$. Therefore, $\widetilde{\Phi}_2$ maps bounded sets into an family of equicontinuous family.

Step 4: $\widetilde{\Phi}_2\mathscr{B}_r$ is relatively compact.

Let $\iota \in l$ be fixed and $\epsilon \in \mathbb{R}$ be a number satisfying $0 < \epsilon < \iota$.

For any $\zeta \in \mathscr{B}_r$, we define

$$
\begin{aligned}
(\widetilde{\Phi}^{\epsilon}_2\zeta)(\iota) &= \int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi)\mathbf{B}\mathbf{u}(\chi)\mathrm{d}\chi + \int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) \\
&= \mathscr{R}(\epsilon)\left[\int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi-\epsilon)\mathbf{B}\mathbf{u}(\chi)\mathrm{d}\chi + \int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi-\epsilon)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right] \\
&= \mathscr{R}(\epsilon)(\widetilde{\Phi}^{\epsilon}_2\zeta)(\iota-\epsilon).
\end{aligned}
$$

Since $\mathscr{R}(\epsilon)$ is compact, the set $\mathfrak{G}_\epsilon(\iota) = \left\{(\widetilde{\Phi}_\epsilon(\zeta))(\iota) : \mathrm{z} \in \mathscr{B}_r\right\}$ is the precompact in $\mathbb{H}$ for every ϵ with $0 < \epsilon < \iota$. In addition, $\zeta \in \mathscr{B}_r$, we have

$$
\begin{aligned}
\mathbb{E}\left\|(\widetilde{\Phi}_2\zeta)(\iota) - (\widetilde{\Phi}^{\epsilon}_2\zeta)(\iota)\right\|^2 &\leq \mathbb{E}\Big\|\int_0^{\iota}\mathscr{R}(\iota-\chi)\mathbf{B}\mathbf{u}(\chi)\mathrm{d}\chi + \int_0^{\iota}\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) \\
&\quad - \int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi)\mathbf{B}\mathbf{u}(\chi)\mathrm{d}\chi + \int_0^{\iota-\epsilon}\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\Big\|^2 \\
&\leq \mathbb{E}\left\|\int_{\iota-\epsilon}^{\iota}\mathscr{R}(\iota-\chi)\mathbf{B}\mathbf{u}(\chi)\mathrm{d}\chi + \int_{\iota-\epsilon}^{\iota}\mathscr{R}(\iota-\chi)\sigma(\chi,\widetilde{\phi}(\chi)+\zeta_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right\|^2 \\
&\leq \Theta^2_{\mathscr{R}}\|\mathbf{B}\|^2\|\mathbf{u}\|^2_{\mathcal{L}^0_{\mathfrak{I}}}\epsilon + \Theta^2_{\mathscr{R}}\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\int_{\iota-\epsilon}^{\iota}\mathrm{d}\chi \\
&\leq \Theta^2_{\mathscr{R}}\|\mathbf{B}\|^2\|\mathbf{u}\|^2_{\mathcal{L}^0_{\mathfrak{I}}}\epsilon + \Theta^2_{\mathscr{R}}\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\widetilde{r}\epsilon.
\end{aligned}
$$

As ϵ approaches to 0, the right-hand side of the preceding inequality tends to 0. These are arbitrarily close precompact sets to the sets $\left\{(\widetilde{\Phi}_2\zeta)(\iota) : \zeta \in \mathscr{B}_r\right\}$. Hence, the set $\left\{(\widetilde{\Phi}_\epsilon(\zeta))(\iota) : \zeta \in \mathscr{B}_r\right\}$ is precompact in $\mathbb{H}$.

Step 5: The operator $\mathscr{B}_{\mathrm{T}}^0 \to \mathscr{B}_{\mathrm{T}}^0$ is continuous.

Let $\{\zeta\}_{n=1}^{\infty}$ in $\mathscr{B}_{\mathrm{T}}^0$ with $\zeta_n \to \zeta \in \mathscr{B}_{\mathrm{T}}^0$. Then, $r > 0$ s.t $\|\zeta_n(\iota)\| < r$, $\forall n$ and $\iota \in l$ a.e., so $\zeta, \zeta_n \in \Lambda_r$

$$\sigma(\iota, \widetilde{\phi}_\iota + \zeta_{n\iota}) \to \sigma(\iota, \widetilde{\phi}_\iota + \zeta_\iota) \text{ as } n \to \infty,\ \iota \in l.$$

Since

$$\mathbb{E}\left\|\sigma(\iota, \widetilde{\phi}_\iota + \zeta_{n\iota}) \to \sigma(\iota, \widetilde{\phi}_\iota + \zeta_\iota)\right\|^2 \le \Theta_\sigma \mathbb{E}\|\zeta_{n\iota} - \zeta_\iota\|_{\mathscr{B}_h}^2,$$

by dominated convergence theorem,

$$\begin{aligned}\mathbb{E}\left\|(\widetilde{\phi}_\iota + \zeta_{n\iota})(\iota) - (\widetilde{\phi}_\iota + \zeta_\iota)(\iota)\right\|^2 &\le \mathbb{E}\left\|\int_0^\iota \mathscr{R}(\iota-\chi)\left[\sigma(\iota, \widetilde{\phi}_\iota + \zeta_{n\iota})\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi) - \sigma(\iota, \widetilde{\phi}_\iota + \zeta_\iota)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right]\right\|^2 \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus, $\widetilde{\Phi}_2$ is continuous. Therefore, $\widetilde{\Phi}_2$ is completely continuous.

Step 6: The set $\Delta = \left\{\zeta \in \mathscr{B}_{\mathrm{T}}^0 : \mathfrak{x} = \lambda\widetilde{\Phi}_1\frac{\zeta}{\lambda} + \lambda\widetilde{\Phi}_2\zeta\right\}$, $\lambda \in (0,1)$ is bounded on l. Let $\zeta \in \Delta$ and for some $0 < \lambda < 1$,

$$\begin{aligned}\mathbb{E}\|\zeta(\iota)\|^2 &\le 5\mathbb{E}\Bigg\{\|\mathscr{R}(\iota)\mathfrak{h}(0,\phi(0))\|^2 + \left\|\mathfrak{h}(\iota, \widetilde{\phi}(\iota) + \zeta(\iota))\right\|^2 \\ &+ \left\|\int_0^\iota \mathscr{R}(\iota-\chi)\mathfrak{f}(\chi, \phi(\chi) + \zeta(\chi), (\widetilde{\phi}(\chi) + \zeta(\chi))(\varpi(\widetilde{\phi}(\chi) + \zeta(\chi), \chi)))\right\|^2 \mathrm{d}\chi \\ &+ \left\|\int_0^\iota \mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}(\chi)\right\|^2 \mathrm{d}\chi + \left\|\int_0^\iota \mathscr{R}(\iota-\chi)\sigma(\chi, \widetilde{\phi}(\chi) + \zeta(\chi))\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\right\|^2\Bigg\} \\ &\le 5\Bigg[\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \|\phi(0)\|^2) + \Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \widetilde{r}) \\ &+ \mathrm{T}\Theta_{\mathscr{R}}^2\int_0^\iota\Bigg\{\mathbb{E}\Big\|\mathfrak{f}(\chi, \phi(\chi) + \zeta(\chi), (\widetilde{\phi}(\chi) + \zeta(\chi))(\varpi(\widetilde{\phi}(\chi) + \zeta(\chi), \chi))) \\ &- \mathfrak{f}(\chi, 0, (\widetilde{\phi}(\chi) + \zeta(\chi))(\varpi(\widetilde{\phi}(\chi) + \zeta(\chi), 0)))\Big\|^2 + \left\|\mathfrak{f}(\chi, 0, (\widetilde{\phi}(\chi) + \zeta(\chi))(\varpi(\widetilde{\phi}(\chi) + \zeta(\chi), 0)))\right\|^2\Bigg\}\mathrm{d}\chi \\ &+ \mathrm{T}\Theta_{\mathscr{R}}^2\|\mathbf{B}\|^2\int_0^\iota\|\mathbf{u}(\chi)\|^2\mathrm{d}\chi + \Theta_{\mathscr{R}}^2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\int_0^\iota\mathbb{E}\left\|\widetilde{\phi}(\chi) + \zeta(\chi)\right\|^2\mathrm{d}\chi\Bigg] \\ &\le \Bigg[\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \|\phi(0)\|^2) + \Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \widetilde{r}) \\ &+ \mathrm{T}\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{f}}(1 + \Theta_\varpi\widetilde{r})\int_0^{\mathrm{t}}\mathbb{E}\left\|\widetilde{\phi}(\chi) + \zeta(\chi)\right\|^2\mathrm{d}\chi + \mathrm{T}\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{f}}\int_0^\iota\mathfrak{f}_0\mathrm{d}\chi + \mathrm{T}\Theta_{\mathscr{R}}^2\|\mathbf{B}\|^2\|\mathbf{u}\|_{\mathcal{L}_{\mathfrak{I}}}^2 \\ &+ \Theta_{\mathscr{R}}^2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma\int_0^\iota\mathbb{E}\left\|\widetilde{\phi}(\chi) + \zeta(\chi)\right\|^2\mathrm{d}\chi\Bigg] \\ &\le 5\Bigg[\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \|\phi(0)\|^2) + \Theta_{\mathscr{R}}^2\Theta_{\mathfrak{h}}(1 + \widetilde{r}) + \mathrm{T}^2\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{f}}\mathfrak{f}_0 + \mathrm{T}\Theta_{\mathscr{R}}^2\|\mathbf{B}\|^2\|\mathbf{u}\|_{\mathcal{L}_{\mathfrak{I}}}^2 \\ &+ \left\{\mathrm{T}\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{f}}(1 + \Theta_\varpi\widetilde{r}) + \Theta_{\mathscr{R}}^2\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma +\right\}\mathrm{T}\widetilde{r}\Bigg] := \mathfrak{D}_2.\end{aligned}$$

Thus, Δ is bounded on l. Hence, by Krasnoselskii-Schaefer fixed point theorem [26], $\exists$ a mild solution for (1.2) on l.

□

4 Optimal Control

Using the Balder's theorem, this section investigates the existence of optimum control for (1.1). Here $\mathbf{u}(\cdot)$ is the control function which takes values in a separable Hilbert space $\mathbb{Y}$, $\mathbf{B} \in \mathcal{L}_\infty(l, \mathcal{L}(\mathbb{Y}, \mathbb{H}))$, $\|\mathbf{B}\|_\infty$ stands for the norm of operator $\mathbf{B}$ on Banach space $\mathcal{L}_\infty(l, \mathcal{L}(\mathbb{Y}, \mathbb{H}))$, where $\mathcal{L}_\infty(l, \mathcal{L}(\mathbb{Y}, \mathbb{H}))$ denote the space of operator valued functions that are measurable in the strong operator topology and uniformly bounded in l. Let $\mathcal{L}^2_{\Im}(l, \mathbb{Y})$ be the closed subspace of $\mathbb{Y}$ consisting of all measurable and $\Im_\iota$-adapted, $\mathbb{Y}$-valued stochastic processes satisfies $\int_0^\iota \|\mathbf{u}(\iota)\|^2_{\mathbb{Y}}\, d\iota < \infty$ and with the norm

$$\|\mathbf{u}(\iota)\|_{\mathcal{L}^2_{\Im}(l,\mathbb{Y})} = \left[\int_0^\iota \|\mathbf{u}(\iota)\|^2_{\mathbb{Y}}\, d\iota\right]^{\frac{1}{2}}$$

Let $\mathrm{v}(\cdot)$ be a nonempty closed bounded convex subset of $\mathbb{U}$. Define $\mathscr{G}_{ad} = \{\mathbf{u}(\cdot) \in \mathcal{L}^2_{\Im}(l, \mathbb{U}) : \mathbf{u}(\iota) \in \mathscr{G}_{ad}, a.e, \iota \in l\}$.

(H4) $\mathbf{B} \in \mathbb{U}_\infty(l, \mathcal{L}(\mathbb{U}, \mathbb{H}))$.

Consider the Bolza problem $\tilde{\mathrm{P}}$ (see [22]), which is to find an optimal pair $(\mathfrak{x}^0, \mathbf{u}^0) \in \mathscr{B} \times \mathscr{G}_{ad}$ such that $\mathcal{G}(\mathfrak{x}^0, \mathbf{u}^0) \leq \mathcal{G}(\mathfrak{x}^{\mathbf{u}}, \mathbf{u})$, $\forall \mathbf{u} \in \mathscr{G}_{ad}$ where the cost functional

$$\mathcal{G}(\mathfrak{x}^{\mathbf{u}}, \mathbf{u}) \;=\; \mathbb{E}\int_0^\iota \mathcal{J}(\iota, \mathfrak{x}^{\mathbf{u}}_\iota, \mathfrak{x}^{\mathbf{u}}(\iota), \mathbf{u}(\iota))d\iota + \mathbb{E}\Theta(\mathfrak{x}^{\mathbf{u}}(\mathrm{T})).$$

We introduce the following hypotheses:

(1) The functional $\mathcal{J} : l \times \mathscr{B} \times \mathbb{H} \times \mathbb{U} \to \mathbb{R} \cup \{\infty\}$ is Borel measurable.

(2) $\mathcal{J}(\iota, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathscr{B} \times \mathbb{H} \times \mathbb{U}$ for almost all $\iota \in [0, \mathrm{T}]$.

(3) $\mathcal{J}(\iota, \mathfrak{x}, \mathfrak{x}_\iota, \cdot)$ is convex on $\mathbb{U}$ for each $\mathfrak{x} \in \mathscr{B}$ and almost all $\iota \in l$.

(4) There exist constants $d, e \geq 0$, $\mathfrak{j} > 0$, μ_0 is non negative and $\mu_0 \in \mathcal{L}^1([0, \mathrm{T}]; \mathbb{R})$ such that

$$\mu_0(\iota) + d\mathbb{E}\|\mathfrak{x}\|^2 + e\mathbb{E}\|\mathfrak{x}_\iota\|^2 + \mathfrak{j}\mathbb{E}\|\mathbf{u}\|^2_{\mathbb{U}} \;\leq\; \mathcal{J}(\iota, \mathfrak{x}(\iota), \mathfrak{x}_\iota, \mathbf{u}(\iota)).$$

Theorem 4.1. *Assume that the assumptions in Theorem 3.1 and (H1)-(H6) are fulfilled and* $\mathbf{B}$ *is strongly continuous operator, then the Bolza problem has at least one optimal pair on* $\mathscr{B} \times \mathscr{G}_{ad}$.

Proof. If $\inf\{\mathcal{G}(\mathbf{u}) : \mathbf{u} \in \mathscr{G}_{ad}\} = +\infty$. Then there is nothing to prove. Assume that $\inf\{\mathcal{G}(\mathbf{u}) : \mathbf{u} \in \mathscr{G}_{ad}\} = \epsilon < \infty$. Using (**H5**), we have $\epsilon > -\infty$. By definition of infimum, there exists a minimizing sequence feasible pair $\{(\mathfrak{x}^n, \mathbf{u}^n)\} \subset \mathscr{G}_{ad} \equiv \{(\mathfrak{x}, \mathbf{u}) : \mathfrak{x}$ is a mild solution of the system (1.1) corresponding to $\mathbf{u}^n \in \mathscr{G}_{ad}\}$, such that $\mathcal{G}(\mathfrak{x}^n, \mathbf{u}^n) \to \epsilon$ as $n \to +\infty$. Since $\mathbf{u}^n \in \mathscr{G}_{ad}$, $\{\mathbf{u}^n\}_{n\geq 1} \subset \mathcal{L}_2(l, \mathbb{U})$ is bounded. Thus, $\exists\, \mathbf{u}^0 \in \mathcal{L}_2(J, \mathcal{K})$ and a subsequence extracted from $\mathbf{u}^n$ such that $\mathbf{u}^n \stackrel{w}{\to} \mathbf{u}^0$ weakly in $\mathcal{L}_2(l, \mathbb{U})$. Since $\mathscr{U}_{ad}$ is closed and convex, the Mazur lemma forces us to conclude that $\mathbf{u}^0 \in \mathscr{G}_{ad}$. Suppose that x^n and x^0 are the mild solutions of (1.2) corresponding to $\mathbf{u}^n$

and $\mathbf{u}^0$, respectively. Let $\mathfrak{x}^n$ and $\mathfrak{x}^0$ satisfy the equations

$$\mathfrak{x}^n(\iota)=\begin{cases}0, \quad \iota\in(-\infty,0],\\ \mathscr{R}(\iota)\left[\phi(0)+\mathfrak{h}(0,\phi(0))\right]-\mathfrak{h}\left(\iota,\mathfrak{x}^n_\iota\right)+\int_0^\iota\mathscr{R}(\iota-\chi)\mathfrak{f}(\chi,\mathfrak{x}^n_\chi,\mathfrak{x}^n(\varpi(\mathfrak{x}^n_\chi,\chi)))\mathrm{d}\chi\\ +\int_0^\iota\mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}^n(\chi)\mathrm{d}\chi+\int_0^\iota\mathscr{R}(\iota-\chi)\sigma(\chi,\mathfrak{x}^n_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi), \quad \iota\in l.\end{cases} \tag{4.1}$$

Similarly, corresponding to $\mathfrak{x}^0$, $\exists$ a mild solution $\mathfrak{x}^0$ of (1.2),

$$\mathfrak{x}^0(\iota)=\begin{cases}0, \quad \iota\in(-\infty,0],\\ \mathscr{R}(\iota)\left[\phi(0)+\mathfrak{h}(0,\phi(0))\right]-\mathfrak{h}\left(\iota,\mathfrak{x}^0_\iota\right)+\int_0^\iota\mathscr{R}(\iota-\chi)\mathfrak{f}(\chi,\mathfrak{x}^0_\chi,\mathfrak{x}^0(\varpi(\mathfrak{x}^0_\chi,\chi)))\mathrm{d}\chi\\ +\int_0^\iota\mathscr{R}(\iota-\chi)\mathbf{B}(\chi)\mathbf{u}^0(\chi)\mathrm{d}\chi+\int_0^\iota\mathscr{R}(\iota-\chi)\sigma(\chi,\mathfrak{x}^0_\chi)\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi), \quad \iota\in l.\end{cases} \tag{4.2}$$

By the properties of boundedness of $\{\mathbf{u}^n\}$ and $\{\mathbf{u}^0\}$, we can show that $\exists$ a +ve number β such that $\mathbb{E}\left\|\mathfrak{x}^n\right\|^2,\mathbb{E}\left\|\mathfrak{x}^0\right\|^2\le\beta$. For every $\iota\in l$, we get

$$\begin{aligned}
\mathbb{E}\left\|\mathfrak{x}^n(\iota)-\mathfrak{x}^0(\iota)\right\|^2 &\le 4\Big\{\mathbb{E}\left\|\mathfrak{h}\left(\iota,\mathfrak{x}^n_\iota\right)-\mathfrak{h}\left(\iota,\mathfrak{x}^0_\iota\right)\right\|^2\\
&+ \mathrm{T}\int_0^\iota\|\mathscr{R}(\iota-\chi)\|^2\,\mathbb{E}\left\|\mathfrak{f}(\chi,\mathfrak{x}^n_\chi,\mathfrak{x}^n(a(\mathfrak{x}^n_\chi,\chi)))-\mathfrak{f}(\chi,\mathfrak{x}^0_\chi,\mathfrak{x}^0(\varpi(\mathfrak{x}^0_\chi,\chi)))\right\|^2\mathrm{d}\chi\\
&+ \mathrm{T}\int_0^\iota\|\mathscr{R}(\iota-\chi)\|^2\,\mathbb{E}\left\|\mathbf{B}(\chi)\mathbf{u}^n(\chi)-\mathbf{B}(\chi)\mathbf{u}^0(\chi)\right\|^2\mathrm{d}\chi\\
&+ \int_0^\iota\|\mathscr{R}(\iota-\chi)\|^2\,\mathbb{E}\left\|\sigma(\chi,\mathfrak{x}^n_\chi)-\sigma(\chi,\mathfrak{x}^0_\chi)\right\|^2\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\chi)\Big\}\\
&\le 4\Big\{\mathbb{E}\left\|\mathfrak{h}\left(\iota,\mathfrak{x}^n_\iota\right)-\mathfrak{h}\left(\iota,\mathfrak{x}^0_\iota\right)\right\|^2\\
&+ \mathrm{T}\Theta^2_{\mathscr{R}}\int_0^\iota\mathbb{E}\left\|\mathfrak{f}(\chi,\mathfrak{x}^n_\chi,\mathfrak{x}^n(\varpi(\mathfrak{x}^n_\chi,\chi)))-\mathfrak{f}(\chi,\mathfrak{x}^0_\chi,\mathfrak{x}^0(\varpi(\mathfrak{x}^0_\chi,\chi)))\right\|^2\mathrm{d}\chi\\
&+ \mathrm{T}\Theta^2_{\mathscr{R}}\int_0^\iota\left\|\mathbf{B}(\chi)\mathbf{u}^n(\chi)-\mathbf{B}(\chi)\mathbf{u}^0(\chi)\right\|^2\mathrm{d}\chi\\
&+ \Theta^2_{\mathscr{R}}\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\int_0^\iota\mathbb{E}\left\|\sigma(\chi,\mathfrak{x}^n_\chi)-\sigma(\chi,\mathfrak{x}^0_\chi)\right\|^2\mathrm{d}\chi\Big\}\\
&\le 4\Big\{\Theta_{\mathfrak{h}}\mathbb{E}\left\|\mathfrak{x}^n_\iota-\mathfrak{x}^0_\iota\right\|^2+\Theta^2_{\mathscr{R}}\left[\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma+\mathrm{T}\Theta_\varpi\right]\int_0^\iota\mathbb{E}\left\|\mathfrak{x}^n_\chi-\mathfrak{x}^0_\chi\right\|^2\mathrm{d}\chi\\
&+ \mathrm{T}\Theta^2_{\mathscr{R}}\Big[\Theta_{\mathfrak{f}}\int_0^\iota\mathbb{E}\left\|\mathfrak{x}^n(\varpi(\mathfrak{x}^n(\chi),\chi))-\mathfrak{x}^n(\varpi(\mathfrak{x}^0(\chi),\chi))\right\|^2\\
&- \mathbb{E}\left\|\mathfrak{x}^n(\varpi(\mathfrak{x}^0(\chi),\chi))-\mathfrak{x}^0(\varpi(\mathfrak{x}^0(\chi),\chi))\right\|^2\Big]\mathrm{d}\chi+\mathrm{T}\Theta^2_{\mathscr{R}}\int_0^\iota\left\|\mathbf{B}(\chi)\left[\mathbf{u}^n(\chi)-\mathbf{u}^0(\chi)\right]\right\|^2\mathrm{d}\chi\Big\}\\
&\le 4\Big\{\Theta_{\mathfrak{h}}\mathbb{E}\left\|\mathfrak{x}^n_\iota-\mathfrak{x}^0_\iota\right\|^2+\Theta^2_{\mathscr{R}}\left[\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma+\mathrm{T}\Theta_\varpi\right]\int_0^\iota\left\|\mathfrak{x}^n_\iota-\mathfrak{x}^0_\iota\right\|^2\mathrm{d}\chi\\
&+ \mathrm{T}\Theta^2_{\mathscr{R}}\Theta_{\mathfrak{f}}\left[r\Theta_\varpi+\mathrm{M_T}+1\right]\int_0^\iota\mathbb{E}\left\|\mathfrak{x}^n(\chi)-\mathfrak{x}^0(\chi)\right\|^2\mathrm{d}\chi+\mathrm{T}\Theta^2_{\mathscr{R}}\int_0^\iota\left\|\mathbf{B}(\chi)\left[\mathbf{u}^n(\chi)-\mathbf{u}^0(\chi)\right]\right\|^2\mathrm{d}\chi\Big\}\\
&\le 4\mathrm{M_T}\Theta_{\mathfrak{h}}\sup_{\iota\in l}\mathbb{E}\left\|\mathfrak{x}^n_\iota-\mathfrak{x}^0_\iota\right\|^2+4\mathrm{T}\Theta^2_{\mathscr{R}}\left\|\mathbf{B}(\chi)\left[\mathbf{u}^n(\chi)-\mathbf{u}^0(\chi)\right]\right\|^2\mathrm{d}\chi\\
&+ 4\Big\{\Theta^2_{\mathscr{R}}\mathrm{M_T}\left[\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma+\mathrm{T}\Theta_\varpi\right]+\mathrm{T}\Theta^2_{\mathscr{R}}\Theta_{\mathfrak{f}}\left[r\Theta_\varpi+\mathrm{M_T}+1\right]\Big\}\int_0^\iota\mathbb{E}\left\|\mathbf{B}(\chi)\left[\mathbf{u}^n(\chi)-\mathbf{u}^0(\chi)\right]\right\|^2\mathrm{d}
\end{aligned}$$

$$\leq \frac{1}{1-\nabla_1}\left\{\nabla_2 \int_0^\iota \mathbb{E}\left\|\mathfrak{x}^n(\chi)-\mathfrak{x}^0(\chi)\right\|^2 \mathrm{d}\chi + 4\mathrm{T}\Theta_{\mathscr{R}}^2 \int_0^\iota \mathbb{E}\left\|\mathbf{B}(\chi)\left[\mathbf{u}^n(\chi)-\mathbf{u}^0(\chi)\right]\right\|^2 \mathrm{d}\chi\right\}$$

where

$$\nabla_1 = 4\mathrm{M_T}\Theta_{\mathfrak{h}} \sup_{\iota\in l} \mathbb{E}\left\|\mathfrak{x}^n(\iota)-\mathfrak{x}^0(\iota)\right\|^2, \qquad r = \mathrm{M_T^2} \sup_{\iota\in l} \mathbb{E}\left\|\mathfrak{x}^n(\iota)\right\|^2$$

$$\nabla_2 = 4\left\{\Theta_{\mathscr{R}}^2 \mathrm{M_T}\left[\mathrm{C}_{\mathscr{H}}\iota^{2\mathscr{H}-1}\Theta_\sigma + \mathrm{T}\Theta_\varpi\right] + \mathrm{T}\Theta_{\mathscr{R}}^2\Theta_{\mathfrak{f}}\left[r\Theta_\varpi + \mathrm{M_T}+1\right]\right\},\ 1-\nabla_1 \neq 0.$$

By applying Gronwall's inequality, there exists $\nabla_3 > 0$ such that

$$\mathbb{E}\left\|\mathfrak{x}^n(\chi)-\mathfrak{x}^0(\chi)\right\|^2 \leq \nabla_3\left\|\mathbf{B}\mathbf{u}^n-\mathbf{B}\mathbf{u}^0\right\|^2$$

Since $\mathbf{B}$ is strongly continuous, we have $\left\|\mathbf{B}\mathbf{u}^n-\mathbf{B}\mathbf{u}^0\right\|^2 \to 0$ as $n\to\infty$. Therefore

$$\mathbb{E}\left\|\mathfrak{x}^n(\chi)-\mathfrak{x}^0(\chi)\right\|^2 \to 0 \text{ as } n\to\infty.$$

Note that (1)-(4) implies that the assumptions of Balder's theorem [26] are fulfilled. Thus, we can deduce that $(\mathfrak{x}_\iota, \mathfrak{x}, \mathbf{u}) \to \int_0^\iota \mathcal{J}(\iota, \mathfrak{x}_\iota, \mathfrak{x}(\iota), \mathbf{u}(\iota))\mathrm{d}\iota$ is sequentially lower semicontinuous on $\mathcal{L}^2_{\mathfrak{I}}(l,\mathbb{U}) \subset \mathcal{L}^1_{\mathfrak{I}}(l,\mathbb{U})$. Hence, $\mathcal{G}$ is weakly lower semicontinuous on $\mathcal{L}^2_{\mathfrak{I}}(l,\mathbb{U})$ and $\mathcal{G} > -\infty$; thus, $\mathcal{G}$ attains its infimum at $\mathbf{u}^0 \in \mathscr{G}_{ad}$, that is

$$\begin{aligned}
\epsilon &= \lim_{n\to\infty} \mathbb{E}\int_0^\iota \mathcal{J}(\iota, \mathfrak{x}^n(\iota), \mathfrak{x}^n_\iota, \mathbf{u^n}(\iota))\mathrm{d}\iota + \lim_{n\to\infty} \mathbb{E}\Xi(\mathfrak{x}^n(\mathrm{T}))\\
&\geq \mathbb{E}\int_0^\iota \mathcal{J}(\mathrm{t}, \mathfrak{x}^0(\mathrm{t}), \mathfrak{x}^0_{\mathrm{t}}, \mathbf{u^0}(\iota))\mathrm{d}\iota + \mathbb{E}\Xi(\mathfrak{x}^n(\mathrm{T}))\\
&= \mathcal{J}(\mathfrak{x}^0, \mathbf{u^0}) \geq \epsilon.
\end{aligned}$$

This proves that $\mathcal{G}$ attains its minimum at $\mathbf{u^0}$ and that $(\mathfrak{x}^0, \mathbf{u^0})$ is the required optimal control pair.

□

5 Applications

Let us consider the controlled neutral partial SIDEs of the form

$$\begin{aligned}
\mathrm{d}\left[\mathfrak{x}(\iota,\mathrm{y}) + 2\mathrm{e}^{3\iota}\mathfrak{x}(\iota-\phi,\mathrm{y})\right] &= \frac{\partial^2}{\partial \mathrm{y}^2}\left[\mathfrak{x}(\iota,\mathrm{y}) + 2\mathrm{e}^{3\iota}\mathfrak{x}(\iota-\phi,\mathrm{y})\right]\mathrm{d}\iota + \left[\int_0^\iota \widetilde{\Lambda}(\iota-\chi)\frac{\partial^2}{\partial \mathrm{y}^2}\left[\mathfrak{x}(\iota,\mathrm{y}) + 2\mathrm{e}^{3\iota}\mathfrak{x}(\iota-\phi,\mathrm{y})\right]\mathrm{d}\chi\right]\\
&+ \int_0^\iota \mathrm{G}(\chi,\mathrm{y})\mathbf{u}(\chi,\mathrm{y})\mathrm{d}\chi + \frac{\mathfrak{x}(\iota, \sin\iota\,|\mathfrak{x}(\iota,\mathrm{y})|) + \iota^2\mathfrak{x}(\iota,\mathrm{y})}{5\pi}\mathrm{d}\iota\\
&+ \frac{\mathrm{e}^\iota\,|\mathfrak{x}(\iota-\phi,\mathrm{y})|}{8}\mathrm{d}\mathbb{Z}_{\mathscr{H}}(\iota),\ \iota\in[0,1],\ \mathrm{y}\in[0,\pi],\\
\mathfrak{x}(\iota,0) &= \mathfrak{x}(\iota,\mathrm{y}) = 0,\ \iota\in[0,1],\\
\mathfrak{x}(\iota,\mathrm{y}) &= \phi(\iota,\mathrm{y}),\ \iota\in(-\infty,0],\ \mathrm{y}\in[0,\pi].
\end{aligned} \tag{5.1}$$

Let $\mathrm{G} : [0,1]\times[0,\pi] \to \mathbb{R}$ be a continuous function. Let $\mathbb{H} = \mathbb{U} = \mathcal{L}^2([0,1],[0,\pi])$ and the operator $\mathfrak{A} : \mathfrak{D}(\mathfrak{A}) \subseteq \mathbb{H} \to \mathbb{H}$ is defined by $\mathfrak{A}\mathfrak{x} = \frac{\partial^2\mathfrak{x}}{\partial \mathrm{y}^2}$, $\mathfrak{D}(\mathfrak{A}) = \{\mathfrak{x} \in \mathbb{H} : \mathfrak{x}, \mathfrak{x}'$ are absolutely continuous,

$\mathfrak{x}'' \in \mathbb{H}, \mathfrak{x}(0) = \mathfrak{x}(\pi) = 0\}$. Then, $\mathfrak{A}$ generates a strongly continuous semigroup $\{\mathfrak{S}(\iota,\ \iota > 0)\}$ which is analytic, compact, and self-adjoint. Moreover, $\mathfrak{A}$ has a discrete spectrum with the eigenvalues $-n^2, n \in \mathbb{N}$ with the corresponding normalized eigenvector $e_n(\mathfrak{x}) = \sqrt{\frac{2}{\pi}}\sin n\mathfrak{x},\ n = 1, 2, \cdots$, where e_n is an orthonormal base. For $\mathfrak{x} \in \mathfrak{D}(\mathfrak{A})$, $\sum_{n=1}^{\infty} < \mathfrak{x}, e_n > e_n$ and $\mathfrak{A}\mathfrak{x} = -\sum_{n=1}^{\infty} n^2 < \mathfrak{x}, e_n > e_n,\ \mathfrak{x} \in \mathfrak{D}(\mathfrak{A})$. Here, the nonlinear functions are

$$\mathfrak{f}(\iota, \mathfrak{x}(\varpi(\mathfrak{x}(\iota, \mathrm{y}), \iota))) = \frac{\mathfrak{x}(\iota, \sin\iota\,|\mathfrak{x}(\iota, \mathrm{y})|) + \iota^2\mathfrak{x}(\iota, \mathrm{y})}{5\pi}, \qquad \sigma(\iota, \mathfrak{x}(\iota, \mathrm{y})) = \frac{\mathrm{e}^{\iota}\,|\mathfrak{x}(\iota - \phi, \mathrm{y})|}{8},$$

$$\mathfrak{h}(\iota, \mathfrak{x}(\iota, \mathrm{y})) = 2e^{3\iota\mathfrak{x}(\iota-\phi,\mathrm{y})}.$$

The non-linear function $\mathfrak{h} : [0, 1] \times \mathscr{B}_h \to \mathbb{H}$ and

$$\begin{aligned}\mathbb{E}\,\|\mathfrak{h}(\iota, \mathfrak{x}_1(\iota, \mathrm{y})) - \mathfrak{h}(\iota, \mathfrak{x}_2(\iota, \mathrm{y}))\|^2 &\leq \int_{-\infty}^{0} \mathbb{E}\left\|\sqrt{2}e^{\chi}\mathfrak{x}_1(\chi - \phi, \mathrm{y}) - \sqrt{2}e^{\chi}\mathfrak{x}_2(\chi - \phi, \mathrm{y})\right\|^2 \mathrm{d}\chi \\ &\leq 2\mathbb{E}\,\|\mathfrak{x}_1(\chi, \mathrm{y}) - \mathfrak{x}_2(\chi, \mathrm{y})\|^2\end{aligned}$$

Thus, (H6) is fulfilled with $\Theta_{\mathfrak{h}} = 2$.

Consider $\mathfrak{f} : [0, 1] \times \mathscr{B}_h \to \mathbb{H}$,

$$\begin{aligned}\mathbb{E}\,\|\mathfrak{f}(\iota, \mathfrak{x}_1(\varpi(\mathfrak{x}_1(\iota, \mathrm{y}), \iota))) - \mathfrak{f}(\iota, \mathfrak{x}_2(\varpi(\mathfrak{x}_2(\iota, \mathrm{y}), \iota)))\|^2 &\leq \mathbb{E}\Bigg\|\frac{\mathfrak{x}_1(\iota, \sin\iota\,|\mathfrak{x}_1(\iota, \mathrm{y})|) + \iota^2\mathfrak{x}_1(\iota, \mathrm{y})}{5\pi} \\ &- \frac{\mathfrak{x}_2(\iota, \sin\iota\,|\mathfrak{x}_2(\iota, \mathrm{y})|) + \iota^2\mathfrak{x}_2(\iota, \mathrm{y})}{5\pi}\Bigg\|^2 \\ &\leq \frac{1}{25}\mathbb{E}\,\|\mathfrak{x}_1(\iota, \mathrm{y}) - \mathfrak{x}_2(\iota, \mathfrak{y})\|^2\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\,\|\varpi(\mathfrak{x}_1(\iota, \mathfrak{x}), \iota) - \varpi(\mathfrak{x}_2(\iota, \mathfrak{x}), \iota)\|^2 &\leq \mathbb{E}\,\|\sin\iota\,|\mathfrak{x}_1(\iota, \mathrm{y})| - \sin\iota\,|\mathfrak{x}_2(\iota, \mathrm{y})|\|^2 \\ &\leq \mathbb{E}\,\|\mathfrak{x}_1(\iota, \mathrm{y}) - \mathfrak{x}_2(\iota, \mathrm{y})\|^2.\end{aligned}$$

As a result, with $\Theta_{\mathfrak{f}} = \frac{1}{25}$, $\Theta_a = 1$, (H4) and (H5) are satisfied

Next, $\sigma : [0, 1] \times \mathscr{B}_h \to \mathcal{L}_2^0$,

$$\begin{aligned}\mathbb{E}\,\|\sigma(\iota, \mathfrak{x}_1(\iota, \mathfrak{x})) - \sigma(\iota, \mathfrak{x}_2(\iota, \mathfrak{x}))\|^2 &\leq \frac{1}{64}\mathbb{E}\,\|e^{\iota}\mathfrak{x}_1(\iota - \phi, \mathrm{y}) - e^{\iota}\mathfrak{x}_2(\iota - \phi, \mathrm{y})\|^2 \\ &\leq \frac{1}{64}\int_{-\infty}^{0} e^{2\chi}\mathbb{E}\,\|e^{\chi}\mathfrak{x}_1(\chi - \phi, \mathrm{y}) - e^{\chi}\mathfrak{x}_2(\chi - \phi, \mathrm{y})\|^2 \mathrm{d}\chi \\ &\leq \frac{1}{64}\mathbb{E}\,\|\mathfrak{x}_1(\iota, \mathrm{y}) - \mathfrak{x}_2(\iota, \mathrm{y})\|^2.\end{aligned}$$

Thus, (H6) holds with $\Theta_\sigma = \frac{1}{64}$.

As a result of combining the values obtained above, we have

By Theorem 3.1, the solution of (5.1) exists if $\mathfrak{D} < 1$. Consider the cost function as follows:

$$\mathcal{G}(\mathfrak{x}^{\mathbf{u}}, \mathbf{u})(\mathrm{y}) = \int_0^{\pi}\int_0^1 \mathbb{E}\,\|\mathfrak{x}^{\mathbf{u}}(\iota, \mathrm{y})\|^2 \,\mathrm{d}\iota\mathrm{d}\mathrm{y} + \int_0^1 \|\mathbf{u}(\iota, \mathrm{y})\|^2 \,\mathrm{d}\iota\mathrm{d}\mathrm{y}.$$

Now,

$$\int_0^{\pi}\int_0^{\iota} \|\mathbf{Bu}(\iota, \mathrm{y})\|^2 \,\mathrm{d}\iota\mathrm{d}\mathrm{y} \quad \leq \quad \Theta_{\mathrm{G}}^2 \times mes(\pi) \times \|\mathbf{u}(\chi, \mathrm{y})\|^2_{\mathcal{L}^2[0,1],[0,\pi]},$$

where $\Theta_{\mathrm{G}}^2 = \max\left\{\|\mathrm{G}(\iota, \mathrm{y})\|^2 : \ \iota \in [0,1]\right\}$. Then, we can conclude that $\mathbf{B}$ is a bounded linear operator in $\mathcal{L}^2[0,1],[0,\pi]$. Furthermore, the Theorem's assumptions are met, implying that at least one optimum pair exists $(\mathfrak{x}^{\mathbf{u}}(\iota, \mathrm{y}), \mathbf{u}(\iota, \mathrm{y}))$.

6 Conclusions

This article we have studied the optimal control for a class of infinite time delay neutral SIDEs with deviated arguments driven by Rosenblatt process in Hilbert space using Balder's theorem. The practical importance of Rossenblatt process has been mentioned in the Introduction. The axiomatic phase space definition has been proven in the Preliminary section. Optimal control with deviated argument driven by Rossenblatt process has been an untreated article in the literature.
One can extend system (1.2) with the second order and study the optimal control using sine and cosine operators. Also, the optimal control for a class of neutral Stochastic Fractional Integro-Differential Equations (SFIDEs) with deviated arguments driven by Rosenblatt process for infinite time delay and state delay are future works.

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