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Article *in* Stochastics An International Journal of Probability and Stochastic Processes · August 2024

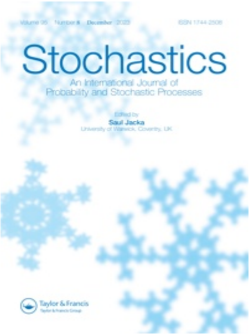
DOI: 10.1080/17442508.2024.2387092

4 authors, including:

Papa Ali Thiam
Gaston Berger University

Mamadou Abdoul Diop
Gaston Berger University

Ramkumar Kasinathan
PSG College of Arts and Science



Stochastics

An International Journal of Probability and Stochastic Processes

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/gssr20

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To cite this article: Papa Ali Thiam, Mamadou Abdoul Diop, Ramkumar Kasinathan & Ravikumar Kasinathan (07 Aug 2024): Existence and exponential stability in *p*th moment of non-autonomous stochastic integro-differential equations, Stochastics, DOI: 10.1080/17442508.2024.2387092

To link to this article: https://doi.org/10.1080/17442508.2024.2387092

Published online: 07 Aug 2024.

STOCHASTICS
https://doi.org/10.1080/17442508.2024.2387092

Existence and exponential stability in *p*th moment of non-autonomous stochastic integro-differential equations

Papa Ali Thiam[a], Mamadou Abdoul Diop[a,b], Ramkumar Kasinathan[c] and Ravikumar Kasinathan [c]

[a]Université Gaston Berger de Saint-Louis, UFR SAT Département de Mathématiques, Saint-Louis, Sénégal; [b]UMMISCO UMI 209 IRD/UPMC, Bondy, France; [c]Department of Mathematics, PSG College of Arts & Science, Coimbatore, India

ABSTRACT

This study is concerned with the existence and exponential stability of solutions of non-autonomous neutral stochastic integro-differential equations with delay; the linear part of this equation is dependent on time and generates a linear evolution system. The obtained results are applied to some neutral stochastic integro-differential equations. These kinds of equations arise in systems related to couple oscillators in a noisy environment or in viscoelastic materials under random or stochastic influences. Finally, we provide an example to illustrate the results.

ARTICLE HISTORY

Received 18 January 2023
Accepted 16 July 2024

KEYWORDS

Stochastic integro-differential equations; resolvent operator; mild solutions; stability in the *p*th moment

2020 MATHEMATICS SUBJECT CLASSIFICATIONS

34K50; 47A10; 60J65; 93D23

1. Introduction

In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. It is well known that these topics have been developed mainly by using two different methods, that is,

- (i) The semigroup approach [10,11]
- (ii) The variational method [31].

On the other hand, although stochastic partial differential equations with finite delays also seem very important as stochastic models of biological, chemical, physical and economical systems, the corresponding properties of these systems have not been studied in great details [2,6,17]. As a matter of fact, there exist extensive literature on the related topics for deterministic partial functional differential equations with finite delays (for example, see [41]. We would also like to mention that some similar topics to the above for stochastic ordinary functional differential equations with finite delays have already been investigated by various authors [2,6,27] In [39] Taniguchi et al. examined the asymptotic behaviour and

CONTACT Mamadou Abdoul Diop ordydiop@gmail.com Université Gaston Berger de Saint-Louis, UFR SAT Département de Mathématiques, Saint-Louis B.P234, Sénégal; UMMISCO UMI 209 IRD/UPMC, Bondy, France

existence of solutions for the stochastic functional differential equation presented below:

$$\begin{cases} du(t) = [-\mathbb{A}(t)u(t) + \sigma(t, u_t)]\,dt + h(t, u_t)\,d\mathrm{B}(t), \quad t \in J = [0, T], \\ u_0 = \varphi \in L^p(\Omega, \mathcal{C}_\alpha), \end{cases} \tag{1}$$

by using analytic semigroups approach and fractional power operator arguments. Balasubramaniam et al. [5] presented the exponential stability of nonlinear fractional order stochastic system with Poisson jumps is studied in finite dimensional space. In this work as well as other related literature like [10,11,27,28], the linear part of the discussed equation is an operator independent of time t and generates a strongly continuous one parameter semigroup or analytic semigroup so that the semigroup approach can be employed. However, when treating some parabolic evolution equations, due to the frequent occurrence of such operators related to time t in applications, the partial differential operators depend on time t; for the details, please see [1,16,18,30,42]. Fu in [21] established the results on existence and exponential stability of solutions for a semilinear non-autonomous neutral stochastic evolution equation with finite delay:

$$\begin{cases} d\left[u(t) - \mathrm{g}(t, u_t)\right] = \left[-\mathbb{A}(t)\left[u(t) - \mathrm{g}(t, u_t)\right] + \sigma(t, u_t)\right]dt \\ \qquad\qquad\qquad\qquad\qquad + h(t, u_t)\,d\mathrm{B}(t), \quad t \in J = [0, T], \\ u_0 = \varphi \in L^p(\Omega, \mathcal{C}_\alpha), \end{cases} \tag{2}$$

where $\mathbb{A}(t)$ generates a linear evolution operator $\{\mathcal{U}(t,s) : 0 \le s \le t\}$ on a separable Hilbert space $\mathbb{X}$ with inner product $\langle \cdot, \cdot \rangle_{\mathbb{X}}$ and $\| \cdot \|_{\mathbb{X}}$. g, σ and h are given functions.Studying the non-autonomous evolution equation, in which the differential operators of the significant component depend on time t, is hence a matter of significance.

The concept of exponential stability plays a crucial role in the dynamical system and its convergence rate is faster than the asymptotic stability. Further, the existence and uniqueness of solutions for stochastic system with Poisson jumps [8], exponential stability for stochastic partial differential equation [34], existence and stability results for second-order stochastic system with fractional Brownian motion [13], existence, uniqueness of mild solution and stability results for second-order stochastic system with delay and Poisson jumps [35] have emerged in the literature.

Integro-differential equations are ubiquitous in various scientific fields, such as modelling the expansion of a parasite population, Lotka-Volterra predator-prey systems, and reaction-diffusion models with memory. On the other hand, the theory of integrodifferential equations with resolvent operators has been the subject of much research because it describes various occurrences that occur in the ordinary world. For example, see [2,6,12,17,19] and the references therein. In the past few years, several authors have researched the existence of solutions to functional integro-differential equations in infinite-dimensional spaces and the qualitative characteristics of those solutions. We refer the reader to [4,23,26,32,34]. It is critical to conduct a stability study to determine whether the available solution significantly differs from the desired behaviour. One well-built type of stability is exponential stability. Because it is resistant to disturbances, exponential stability convergence has emerged as a significant application in its entirety. The topic of exponential stability has been the focus of a number of studies that are particularly significant [7,24,29,36,40].

However, it should be further emphasized that the existence and exponential stability of solutions of non-autonomous neutral stochastic integro-differential equations with delay is yet to be elaborated, compared to that of ordinary differential system. In this regard, it is necessary and important to study the exponential stability of nonlinear stochastic integro-differential system with delay. To the best of authors knowledge, only a few steps are taken to use the fixed point theorems to investigate the stability of stochastic integro-differential system. There is no work reported in the literature to study the problem of existence and exponential stability of solutions of non-autonomous neutral stochastic integro-differential equations with delay. In order to fill up this gap, in this paper, we study the following non-autonomous neutral stochastic integro-differential equations :

$$\begin{cases} d\left[u(t)-\mathrm{g}(t,u_t)\right]=\Big[\mathbb{A}(t)\left[u(t)-\mathrm{g}(t,u_t)\right] \\ \quad +\int_0^t\Gamma(t,s)\left[u(s)-\mathrm{g}(s,u_s)\right]\mathrm{d}s+\sigma\,(t,u_t)\Big]\,\mathrm{d}t+h\,(t,u_t)\,d\mathrm{B}\,(t)\,, \quad t\in J=[0,T], \\ u_0=\varphi\in L^p(\Omega,\mathcal{C}), \end{cases} \tag{3}$$

where $\{\mathbb{A}(t) : t\in\mathbb{R}^+\}$ and $\{\Gamma(t,s) : 0\le s\le t,\ t\in\mathbb{R}^+\}$ are families of closed unbounded linear operators on $\mathbb{X}$ with a fixed domain $\mathcal{D}(\mathbb{A})$. The state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{X}$, and the history u_t is defined in the usual way by $u_t(\theta)=u(t+\theta)$ for $\theta\in[-b,0]$ and belongs to an abstract space $\mathcal{C}$ that will be specified later. In addition to this, $\mathrm{B}(t)$ is a $\mathbb{V}$-valued Wiener process with a finite trace nuclear covariance operator $\mathrm{Q}>0$, where $\mathbb{V}$ is another separable Hilbert space. The functions g,σ and h are appropriate functions that will be described later. However, it occurs very often that the linear part of (1) is dependent on time t. Indeed, a lot of stochastic partial functional differential equations can be rewritten to semilinear non-autonomous equations having the form of (1) with $\mathbb{A}=\mathbb{A}(t)$. There exists much work on existence, asymptotic behaviour, and controllability for deterministic non-autonomous partial (functional) differential equations with finite or infinite delays. But little is known to us for non-autonomous stochastic differential equations in abstract space, especially for the case that $\mathbb{A}(t)$ is a family of unbounded operators.

Remark 1.1: Our purpose in the present paper is to obtain results concerning existence, uniqueness, and stability of the solutions of the non-autonomous stochastic integro-differential Equation (3). A motivation example for this class of equation is the following non-autonomous boundary problem:

$$\begin{aligned} d\left[\mathrm{Z}(t,x)-b\mathrm{Z}(t-r,x)\right] &= \left[a(t,x)\frac{\partial^2}{\partial^2}\left[\mathrm{Z}(t,x)-b\mathrm{Z}(t-r,x)\right]+f(t,\mathrm{Z}(t+r_1(t),x))\right]dt \\ &\quad + g(t,\mathrm{Z}(t+r_1(t),x))d\beta(t), \\ \mathrm{Z}(t,0) &= \mathrm{Z}(t,\pi)=0,\ t\ge 0, \\ \mathrm{Z}(\theta,x) &= \phi(x),\ \theta\in[-r,0],\ 0\le x\le\pi. \end{aligned} \tag{4}$$

these problems arise in systems related to couple oscillators in a noisy environment or in viscoelastic materials under random or stochastic influences. Therefore, it is meaningful to deal with (3) to acquire some results applicable to problem (4). As is common knowledge, the non-autonomous evolution equations are far more challenging to solve than those of autonomous evolution. Our approach here is inspired by the work in paper [15,22].

The original contributions of this paper are stated as follows:

- (i) Existence and exponential stability in pth moment of non-autonomous stochastic integro-differential equations is investigated.
- (ii) We introduce a relation between the resolvent operator, generated by the linear part of (3), and the evolution family generated by the families operator $\mathbb{A}(t)$.
- (iii) This relation enables us to find conditions that guarantee the existence of mild solutions of Equation (3) without any assumptions on the compactness of g, σ, h.
- (iv) We establish our results using the theory of resolvent operator in the sense of Grimmer. We use the fact that the norm-continuity of the resolvent operator yields from the evolution family. Our study may be considered an extension and advancement of the relevant works cited earlier.

The following is the overall organization of this paper. In Section 2, we provide a brief overview of some fundamental notions and preliminary results, useful for the proof of our results. In Section 3, we are concerned with the existence of mild solutions for the system (3). In Section 4, we are interested with the exponential stability in the pth moment. Finally, Section 5 provides an illustration of the theory through the use of an example.

2. Preliminaries

This section provides a quick overview of certain fundamental ideas, concepts, and lemmas that are required for the remainder of this article.

2.1. The wiener process

Throughout this work, $\mathbb{X}$ and $\mathbb{V}$ are two real separable Hilbert spaces; we will identify by $\langle\cdot,\cdot\rangle_{\mathbb{X}}$, $\langle\cdot,\cdot\rangle_{\mathbb{V}}$ their inner products and by $\|\cdot\|_{\mathbb{X}}$, $\|\cdot\|_{\mathbb{V}}$ their associated norms, respectively. Let $\mathcal{L}(\mathbb{V},\mathbb{X})$ be the space of all bounded linear operators from $\mathbb{V}$ into $\mathbb{X}$ endowed with the norm $\|\cdot\|_{\mathcal{L}(\mathbb{V},\mathbb{X})}$. Whenever $\mathbb{X}=\mathbb{V}$, we indicate it with $\mathcal{L}(\mathbb{X})$. To keep things simple, we refer the norms of $\mathbb{V}$ and $\mathbb{X}$ with the same symbol $\|\cdot\|$ when there is no possibility of confusion. Further, let $\Omega := (\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ fulfilling the standard requirements (i.e it is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Assume that $\mathrm{B}(t)$ is a given $\mathbb{V}$-valued Wiener process with a finite trace nuclear covariance operator $\mathrm{Q} > 0$. Let $\alpha_k(t)$, $k = 1,2,\ldots$, be a sequence of real-valued one dimensional standard Brownian motion mutually independent over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$. One can construct $\mathrm{B}(t)$ as follows

$$\mathrm{B}\,(t) = \sum_{k=1}^{+\infty} \sqrt{\rho_k}\alpha_k\,(t)\,e_k, \quad t \geq 0,$$

where $\rho_k \geq 0$, $k = 1,2,\ldots$, are nonnegative real numbers, and $\{e_k\}$, $(k = 1,2,\ldots,)$ is a complete orthonormal basis in $\mathbb{V}$. Let $\mathrm{Q} \in \mathcal{L}(\mathbb{V})$ be an operator defined by $\mathrm{Q}e_k = \rho_k e_k$ with finite trace $tr\mathrm{Q} = \sum_{k=1}^{+\infty} \rho_k < \infty$. Thus, the above $\mathbb{V}$-valued stochastic process is referred to as a Q-Wiener process.

Definition 2.1: Let $\mu \in \mathcal{L}(\mathbb{V}, \mathbb{X})$ and define

$$\|\mu\|^2_{\mathcal{L}^0_2} := tr\left(\mu \mathrm{Q} \mu^*\right) = \sum_{k=1}^{+\infty} \left\|\sqrt{\rho_k}\mu e_k\right\|^2.$$

If $\|\mu\|^2_{\mathcal{L}^0_2} < \infty$, then μ is called a Q-Hilbert-Schmidt operator, and let $\mathcal{L}^0_2(\mathbb{V}, \mathbb{X})$ denote the space of all Q-Hilbert-Schmidt operators $\mu : mathbbV \to \mathbb{X}$.

The following important result will be helpful throughout the paper.

Lemma 2.2 ([10, Lemma 7.2]): *For any $p \geq 2$ and for arbitrary $\mathcal{L}^0_2$-valued predicable process $\psi(t)$, $t \in [0, T]$, we have the following inequality*

$$\mathbb{E}\left(\sup_{\ell \in [0,t]} \left\|\int_0^{\ell} \psi(r)\, \mathrm{d}B(r)\right\|^p\right) \leq \nu_p \mathbb{E}\left(\int_0^t \|\psi(\ell)\|^2_{\mathcal{L}^0_2}\, \mathrm{d}\ell\right)^{p/2},$$

for some constant $\nu_p > 0$.

2.2. Resolvent operators

The resolvent operator is a valuable tool in the study of the existence of solutions and gives a variation of constants formula for nonlinear systems. In this subsection, we recall some tools and facts that play an important role in the existence of the resolvent operators. For more details, we refer to [25]. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be Banach spaces. Then $\mathcal{L}(\mathcal{Z}_1)$ and $\mathcal{L}(\mathcal{Z}_1, \mathcal{Z}_2)$ denote respectively the space of bounded linear operators on $\mathcal{Z}_1$ and the space of bounded linear operators from $\mathcal{Z}_1$ to $\mathcal{Z}_2$.

Definition 2.3 ([38, Page: 93]): Let $\{\mathbb{A}(t) : t \geq 0\}$ be a family of generators of C_0-semigroups. $\{\mathbb{A}(t) : t \geq 0\}$ is called stable if there exist constants $N_{\mathbb{A}} \geq 1$ and λ_0 such that

$$\left\|(\mathbb{A}(t_i) - \omega I)^{-1}(\mathbb{A}(t_{i-1}) - \omega I)^{-1} \cdots (\mathbb{A}(t_1) - \omega I)^{-1}\right\| \leq \frac{N_{\mathbb{A}}}{(\omega - \lambda_0)^i}$$

for all $\omega > \lambda_0$ and for every finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_i$, $i \in \mathbb{N}^*$.

Let $\mathcal{E} = \mathrm{BU}(\mathbb{R}^+, \mathcal{Z}_1)$ be the space of all bounded uniformly continuous functions defined from $\mathbb{R}^+$ to $\mathcal{Z}_1$. We denote $\mathcal{H}$ as the Banach space formed by $\mathcal{D}(\mathbb{A})$ equipped with the graph norm $\|y\|_{\mathcal{H}} = \|\mathbb{A}(0)y\|_{\mathcal{Z}_1} + \|y\|_{\mathcal{Z}_1}$ for $y \in \mathcal{D}(\mathbb{A})$. Let define the linear operator $\tilde{\Gamma}(t)$, $t \geq 0$ from $\mathcal{H}$ into $\mathcal{E}$ by $(\tilde{\Gamma}(t)x)(s) = \Gamma(t+s, t)x$ for $s \geq 0$ and $x \in \mathcal{H}$. Let $\mathbb{D}$ be the differentiation operator defined on $\mathcal{E}$ by $\mathbb{D}f = f'$ on a domain $\mathcal{D}(\mathbb{D}) \subset \mathcal{E}$. Then, $\mathbb{D}$ is the infinitesimal generator of the translation semigroup $(S(t))_{t\geq 0}$ defined on $\mathcal{E}$ given by

$$\left(S(t)f\right)(s) = f(t+s) \quad \text{for } f \in \mathcal{E} \text{ and } t, s \geq 0.$$

Throughout this study, the following assumptions are necessary.

$(\mathbf{W}_1)$ The family $\{\mathbb{A}(t) : t \geq 0\}$ is a stable family of generators such that $\mathbb{A}(t)x$ is strongly continuously differentiable on $\mathbb{R}^+$ for $x \in \mathcal{H}$. In addition, $\tilde{\Gamma}(t)x$ is strongly continuously differentiable on $\mathbb{R}^+$ for $x \in \mathcal{H}$.

$(\mathbf{W}_2)$ $\tilde{\Gamma}(t)$ is continuous on $\mathbb{R}^+$ into $\mathcal{L}(\mathcal{H}, \mathcal{E})$.

$(\mathbf{W}_3)$ $\tilde{\Gamma}(t)x \subset \mathcal{D}(\mathbb{D})$ for each $t \geq 0$ and $\mathbb{D}\tilde{\Gamma}(t)$ is continuous on $\mathbb{R}^+$ into $\mathcal{L}(\mathcal{H}, \mathcal{E})$.

If all the above tree conditions are satisfied, we say that condition **(W)** is verified.

Remark 2.1: If $\mathbb{A}(t) = \mathbb{A}$ and $\Gamma(t,s) = \Gamma(t-s)$, then $\tilde{\Gamma}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{Z}_1)$ is constant, which implies that assumptions $(\mathbf{W}_1) - (\mathbf{W}_2)$ are fulfilled if $\mathbb{A}$ generates a C_0-semigroup.

The author in [25], studied the existence of the resolvent operator of the following non-autonomous equation:

$$\begin{cases} \vartheta'(t) = \mathbb{A}(t)\vartheta(t) + \int_0^t \Gamma(t,s)\vartheta(s)\,\mathrm{d}s for\ t \geq 0, \\ \vartheta(0) = \vartheta_0 \in \mathcal{Z}_1. \end{cases} \tag{5}$$

In the next, we recall some useful properties on this theory.

Definition 2.4 ([25, Definition 2.2]): A family $\{\mathscr{R}(t,s) : 0 \leq s \leq t\}$ of bounded linear operator on $\mathcal{Z}_1$ is called resolvent operator for Equation (5) if the following properties are verified :

(i) $\mathscr{R}(t,s)$ is strongly continuous in s and t, $\mathscr{R}(s,s) = I$, for $s \geq 0$ and $\|\mathscr{R}(t,s)\|_{\mathcal{L}(\mathcal{Z}_1)} \leq N\,\mathrm{e}^{\eta(t-s)}$, $0 \leq s \leq t$, for some constants $N \geq 1$ and $\eta \in \mathbb{R}$.

(ii) $\mathscr{R}(t,s)\mathcal{H} \subset \mathcal{H}$ and $\mathscr{R}(t,s)$ is strongly continuous in s and t on $\mathcal{H}$.

(iii) For every $x \in \mathcal{H}$, $\mathscr{R}(t,s)x$ is strongly continuously differentiable in t and s. Moreover,

$$\begin{cases} \dfrac{\partial \mathscr{R}(t,s)\,x}{\partial t} = \mathbb{A}(t)\mathscr{R}(t,s)x + \displaystyle\int_s^t \Gamma(t,\tau)\mathscr{R}(\tau,s)x\,\mathrm{d}\tau, \\ \dfrac{\partial \mathscr{R}(t,s)\,x}{\partial s} = -\mathscr{R}(t,s)\mathbb{A}(s)x - \displaystyle\int_s^t \mathscr{R}(t,\tau)\Gamma(\tau,s)x\,\mathrm{d}\tau. \end{cases} \tag{6}$$

Lemma 2.5 ([25, Theorem 2.4]): *If $\mathbb{A}(t) = \mathbb{A}$ and $\Gamma(t,s) = \Gamma(t-s)$ for $t \geq 0$ and $t \geq s \geq 0$ respectively, then $\mathscr{R}(t,s) = \mathscr{R}(t-s)$ for each $t \geq s \geq 0$.*

The following Theorem ensures the existence of a resolvent operator for Equation (5).

Lemma 2.6 ([25]): *Assume that $(\mathbf{W}_1) - (\mathbf{W}_3)$ hold. Then, Equation (5) has a unique resolvent operator.*

It is assumed throughout the entirety of this work that the assumptions $(\mathbf{W}_1) - (\mathbf{W}_3)$ hold. We consider the following equation:

$$\begin{cases} \vartheta'(t) = \mathbb{A}(t)\vartheta(t) + \int_0^t \Gamma(t,s)\vartheta(s)\,\mathrm{d}s + \zeta(t)\ \ for\ t \geq 0, \\ \vartheta(0) = \vartheta_0 \in \mathcal{Z}_1, \end{cases} \tag{7}$$

where $\zeta \in L_{loc}(\mathbb{R}^+; \mathcal{Z}_1)$.

Definition 2.7 ([25]): A function $\vartheta : [0, +\infty[\to \mathcal{Z}_1$ is called a strict solution of Equation (7) if

(i) $\vartheta \in \mathcal{C}^1([0, +\infty[; \mathcal{Z}_1) \cap \mathcal{C}([0, +\infty[; \mathcal{D}(\mathtt{A}))$.
(ii) ϑ satisfies the following equation:

$$\vartheta(t) = \mathscr{R}(t, 0)\, \vartheta_0 + \int_0^t \mathscr{R}(t, s)\, \zeta(s)\, \mathrm{d}s, \quad t \geq 0, \tag{8}$$

where $\{\mathscr{R}(t, s) : 0 \leq s \leq t\}$ is the resolvent operator of Equation (5).

The existence of a strict solution for Equation (7) is guaranteed by the Theorem below.

Theorem 2.8 ([25, Corollary 3.8]): *Assume that* $(\mathbf{W}_1) - (\mathbf{W}_3)$ *hold. If* $\vartheta_0 \in \mathcal{D}(\mathtt{A})$ *and* $\zeta \in \mathcal{C}^1(\mathbb{R}^+; \mathcal{Z}_2)$, *then Equation* (7) *has a strict solution given by* (8).

Definition 2.9: A mild solution of Equation (7) is defined as a function $\vartheta : [0, +\infty[\to \mathcal{Z}_1$ that satisfies (8).

When studying the existence of mild solutions of non-autonomous systems, the theory of evolution family is an absolute necessity. To begin, let us define the evolution family.

Definition 2.10 ([33]): A family $\{\mathcal{U}(t, s) : 0 \leq s \leq t \leq T\}$ of bounded linear operators is called an evolution family if

(i) $\mathcal{U}(t, t) = I$ and $\mathcal{U}(t, r)\mathcal{U}(r, s) = \mathcal{U}(t, s)$ for $0 \leq s \leq r \leq t \leq T$.
(ii) The mapping $(t, s) \mapsto \mathcal{U}(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

To construct an evolution family, let us impose the following assumptions on the family of linear operators $\{\mathtt{A}(t), t \in J\}$ see [33, Chapter 5, Theorem 6.1]. We suppose that:

$(\mathbf{R}_1)$ $\mathtt{A}(t)$ is closed and the domain $\mathcal{D}(\mathtt{A}(t)) = \mathcal{D}(\mathtt{A})$ is independent of t and is dense in $\mathbb{X}$.
$(\mathbf{R}_2)$ For each $t \geq 0$, the map $R(\omega, \mathtt{A}(t)) = (\omega I - \mathtt{A}(t))^{-1}$ exists for all $\omega \in \mathbb{C}$ with $Re(\omega) \leq 0$ and there exists $\alpha_1 > 0$ such that

$$\|R(\omega, \mathtt{A}(t))\|_{\mathcal{L}(\mathbb{X})} \leq \frac{\alpha_1}{|\omega| + 1}.$$

$(\mathbf{R}_3)$ There exist constants $\alpha_2 > 0$ and $0 < \kappa \leq 1$ such that

$$\left\|(\mathtt{A}(t_1) - \mathtt{A}(t_2))\, \mathtt{A}(t_3)^{-1}\right\|_{\mathcal{L}(\mathbb{X})} \leq \alpha_2 |t_1 - t_2|^{\kappa} \quad \text{for } t_1 \geq 0,\ t_2 \geq 0 \quad \text{and } t_3 \geq 0.$$

$(\mathbf{R}_4)$ The operator $R(\omega, \mathtt{A}(t)), t \in J$ is compact for some $\omega \in \rho(\mathtt{A}(t))$ where $\rho(\mathtt{A}(t))$ is the resolvent set of $\mathtt{A}(t)$.

Lemma 2.11 ([33, Chapter 5, Theorem 6.1]): *Assume that* $(\mathbf{R}_1) - (\mathbf{R}_3)$ *hold. Then, there exists a unique evolution system* $\{\mathcal{U}(t, s) : 0 \leq s \leq t\}$ *generated by the family* $\{\mathtt{A}(t) : t \geq 0\}$.

Lemma 2.12 ([20, Proposition 2.1]): *Suppose $\{\mathbb{A}(t) : t \in J\}$ satisfies the Assumptions* $(\mathbf{R}_1) - (\mathbf{R}_4)$. *Let $\{\mathcal{U}(t,s) : 0 \le s \le t\}$ be the linear evolution family generated by the family $\{\mathbb{A}(t) : t \ge 0\}$, then $\{\mathcal{U}(t,s) : 0 \le s \le t\}$ is a compact operator whenever $t-s > 0$.*

Theorem 2.13 ([15]): *Assume that* $(\mathbf{R}_1) - (\mathbf{R}_3)$ *are satisfied. Let $\{\mathcal{U}(t,s) : 0 \le s \le t\}$ be the unique evolution system generated by $\{\mathbb{A}(t) : t \ge 0\}$ and $\{\mathscr{R}(t,s) : 0 \le s \le t\}$ be the unique resolvent operator of Equation* (5). *Then, $\mathcal{U}(t,s)$ is compact for $t-s > 0$ if and only if $\mathscr{R}(t,s)$ is compact for $t-s > 0$.*

Theorem 2.14 ([15, Theorem 2.16]): *Assume that* $(\mathbf{W}_1) - (\mathbf{W}_3)$ *and* $(\mathbf{R}_1) - (\mathbf{R}_3)$ *are valid.*

Let $\{\mathscr{R}(t,s) : 0 \le s \le t\}$ be the unique resolvent operator of Equation (5) *and $\{\mathcal{U}(t,s) : 0 \le s \le t\}$ the evolution system generated by the family $\{\mathbb{A}(t) : t \ge 0\}$. Then, $\mathscr{R}(t,s)$ is norm-continuous for $t-s > 0$ if only if $\mathcal{U}(t,s)$ is norm-continuous for $t-s > 0$.*

In the sequel, we denote $\mathcal{C} := \mathcal{C}([-b,0],\mathbb{X})$, the space of all continuous functions from $[-b,0]$ to $\mathbb{X}$. Suppose that $u(t) : Omega \to \mathbb{X},\ t \ge -b$, is a continuous, $\mathcal{F}_t$-adapted, $\mathbb{X}$-valued stochastic process, we can associate with another process $u_t : Omega \to \mathcal{C},\ t \ge 0$, by setting $u_t(\theta)(\omega) = u(t+\theta)(\omega),\ \theta \in [-b,0]$. Thus, we say that the process u_t is induced by the process $u(t)$. Let $\mathcal{MC}(p), p > 2$, denote the space of all $\mathcal{F}_t$-mesurable $\mathcal{C}$-valued functions $\mathcal{K} : Omega \mapsto \mathcal{C}$ with norm $\mathbb{E}\|\mathcal{K}\|_{\mathcal{C}}^p := \mathbb{E}(\sup_{\theta\in[-b,0]} \|\mathcal{K}(\theta)\|^p) < \infty$.

3. Existence and uniqueness of mild solutions

In the following part, we will investigate the existence and uniqueness of mild solutions for Equation (3). To solve this problem, we will assume that the following conditions hold :

- $(\mathbf{C}_1)$ The resolvent operator $\{\mathscr{R}(t,s) : 0 \le s \le t\}$ associated to the system (3) is norm-continuous.
- $(\mathbf{C}_2)$ The function $\mathrm{g} : [0,T] \times \mathcal{C} \to \mathbb{X}$ satisfies the following Lipschitz conditions: that is, there exists a constant $C_{\mathrm{g}} > 0$ such that, for any and $\xi \in \mathcal{C}, t \in [0,T]$,

$$\left\|\mathrm{g}(t,\xi_1) - \mathrm{g}(t,\xi_2)\right\|^p \le C_{\mathrm{g}} \|\xi_1 - \xi_2\|_{\mathcal{C}}^p,$$
$$\left\|\mathrm{g}(t,\xi)\right\|^p \le C_{\mathrm{g}} \|\xi\|_{\mathcal{C}}^p,$$

  and $3^{p-1}C_{\mathrm{g}} < 1$.
- $(\mathbf{C}_3)$ The function $\sigma : [0,T] \times \mathcal{C} \to \mathbb{X}$ satisfies the following Lipschitz conditions: that is, there exists a constant $C_\sigma > 0$ such that for any $\xi_1, \xi_2 \in \mathcal{C}$ and $t \in [0,T]$,

$$\|\sigma(t,\xi_1) - \sigma(t,\xi_2)\|^p \le C_\sigma \|\xi_1 - \xi_2\|_{\mathcal{C}}^p.$$

- $(\mathbf{C}_4)$ For function $h : [0,T] \times \mathcal{C} \to \mathcal{L}_2^0(\mathbb{V},\mathbb{X})$, there exists a constant $C_h > 0$ such that

$$\|h(t,\xi_1) - h(t,\xi_2)\|_{\mathcal{L}_2^0}^p \le C_h \|\xi_1 - \xi_2\|_{\mathcal{C}}^p,$$

  for any $\xi_1, \xi_2 \in \mathcal{C}$ and $t \in [0,T]$.

($\mathbf{C}_5$) There exists a constant $C_0 > 0$ such that

$$\|\sigma(t,\xi)\|^p + \|h(t,\xi)\|^p_{\mathcal{L}_2^0} \le C_0\left(\|\xi\|^p_{\mathcal{C}} + 1\right)$$

for any $\xi \in \mathcal{C}$ and $t \in [0,T]$.

Additionally, we always suppose that for any $t \ge 0$, $\mathrm{g}(t,0)=0$, $\sigma(t,0)=0$ and $h(t,0)=0$. Thus, when $\varphi = 0$, it is obvious that system (3) has a trivial solution. Let

$$M_T = \sup\left\{\left\|\mathscr{R}(t,s)\right\|_{\mathcal{L}(\mathbb{X})} : 0 \le s \le t \le T\right\}.$$

Definition 3.1: A continuous process $u : [-b,T]\times\Omega \to \mathbb{X}$ is called a mild solution of (3) if

(i) $u(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \in [0,T]$;
(ii) $\int_0^T \|u_s\|^p \,\mathrm{d}s < \infty$, a.s.;
(iii) $u(t)$ satisfies the following stochastic integral

$$u(t) = \mathscr{R}(t,0)\left[\varphi(0) - \mathrm{g}(0,\varphi)\right] + \mathrm{g}(t,u_t) + \int_0^t \mathscr{R}(t,s)\,\sigma(s,u_s)\,\mathrm{d}s + \int_0^t \mathscr{R}(t,s)\,h(s,u_s)\,\mathrm{dB}(s) \tag{9}$$

on the interval $[0,T]$, and $u(t) = \varphi(t)$ for $t \in [-b,0]$.

Theorem 3.2: *Assume that* ($\mathbf{C}_1$)-($\mathbf{C}_5$) *hold. If the resolvent operator* $\{\mathscr{R}(t,s) : 0 \le s \le t\}$ *is norm-continuous, then system (3) has a unique mild solution for any initial value* $\varphi \in \mathcal{MC}(p)$.

Proof: Let $\mathcal{N}_T$ denote the Banach space of all continuous processes $u(t)$ which belong to the space $\mathcal{C}([-b,T], L^p(\Omega,\mathbb{X}))$ with $\|u\|_{\mathcal{N}_T} < \infty$, where

$$\|u\|_{\mathcal{N}_T} := \sup_{t\in[0,T]}\left(\mathbb{E}\,\|u_t\|^p_{\mathcal{C}}\right)^{1/p},$$

$$\mathbb{E}\,\|u_t\|^p_{\mathcal{C}} := \mathbb{E}\left(\sup_{\theta\in[-b,0]}\|u_t(\theta)\|^p\right).$$

Define the operator $\mathbb{F}$ on $\mathcal{N}_T$ by $(\mathbb{F}u)(t) = \varphi(t)$ for $t \in [-b,0]$, and

$$(\mathbb{F}u)(t) = \mathscr{R}(t,0)\left[\varphi(0) - \mathrm{g}(0,\varphi)\right] + \mathrm{g}(t,u_t) + \int_0^t \mathscr{R}(t,s)\,\sigma(s,u_s)\,\mathrm{d}s + \int_0^t \mathscr{R}(t,s)\,h(s,u_s)\,\mathrm{dB}(s) \quad \text{for } t \in [0,T]. \tag{10}$$

Clearly, any fixed point of $\mathbb{F}$ is a mild solution of (3). Then, demonstrating that $\mathbb{F}$ has a fixed point suffices to establish the existence of a mild solution. We split the proof in three steps.

Step 1: For randomly $u \in \mathcal{N}_T$, $(\mathbb{F}u)(t)$ is continuous on $[0, T]$ in the L^p-sense.
Let $0 < t < T$ and $|r|$ be sufficiently small. For any fixed $u \in \mathcal{N}_T$, we have

$$\begin{aligned}
&\mathbb{E}\left\|(\mathbb{F}u)(t+r)-(\mathbb{F}u)(t)\right\|^p \\
&\quad\leq 4^{p-1}\mathbb{E}\left\|\left(\mathscr{R}(t+r,0)-\mathscr{R}(t,0)\right)\left[\varphi(0)-\mathrm{g}(0,\varphi)\right]\right\|^p \\
&\qquad+4^{p-1}\mathbb{E}\left\|\mathrm{g}(t+r,u_{t+r})-\mathrm{g}(t,u_t)\right\|^p \\
&\qquad+4^{p-1}\mathbb{E}\left\|\int_0^{t+r}\mathscr{R}(t+r,s)\,\sigma(s,u_s)\,\mathrm{d}s-\int_0^t\mathscr{R}(t,s)\,\sigma(s,u_s)\,\mathrm{d}s\right\|^p \\
&\qquad+4^{p-1}\mathbb{E}\left\|\int_0^{t+r}\mathscr{R}(t+r,s)\,h(s,u_s)\,\mathrm{dB}(s)-\int_0^t\mathscr{R}(t,s)\,h(s,u_s)\,\mathrm{d}B(s)\right\|^p \\
&\quad:=\sum_{k=1}^{4}I_k.
\end{aligned} \tag{11}$$

For the first term I_1, by using the strong continuity of $\{\mathscr{R}(t,s) : 0 \leq s \leq t\}$, we have

$$\left(\mathscr{R}(t+r,0)-\mathscr{R}(t,0)\right)\left[\varphi(0)-\mathrm{g}(0,\varphi)\right]\to 0 \quad \text{as } r\to 0.$$

Further, we clearly have

$$\left\|\left(\mathscr{R}(t+r,0)-\mathscr{R}(t,0)\right)\left[\varphi(0)-\mathrm{g}(0,\varphi)\right]\right\|\leq 2M_T\left\|\varphi(0)-\mathrm{g}(0,\varphi)\right\|.$$

Then, the Lebesgue-dominated theorem allows us to conclude that $\lim_{r\to 0} I_1 = 0$.
For the term I_3, we have

$$\begin{aligned}
I_3 &\leq 8^{p-1}\mathbb{E}\left\|\int_t^{t+r}\mathscr{R}(t+r,s)\,\sigma(s,u_s)\,\mathrm{d}s\right\|^p \\
&\quad+8^{p-1}\mathbb{E}\left\|\int_0^t\left[\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right]\sigma(s,u_s)\,\mathrm{d}s\right\|^p.
\end{aligned} \tag{12}$$

Using η_1 and η_2 to denote the terms of the right-hand side of the above inequality, respectively, we have

$$\begin{aligned}
\eta_1 &= 8^{p-1}\mathbb{E}\left\|\int_t^{t+r}\mathscr{R}(t+r,s)\,\sigma(s,u_s)\,\mathrm{d}s\right\|^p \\
&\leq 8^{p-1}\mathbb{E}\left(\int_t^{t+r}\left\|\mathscr{R}(t+r,s)\,\sigma(s,u_s)\right\|\,\mathrm{d}s\right)^P \\
&\leq 8^{p-1}C_0M_T^p r^{p-1}\int_t^{t+r}\left(1+\mathbb{E}\left\|u_s\right\|_{\mathcal{C}}^p\right)\,\mathrm{d}s \\
&\leq 8^{p-1}C_0M_T^p r^p\left(1+\left\|u\right\|_{\mathcal{N}_T}^p\right),
\end{aligned} \tag{13}$$

and

$$\begin{aligned}\eta_2 &= 8^{p-1}\mathbb{E}\left\|\int_0^t [\mathscr{R}(t+r,s)-\mathscr{R}(t,s)]\,\sigma\,(s,u_s)\,\mathrm{d}s\right\|^p\\ &\le 8^{p-1}\mathbb{E}\left(\int_0^t \left\|[\mathscr{R}(t+r,s)-\mathscr{R}(t,s)]\,\sigma\,(s,u_s)\right\|\,\mathrm{d}s\right)^p\\ &\le 8^{p-1}C_0T\left(1+\|u\|^p_{\mathcal{N}_T}\right)\left(\int_0^t \left\|\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right\|^q_{\mathcal{L}(\mathbb{X})}\,\mathrm{d}s\right)^{p/q}, \qquad (14)\end{aligned}$$

where $q>0$ satisfies $\frac{1}{p}+\frac{1}{q}=1$.

Then replacing (13) and (14) into (12), it follows

$$\begin{aligned}I_3 &\le 8^{p-1}C_0M_T^pr^p\left(1+\|u\|^p_{\mathcal{N}_T}\right)\\ &\quad+8^{p-1}C_0T\left(1+\|u\|^p_{\mathcal{N}_T}\right)\left(\int_0^t \left\|\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right\|^q_{\mathcal{L}(\mathbb{X})}\,\mathrm{d}s\right)^{p/q}. \qquad (15)\end{aligned}$$

Likewise, we have that

$$\begin{aligned}I_4 &\le 8^{p-1}\mathbb{E}\left\|\int_t^{t+r}\mathscr{R}(t+r,s)\,h(s,u_s)\,\mathrm{dB}(s)\right\|^p\\ &\quad+8^{p-1}\mathbb{E}\left\|\int_0^t[\mathscr{R}(t+r,s)-\mathscr{R}(t,s)]\,h(s,u_s)\,\mathrm{dB}(s)\right\|^p\\ &:=\eta_3+\eta_4. \qquad (16)\end{aligned}$$

From $(\mathbf{C}_5)$, Lemma 2.2 and Hölder inequality, we derive that

$$\begin{aligned}\eta_3 &= 8^{p-1}\mathbb{E}\left\|\int_t^{t+r}\mathscr{R}(t+r,s)\,h(s,u_s)\,\mathrm{dB}(s)\right\|^p\\ &\le 8^{p-1}\nu_p\mathbb{E}\left(\int_t^{t+r}\left\|\mathscr{R}(t+r,s)\,h(s,u_s)\right\|^2\,\mathrm{d}s\right)^{p/2}\\ &\le 8^{p-1}\nu_pM_T^pr^{(p-2)/2}C_0\int_t^{t+r}\left(1+\mathbb{E}\,\|u_s\|^p_{C}\right)\mathrm{d}s\\ &\le 8^{p-1}\nu_pM_T^pr^{p/2}C_0\left(1+\|u\|^p_{\mathcal{N}_T}\right), \qquad (17)\end{aligned}$$

and

$$\begin{aligned}\eta_4 &= 8^{p-1}\mathbb{E}\left\|\int_0^t[\mathscr{R}(t+r,s)-\mathscr{R}(t,s)]\,h(s,u_s)\,\mathrm{dB}(s)\right\|^p\\ &\le 8^{p-1}\nu_p\mathbb{E}\left(\int_0^t\left\|\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right\|^2_{\mathcal{L}(\mathbb{X})}\|h(s,u_s)\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\right)^{p/2}\\ &\le 8^{p-1}\nu_pC_0T\left(1+\|u\|^p_{\mathcal{N}_T}\right)\left(\int_0^t\left\|\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right\|^{2p/(p-2)}_{\mathcal{L}(\mathbb{X})}\,\mathrm{d}s\right)^{(p-2)/2}. \qquad (18)\end{aligned}$$

Then, by replacing (17) and (18) into (16), we obtain

$$I_4 \leq 8^{p-1}\nu_p M_T^p r^{p/2} C_0 \left(1+\|u\|_{\mathcal{N}_T}^p\right) + 8^{p-1}\nu_p C_0 T \left(1+\|u\|_{\mathcal{N}_T}^p\right)\left(\int_0^t \left\|\mathscr{R}(t+r,s)-\mathscr{R}(t,s)\right\|_{\mathcal{L}(\mathbb{X})}^{2p/(p-2)} \mathrm{d}s\right)^{(p-2)/2}. \tag{19}$$

From (15) and (19), we deduce that I_3 and I_4 tend to 0 as $r \to 0$ by virtue of the norm-continuity of $\mathscr{R}(t,s)$ for $t-s>0$ and the Lebesgue dominated convergence theorem. Furthermore, I_2 tends to 0 as $r \to 0$ from $(\mathbf{C}_2)$. Hence, $(\mathbb{F}u)(t)$ is continuous on $[0,T]$ in the L^p-sense.

Step 2: We prove that $\mathbb{F}$ maps $\mathcal{N}_T$ into itself.

Let $u \in \mathcal{N}_T$, then we have

$$\begin{aligned}
\mathbb{E}\left\|(\mathbb{F}u)_t\right\|_C^p \leq\ & 4^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\mathscr{R}(t+\theta,0)\left[\varphi(0)-\mathrm{g}(0,\varphi)\right]\right\|^p \\
&+4^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\mathrm{g}(t+\theta,u_{t+\theta})\right\|^p \\
&+4^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}(t+\theta,s)\,\sigma(s,u_s)\,\mathrm{d}s\right\|^p \\
&+4^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}(t+\theta,s)\,h(s,u_s)\,\mathrm{dB}(s)\right\|^p \\
:=\ & \sum_{k=1}^4 J_k.
\end{aligned} \tag{20}$$

By $(\mathbf{C}_1)$, we have that

$$J_1 \leq 4^{p-1}M_T^p\mathbb{E}\left\|\varphi(0)-\mathrm{g}(0,\varphi)\right\|^p. \tag{21}$$

From $(\mathbf{C}_2)$, one clearly has

$$J_2 \leq 4^{p-1}C_{\mathrm{g}}\|u\|_{\mathcal{N}_T}^p. \tag{22}$$

Using Hölder inequality and by $(\mathbf{C}_5)$, we get

$$\begin{aligned}
J_3 &\leq 4^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}\left\|\mathscr{R}(t+\theta,s)\,\sigma(s,u_s)\right\|\,\mathrm{d}s\right)^p \\
&\leq 4^{p-1}M_T^p\mathbb{E}\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}\|\sigma(s,u_s)\|\,\mathrm{d}s\right)^p \\
&\leq 4^{p-1}C_0M_T^pT^p\left(1+\|u\|_{\mathcal{N}_T}^p\right).
\end{aligned} \tag{23}$$

By the equality below

$$\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-\beta}\,\mathrm{d}s = \frac{\pi}{\sin\pi\beta}, \quad \text{for } \tau\leq s\leq t, \beta\in(0,1),$$

proven in [9], it follows that

$$J_4 = 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p \mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}(t+\theta,s)\,h(s,u_s)\right.$$
$$\left.\left(\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{-\beta}\,\mathrm{d}s\right)\mathrm{dB}(s)\right\|^p$$
$$\le 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p \mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}(t+\theta,s)\,(t+\theta-s)^{\beta-1}\,Y(s)\,\mathrm{d}s\right\|^p,$$

where

$$Y(s)=\int_0^s (s-\tau)^{-\beta}\,\mathscr{R}(s,\tau)\,h(\tau,u_\tau)\,\mathrm{dB}(\tau).$$

Using Hölder inequality, we have

$$j_4 \le 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p M_T^p\mathbb{E}\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}(t+\theta-s)^{\beta-1}\,\|Y(s)\|\,\mathrm{d}s\right)^p$$
$$\le 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p \frac{M_T^p T^{p\beta-1}}{(q(\beta-1)+1)^{p/q}}\mathbb{E}\int_0^T \|Y(s)\|^p\,\mathrm{d}s. \tag{24}$$

Following Lemma 2.2, we have

$$\mathbb{E}\int_0^T\|Y(s)\|^p\,\mathrm{d}s = \mathbb{E}\int_0^T\left\|\int_0^s (s-\tau)^{-\beta}\,\mathscr{R}(s,\tau)\,h(\tau,u_\tau)\,\mathrm{d}B(\tau)\right\|^p\mathrm{d}s$$
$$\le \nu_p M_T^p\mathbb{E}\int_0^T\left(\int_0^s (s-\tau)^{-2\beta}\,\|h(\tau,u_\tau)\|^2\,\mathrm{d}\tau\right)^{p/2}\mathrm{d}s.$$

So, by using Young's inequality and $(\mathbf{C}_5)$ in the above inequality, we get

$$\mathbb{E}\int_0^T\|Y(s)\|^p\,\mathrm{d}s \le \nu_p M_T^p\left(\int_0^T s^{-2\beta}\,\mathrm{d}s\right)^{p/2}\mathbb{E}\int_0^T\|h(s,u_s)\|^p\,\mathrm{d}s$$
$$\le \nu_p M_T^p\frac{T^{p/2-p\beta}}{(1-2\beta)^{p/2}}C_0\int_0^T\left(1+\mathbb{E}\|u_s\|_C^p\right)\mathrm{d}s$$
$$\le \frac{\nu_p M_T^p T^{p/2-p\beta+1}}{(1-2\beta)^{p/2}}C_0\left(1+\|u\|_{\mathcal{N}_T}^p\right). \tag{25}$$

Then replacing (25) into (24), we obtain

$$J_4 \le 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p\frac{C_0\nu_p M_T^{2p}T^{p/2}}{(q(\beta-1)+1)^{p/q}(1-2\beta)^{p/2}}\left(1+\|u\|_{\mathcal{N}_T}^p\right). \tag{26}$$

Hence on account of (21), (22), (23) and (26), we have $\|\mathbb{F}u\|_{\mathcal{N}_T}^p<\infty$, and so $\mathbb{F}$ maps $\mathcal{N}_T$ into itself.

Step 3: We show that $\mathbb{F}$ is a contraction on $\mathcal{N}_T$.

Let $u, u^* \in \mathcal{N}_T$, then for any fixed $t \in [0, T]$, we have

$$\begin{aligned}
&\mathbb{E}\left\|(\mathbb{F}u)_t - \left(\mathbb{F}u^*\right)_t\right\|_C^p \\
&\leq 3^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\mathrm{g}\left(t, u_{t+\theta}\right) - \mathrm{g}\left(t, u^*_{t+\theta}\right)\right\|^p \\
&\quad + 3^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}\left(t+\theta, s\right)\left[\sigma\left(s, u_s\right) - \sigma\left(s, u_s^*\right)\right]\mathrm{d}s\right\|^p \\
&\quad + 3^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}\left(t+\theta, s\right)\left[h\left(s, u_s\right) - h\left(s, u_s^*\right)\right]\mathrm{dB}\left(s\right)\right\|^p \\
&:= \sum_{k=1}^{3} G_k.
\end{aligned} \tag{27}$$

It is easily seen that

$$G_1 \leq 3^{p-1}C_{\mathrm{g}}\left\|u - u^*\right\|_{\mathcal{N}_T}^p. \tag{28}$$

Further, we have

$$\begin{aligned}
G_2 &\leq 3^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}\left\|\mathscr{R}\left(t+\theta, s\right)\left[\sigma\left(s, u_s\right) - \sigma\left(s, u_s^*\right)\right]\right\|\,\mathrm{d}s\right)^p \\
&\leq 3^{p-1}M_T^pT^{p-1}\mathbb{E}\sup_{\theta\in[-b,0]}\int_0^{t+\theta}\left\|\sigma\left(s, u_s\right) - \sigma\left(s, u_s^*\right)\right\|^p\,\mathrm{d}s \\
&\leq 3^{p-1}M_T^pT^pC_\sigma\left\|u - u^*\right\|_{\mathcal{N}_T}.
\end{aligned} \tag{29}$$

By setting

$$Y_1\left(s\right) = \int_0^s\left(s-\tau\right)^{-\beta}\mathscr{R}\left(s, \tau\right)\left[h\left(\tau, u_\tau\right) - h\left(\tau, u_\tau^*\right)\right]\mathrm{dB}\left(\tau\right),$$

and similarly to how we did with the term J_4, we obtain

$$\begin{aligned}
G_3 &\leq 3^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p\mathbb{E}\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}\left(t+\theta, s\right)\left(t+\theta-s\right)^{\beta-1}Y_1\left(s\right)\mathrm{d}s\right\|^p \\
&\leq 3^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p\frac{C_h\nu_pM_T^{2p}T^{p/2}}{\left(q\left(\beta-1\right)+1\right)^{p/q}\left(1-2\beta\right)^{p/2}}\left\|u - u^*\right\|_{\mathcal{N}_T}.
\end{aligned} \tag{30}$$

Hence,

$$\left\|\mathbb{F}\left(u\right) - \mathbb{F}\left(u^*\right)\right\|_{\mathcal{N}_T}^p \leq \Pi\left(T\right)\left\|u - u^*\right\|_{\mathcal{N}_T}^p,$$

with

$$\Pi\left(T\right) = \frac{1}{1-3^{p-1}C_{\mathrm{g}}}\left[3^{p-1}M_T^pT^pC_\sigma + 3^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p\frac{C_h\nu_pM_T^{2p}T^{p/2}}{\left(q\left(\beta-1\right)+1\right)^{p/q}\left(1-2\beta\right)^{p/2}}\right]. \tag{31}$$

Thus, one can select a suitable $0 < T_0 < T$ sufficiently small such that $\Pi(T_0) < 1$, which implies that $\mathbb{F}$ is a contraction on $\mathcal{N}_{T_0}$, where $\mathcal{N}_{T_0}$ indicates $\mathcal{N}_T$ with T substituted by T_0.

Then, based on the Banach fixed point theorem, we have a unique fixed point $u \in \mathcal{N}_{T_0}$ for $\mathbb{F}$ corresponding to a mild solution of (3) on $[0, T_0]$. This procedure can be continued to extend the solution over the whole interval $[0, T]$ in finitely many identical steps, thereby finishing the proof for the existence and uniqueness of mild solutions on the entire interval $[0, T]$. ■

4. Exponential stability

Because we are interested in the exponential stability in the pth(p > 2) moment of the mild solutions to (3), we will therefore assume that for every $T > 0$ and every $\varphi \in L^p(\Omega, \mathcal{C})$, the problem (3) has a mild solution by Definition 3.1.We need to state the following conditions:

($\mathbf{E}_1$): The resolvent operator $\{\mathscr{R}(t,s) : 0 \leq s \leq t\}$ associated to the system (3) is exponentially stable; i.e. :, $\|\mathscr{R}(t,s)\|_{\mathcal{L}(\mathbb{X})} \leq M\,\mathrm{e}^{-\lambda(t-s)}$, $\forall\, t, s \geq 0$, where $M \geq 1$, $\lambda > 0$.

The following theorem shows the exponential stability in the pth (p > 2) moment of the mild solutions to (3).

Theorem 4.1: *In addition to* ($\mathbf{E}_1$), *assume that the functions* $\mathrm{g} : [0, +\infty) \times \mathcal{C} \to \mathbb{X}$, $\sigma : [0, +\infty) \times \mathcal{C} \to \mathbb{X}$ *and* $h : [0, +\infty) \times \mathcal{C} \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{X})$ *satisfy the Lipschitz conditions in* ($\mathbf{C}_2$)-($\mathbf{C}_4$), *respectively. Also, assume that there exist nonnegative real numbers* L_{g}, L_σ *and* L_h, *and continuous functions* $Z_i : [0, +\infty) \to \mathbb{R}^+$ *with* $Z_i(t) \leq N_i\,\mathrm{e}^{-\lambda t}(i = 0, 1, 2), N_i > 0$, *such that*

$$
\begin{aligned}
\mathbb{E}\left\|\mathrm{g}\left(t, u_t\right)\right\|^p &\leq L_{\mathrm{g}} \mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p + Z_0(t), \quad t \geq 0, \\
\mathbb{E}\left\|\sigma\left(t, u_t\right)\right\|^p &\leq L_\sigma \mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p + Z_1(t), \quad t \geq 0, \\
\mathbb{E}\left\|h\left(t, u_t\right)\right\|_{\mathcal{L}_2^0}^p &\leq L_h \mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p + Z_2(t), \quad t \geq 0,
\end{aligned} \tag{32}
$$

for any mild solution $u(t)$ *of* (3). *If the constants* L_σ *and* L_h *are small enough such that* $\lambda - \tilde{K}_3/(1 - 4^{p-1}L_{\mathrm{g}}) > 0$ *with* $\tilde{K}_3$ *given by* (35) *below, then the solution is exponentially stable in the pth moment. In other words, there exist positive constant* $\delta > 0$ *and* $\tilde{K}(p, \delta, \varphi) > 0$ *such that, for each* $t \geq 0$,

$$
\mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p \leq \tilde{K}\left(p, \delta, \varphi\right) \mathrm{e}^{-\delta t}.
$$

Proof: Let $u(t)$ be a mild solution of (3), then

$$
\begin{aligned}
\mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p &= \mathbb{E}\left(\sup_{\theta \in [-b, 0]} \left\|u(t+\theta)\right\|^p\right) \\
&\leq 4^{p-1} \mathbb{E}\left(\sup_{\theta \in [-b, 0]} \left\|\mathscr{R}(t+\theta, 0)\left[\varphi(0) - \mathrm{g}(0, \varphi)\right]\right\|^p\right) \\
&\quad + 4^{p-1} \mathbb{E}\left(\sup_{\theta \in [-b, 0]} \left\|\mathrm{g}\left(t+\theta, u_{t+\theta}\right)\right\|^p\right) \\
&\quad + 4^{p-1} \mathbb{E}\left(\sup_{\theta \in [-b, 0]} \left\|\int_0^{t+\theta} \mathscr{R}(t+\theta, s)\, \sigma\left(s, u_s\right) \mathrm{d}s\right\|^p\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4^{p-1}\mathbb{E}\left(\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}\left(t+\theta,s\right)h\left(s,u_s\right)\mathrm{dB}\left(s\right)\right\|^p\right)\\
&:=\sum_{k=1}^{4}\varpi_k. \qquad (33)
\end{aligned}
$$

From $(\mathbf{C}_1)$, we have

$$
\begin{aligned}
\varpi_1 &\le 4^{p-1}\mathbb{E}\left(\sup_{\theta\in[-b,0]}M^p\,\mathrm{e}^{-\lambda p(t+\theta)}\left\|\varphi\left(0\right)-\mathrm{g}\left(0,\varphi\right)\right\|^p\right)\\
&\le 4^{p-1}M^p\,\mathrm{e}^{\lambda pb}\mathbb{E}\left\|\varphi\left(0\right)-\mathrm{g}\left(0,\varphi\right)\right\|^p\cdot\mathrm{e}^{-\lambda t}\\
&:=\tilde{K}_0\cdot\mathrm{e}^{-\lambda t}.
\end{aligned}
$$

For ϖ_2, it is clear that

$$
\varpi_2\le 4^{p-1}\left(L_{\mathrm{g}}\mathbb{E}\left\|u_t\right\|_{\mathcal{C}}^p+Z_0\left(t\right)\right).
$$

For ϖ_3, we have

$$
\begin{aligned}
\varpi_3 &\le 4^{p-1}\mathbb{E}\left(\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}\left\|\mathscr{R}\left(t+\theta,s\right)\sigma\left(s,u_s\right)\right\|\,\mathrm{d}s\right)^p\right)\\
&\le 4^{p-1}M^p\mathbb{E}\left(\sup_{\theta\in[-b,0]}\left(\int_0^{t+\theta}\mathrm{e}^{-\lambda(t+\theta-s)}\,\mathrm{d}s\right)^{p-1}\int_0^{t+\theta}\mathrm{e}^{-\lambda(t+\theta-s)}\left\|\sigma\left(s,u_s\right)\right\|^p\right)\\
&:=\tilde{K}_1\cdot\int_0^t\mathrm{e}^{-\lambda(t-s)}\left[L_\sigma\mathbb{E}\left\|u_s\right\|_{\mathcal{C}}^p+Z_1\left(s\right)\right]\mathrm{d}s.
\end{aligned}
$$

Concerning ϖ_4, one has

$$
\begin{aligned}
\varpi_4 &\le 4^{p-1}\left|\frac{\sin\pi\beta}{\pi}\right|^p\mathbb{E}\left(\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathscr{R}\left(t+\theta,s\right)\left(t+\theta-s\right)^{\beta-1}Y_2\left(s\right)\mathrm{d}s\right\|^p\right)\\
&\le 4^{p-1}M^p\left|\frac{\sin\pi\beta}{\pi}\right|^p\mathbb{E}\left(\sup_{\theta\in[-b,0]}\left\|\int_0^{t+\theta}\mathrm{e}^{-\lambda(t+\theta-s)}\left(t+\theta-s\right)^{\beta-1}Y_2\left(s\right)\mathrm{d}s\right\|^p\right),
\end{aligned}
$$

where

$$
Y_2\left(s\right)=\int_0^s\left(s-\tau\right)^{-\beta}\mathscr{R}\left(s,\tau\right)h\left(\tau,u_\tau\right)\mathrm{dB}\left(\tau\right).
$$

Thus by the Hölder inequality, we have

$$\varpi_4 \leq 4^{p-1} M^p \left|\frac{\sin \pi\beta}{\pi}\right|^p \mathbb{E}\left(\sup_{\theta\in[-b,0]} \left(\int_0^{t+\theta} (t+\theta-s)^{(p/p-1)(\beta-1)} \, \mathrm{e}^{-\lambda(t+\theta-s)} \, \mathrm{d}s\right)^{p-1}\right.$$
$$\left. \times \int_0^{t+\theta} \mathrm{e}^{-\lambda(t+\theta-s)} \left\|Y_2(s)\right\|^p \, \mathrm{d}s\right)$$
$$\leq 4^{p-1} \, \mathrm{e}^{\lambda b} M^p \left|\frac{\sin \pi\beta}{\pi}\right|^p \left[\Gamma\left(\frac{p}{p-1}(\beta-1)+1\right)/\lambda^{(p/p-1)(\beta-1)+1}\right]^{p-1}$$
$$\int_0^t \mathrm{e}^{-\lambda(t-s)} \mathbb{E}\left\|Y_2(s)\right\|^p \, \mathrm{d}s.$$

Using lemma (2.2), we can see that

$$\int_0^t \mathrm{e}^{-\lambda(t-s)} \mathbb{E}\left\|Y_2(s)\right\|^p \, \mathrm{d}s$$
$$= \int_0^t \mathrm{e}^{-\lambda(t-s)} \mathbb{E}\left(\left\|\int_0^s (s-\tau)^{-\beta} \mathscr{R}(s,\tau)\, h(\tau,u_\tau)\, \mathrm{dB}(\tau)\right\|^p\right) \mathrm{d}s$$
$$\leq \nu_p M^p \mathbb{E} \int_0^t \left(\int_0^s (s-\tau)^{-2\beta} \, \mathrm{e}^{-2\lambda(1/q)(s-\tau)} \, \mathrm{e}^{-2\lambda(1/p)(s-\tau)} \left\|h(\tau,u_\tau)\right\|^2_{\mathcal{L}_2^0} \, \mathrm{d}\tau\right)^{p/2} \mathrm{d}s,$$

and the Young inequality allows us to obtain that

$$\int_0^t \mathrm{e}^{-\lambda(t-s)} \mathbb{E}\left\|Y_2(s)\right\|^p \, \mathrm{d}s$$
$$\leq \nu_p M^p \left(\int_0^t (s-\tau)^{-2\beta} \, \mathrm{e}^{-2\lambda(1/q)(s-\tau)} \, \mathrm{d}s\right)^{p/2} \left(\int_0^t \mathrm{e}^{-\lambda(t-s)} \mathbb{E}\left\|h(s,u_s)\right\|^p_{\mathcal{L}_2^0} \, ds\right)$$
$$\leq \nu_p M^p \left(\Gamma(1-2\beta)\left(\frac{2\lambda}{q}\right)^{2\beta-1}\right)^{p/2} \left(\int_0^t \mathrm{e}^{-\lambda(t-s)} \left[L_h \mathbb{E}\left\|u_s\right\|^p_C + Z_2(s)\right] \mathrm{d}s\right).$$

Therefore,

$$\varpi_4 \leq 4^{p-1} \, \mathrm{e}^{\lambda b} M^p \left|\frac{\sin \pi\beta}{\pi}\right|^p \left[\Gamma\left(\frac{p}{p-1}(\beta-1)+1\right)/\lambda^{(p/p-1)(\beta-1)+1}\right]^{p-1}$$
$$\cdot \nu_p M^p \left(\Gamma(1-2\beta)\left(\frac{2\lambda}{q}\right)^{2\beta-1}\right)^{p/2} \left(\int_0^t \mathrm{e}^{-\lambda(t-s)} \left[L_h \mathbb{E}\left\|u_s\right\|^p_C + Z_2(s)\right] \mathrm{d}s\right)$$
$$:= \tilde{K}_2. \int_0^t \mathrm{e}^{-\lambda(t-s)} \left[L_h \mathbb{E}\left\|u_s\right\|^p_C + Z_2(s)\right] \mathrm{d}s.$$

Thus, considering the above inequalities, (33) becomes

$$\left(1-4^{p-1}L_g\right) \mathbb{E}\left\|u_t\right\|^p_C \leq \tilde{K}_0 \, \mathrm{e}^{-\lambda t} + 4^{p-1} Z_0(t) + \tilde{K}_3 \, \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s} \mathbb{E}\left\|u_s\right\|^p_C \, \mathrm{d}s$$
$$+ \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s} \left[\tilde{K}_1 Z_1(s) + \tilde{K}_2 Z_2(s)\right] \mathrm{d}s, \tag{34}$$

where

$$\tilde{K}_3 = \tilde{K}_1 L_\sigma + \tilde{K}_2 L_h. \tag{35}$$

Choosing randomly $\delta \in \mathbb{R}$ such that $0 < \delta < \lambda$ and $T > 0$ sufficiently large, we get that

$$\begin{aligned}\left(1 - 4^{p-1} L_{\mathrm{g}}\right) \int_0^T \mathrm{e}^{\delta t} \mathbb{E} \left\| u_t \right\|_C^p \mathrm{d}t \leq \tilde{K}_0 \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \mathrm{d}t + \int_0^T 4^{p-1} Z_0(t) \mathrm{e}^{\delta t} dt \\ + \tilde{K}_3 \int_0^T \mathrm{e}^{\delta t - \lambda t} \int_0^t \mathrm{e}^{\lambda s} \mathbb{E} \left\| u_s \right\|_C^p \mathrm{d}s \, \mathrm{d}t \\ + \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \int_0^t \mathrm{e}^{\lambda s} \left[\tilde{K}_1 Z_1(s) + \tilde{K}_2 Z_2(s) \right] \mathrm{d}s \, \mathrm{d}t.\end{aligned} \tag{36}$$

Now, by switching the order of integrals, one arrives at

$$\begin{aligned}\int_0^T \mathrm{e}^{\delta t - \lambda t} \int_0^t \mathrm{e}^{\lambda s} \mathbb{E} \left\| u_s \right\|_C^p \mathrm{d}s \, \mathrm{d}t &= \int_0^T \int_s^T \mathrm{e}^{\lambda s} \mathbb{E} \left\| u_s \right\|_C^p \mathrm{e}^{\delta t - \lambda t} \mathrm{d}t \, \mathrm{d}s \\ &\leq \frac{1}{\lambda - \delta} \int_0^T \mathrm{e}^{\delta s} \mathbb{E} \left\| u_s \right\|_C^p \mathrm{d}s.\end{aligned} \tag{37}$$

Then, substituting (37) into (36) yields that

$$\begin{aligned}\left(1 - 4^{p-1} L_{\mathrm{g}}\right) \int_0^T \mathrm{e}^{\delta t} \mathbb{E} \left\| u_t \right\|_C^p \mathrm{d}t \leq \tilde{K}_0 \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \mathrm{d}t + \int_0^T 4^{p-1} Z_0(t) \mathrm{e}^{\delta t} dt \\ + \frac{\tilde{K}_3}{\lambda - \delta} \int_0^T \mathrm{e}^{\delta s} \mathbb{E} \left\| u_s \right\|_C^p \mathrm{d}s \\ + \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \int_0^t \mathrm{e}^{\lambda s} \left[\tilde{K}_1 Z_1(s) + \tilde{K}_2 Z_2(s) \right] \mathrm{d}s \, \mathrm{d}t.\end{aligned}$$

Since $\tilde{K}_3$ can be small enough by assumption, a suitable $\delta \in \mathbb{R}^+$ can be selected with $0 < \delta < \lambda - \tilde{K}_3 / (1 - 4^{p-1} L_{\mathrm{g}})$ such that

$$1 - 4^{p-1} L_{\mathrm{g}} - \frac{\tilde{K}_3}{\lambda - \delta} > 0.$$

Thus, we have

$$\begin{aligned}&\int_0^T \mathrm{e}^{\delta t} \mathbb{E} \left\| u_t \right\|_C^p \mathrm{d}t \\ &\quad \leq \frac{1}{1 - 4^{p-1} L_{\mathrm{g}} - \tilde{K}_3 / (\lambda - \delta)} \left[\tilde{K}_0 \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \mathrm{d}t + \int_0^T 4^{p-1} Z_0(t) \mathrm{e}^{\delta t} \mathrm{d}t \right. \\ &\quad \left. + \int_0^T \mathrm{e}^{-(\lambda-\delta)t} \int_0^t \mathrm{e}^{\lambda s} \left(\tilde{K}_1 Z_1(s) + \tilde{K}_2 Z_2(s) \right) \mathrm{d}s \, \mathrm{d}t \right].\end{aligned} \tag{38}$$

Hence letting $T \to +\infty$ in (38) yields that

$$\begin{aligned}
&\int_0^{+\infty} \mathrm{e}^{\delta t}\mathbb{E}\left\|u_t\right\|_C^p \,\mathrm{d}t \\
&\quad \le \frac{1}{1-4^{p-1}L_g-\tilde{K}_3/(\lambda-\delta)}\left[\tilde{K}_0\int_0^{+\infty} \mathrm{e}^{-(\lambda-\delta)t}\,\mathrm{d}t+\int_0^{+\infty} 4^{p-1}Z_0(t)\,\mathrm{e}^{\delta t}\,\mathrm{d}t\right. \\
&\quad\quad \left.+\int_0^{+\infty} \mathrm{e}^{-(\lambda-\delta)t}\int_0^t \mathrm{e}^{\lambda s}\left(\tilde{K}_1Z_1(s)+\tilde{K}_2Z_2(s)\right)\mathrm{d}s\,\mathrm{d}t\right] \\
&\quad := \tilde{K}'\left(p,\delta,\varphi\right) < \infty. \qquad (39)
\end{aligned}$$

Conversely, it can be easily observed that the inequality (34) remains true when δ is used in place of λ, that is

$$\begin{aligned}
\left(1-4^{p-1}L_g\right)\mathbb{E}\left\|u_t\right\|_C^p \le &\tilde{K}_0\,\mathrm{e}^{-\delta t}+4^{p-1}Z_0(t)+\tilde{K}_3\,\mathrm{e}^{-\delta t}\int_0^t \mathrm{e}^{\delta s}\mathbb{E}\left\|u_s\right\|_C^p\,\mathrm{d}s \\
&+\mathrm{e}^{-\delta t}\int_0^t \mathrm{e}^{\delta s}\left(\tilde{K}_1Z_1(s)+\tilde{K}_2Z_2(s)\right)\mathrm{d}s. \qquad (40)
\end{aligned}$$

Therefore, (39) and (40) imply that

$$\begin{aligned}
\mathbb{E}\left\|u_t\right\|_C^p \le &\left[\tilde{K}_0+4^{p-1}Z_0(t)\,\mathrm{e}^{\delta t}+\tilde{K}_3\tilde{K}'\left(p,\delta,\varphi\right)+\int_0^{+\infty}\mathrm{e}^{\delta s}\left(\tilde{K}_1Z_1(s)+\tilde{K}_2Z_2(s)\right)\mathrm{d}s\right] \\
&\cdot\,\mathrm{e}^{-\delta t} \\
&:= \tilde{K}\left(p,\delta,\varphi\right)\mathrm{e}^{-\delta t},
\end{aligned}$$

which is the inequality we want. Then the proof is over. ■

The following result for almost surely stability is also obtained.

Theorem 4.2: *Assume that all the conditions of Theorem* 4.1 *hold. Then, the mild solution of* (3) *is almost surely exponentially stable. Moreover, there exists a positive constant* $\delta > 0$ *such that*

$$\limsup_{t\to+\infty}\frac{\log\left\|u(t)\right\|}{t} \le -\frac{\delta}{2p} \quad \text{a.s.}$$

Proof: The methods for proving this Theorem are the same as those used in [39] for Theorem 3.3, then we omit it. ■

5. Example

Consider the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[x(t,\eta)-ax(t-b,\eta)\right]=\gamma(t)\dfrac{\partial^2}{\partial\eta^2}\left[x(t,\eta)-ax(t-b,\eta)\right] \\ \quad +\displaystyle\int_0^t \Phi(s,t)\dfrac{\partial^2}{\partial\eta^2}\left[x(s,\eta)-ax(s-b,\eta)\right]\mathrm{d}s+\sigma_1(t,x(t-b,\eta)) \\ \quad +h_1(t,x(t-b,\eta))\,d\mathrm{B}(t), \quad t\geq 0,\ \eta\in[0,\pi], \\ x(t,0)=x(t,\pi)=0, \quad t\geq 0, \\ x(\theta,\eta)=\varphi(\theta,\eta), \quad \theta\in[-b,0],\ \eta\in[0,\pi], \end{cases} \tag{41}$$

where $\gamma(t)$ is Hölder continuous with order $0<\delta\leq 1$, meaning that there exists a positive constant L_γ such that

$$|\gamma(t)-\gamma(s)|\leq L_\gamma\,|t-s|^\delta \quad \text{for } t,s\in\mathbb{R}^+.$$

In addition, $\gamma(\cdot)$ is continuously differentiable and $\gamma(t)<-1$. The function $\Phi(\cdot,\cdot)\in \mathrm{BU}(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$, where $\mathrm{BU}(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ denotes the space of all defined bounded uniformly continuous functions from $\mathbb{R}^+\times\mathbb{R}^+$ into $\mathbb{R}$. Moreover, $t\to\frac{\partial}{\partial t}\Phi(t,s)$ is bounded and continuous from $\mathbb{R}^+$ to $\mathbb{R}$. $\mathrm{B}(t)$ is a one-dimensional Wiener process defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. The initial function $\varphi(\cdot)$ is $\mathcal{F}_0$-measurable, and a is a positive constant. The functions σ_1 and h_1 are continuous functions to be described below. Let $\mathbb{X}=\mathbb{V}=L^2([0,\pi];\mathbb{R})$ and the family of operators $\mathtt{A}(t)$ defined as

$$\begin{cases} \left(\mathtt{A}(t)f\right)(\eta)=\gamma(t)f''(\eta), \quad t\geq 0,\ \eta\in[0,\pi], \\ f\in\mathcal{D}(\mathtt{A}(t)(t))=\mathcal{D}(\mathtt{A})=H_0^1(0,\pi)\cap H^2(0,\pi), \end{cases}$$

We define the operator $\mathtt{A}$ as

$$\begin{cases} \left(\mathtt{A}f\right)(\eta)=f''(\eta), \quad t\geq 0,\ \eta\in[0,\pi], \\ f\in\mathcal{D}(\mathtt{A}(t))=\mathcal{D}(\mathtt{A})=H_0^1(0,\pi)\cap H^2(0,\pi), \end{cases}$$

and

$$\begin{cases} \left(\Gamma(t,s)f\right)(\eta)=\Phi(t,s)f''(\eta), \quad 0\leq s\leq t,\ \eta\in[0,\pi], \\ f\in\mathcal{D}(\mathtt{A}(t))=\mathcal{D}(\mathtt{A})=H_0^1(0,\pi)\cap H^2(0,\pi). \end{cases}$$

We assume that:

- **($\mathbf{E}_1$)** The function $\sigma_1:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $h_1:[0,+\infty)\to\mathbb{R}$ are continuous and global Lipschitz continuous with regard to the second variable.
- **($\mathbf{E}_2$)** There exist continuous functions $\zeta_1(\cdot)$, $\beta_1(\cdot):[0,+\infty)\to\mathbb{R}_+$ and positive constants $\Delta_1,\Delta_2>0$ such that

$$\begin{aligned} |\sigma_1(t,z)|^p &\leq \Delta_1|z|^p+\zeta_1(t), \quad \text{for } t\geq 0,\ z\in\mathbb{R} \\ |h_1(t,z)|^p &\leq \Delta_2|z|^p+\beta_1(t), \quad \text{for } t\geq 0,\ z\in\mathbb{R}, \end{aligned} \tag{42}$$

where ζ_1, β_1 satisfy that $|\zeta_1(t)| \leq N_1 \mathrm{e}^{-\lambda t}$ and $|\beta_1(t)| \leq N_2 \mathrm{e}^{-\lambda t}$ for some $N_k > 0$ small enough $(k = 1, 2)$. functions $\mathrm{g} : [0, +\infty) \times \mathcal{C}([-b, 0]; \mathbb{X}) \to \mathbb{X}$, $\sigma : [0, +\infty) \times \mathcal{C}([-b, 0]; \mathbb{X}) \to \mathbb{X}$ and $h : [0, +\infty) \times \mathcal{C}([-b, 0]; \mathbb{X}) \to \mathcal{L}_2^0(\mathbb{V}, \mathbb{X})$ as

$$\begin{aligned} \mathrm{g}(t, \phi)(\eta) &:= a\phi(-b)(\eta), \\ \sigma(t, \phi)(\eta) &:= \sigma_1(t, \phi(-b))(\eta) \\ h(t, \phi)(\eta) &:= h_1(t, \phi(-b))(\eta). \end{aligned}$$

Let

$$\begin{cases} u(t)(\eta) = x(t, \eta) \text{ for } t \in [0, T], \text{ and } \eta \in [0, \pi], \\ \varphi(\theta)(\eta) = \varphi(\theta)(\eta) \text{ for } \theta \in [-b, 0], \text{ and } \eta \in [0, \pi]. \end{cases} \tag{43}$$

Then, we can rewrite (41) in the following abstract form:

$$\begin{cases} d\left[u(t) - \mathrm{g}(t, u_t)\right] = \Big[\mathbb{A}(t)\left[u(t) - \mathrm{g}(t, u_t)\right] \\ \quad + \int_0^t \Gamma(t, s)\left[u(s) - \mathrm{g}(s, u_s)\right] \mathrm{d}s + \sigma(t, u_t)\Big] \mathrm{d}t \\ \quad + h(t, u_t)\,\mathrm{dB}(t), \quad t \in [0, T], \\ u_0 = \varphi \in L^p(\Omega, \mathcal{C}). \end{cases} \tag{44}$$

In the following, let us verify that our assumptions in Theorems 3.2 and 4.1 are satisfied.

• **Assumptions:** $(\mathbf{W}_1) - (\mathbf{W}_3)$.

- Assumption $(\mathbf{W}_1)$: It is not difficult to observe that for every $t \geq 0$, $\mathbb{A}(t) = \gamma(t)\Delta$ is stable. In addition, we have $(\tilde{\Gamma}(t)x)(\cdot) = \Phi(t + \cdot, t)\mathbb{A}x$ for every $x \in \mathcal{D}(\mathbb{A})$. Since $t \to \frac{\partial \Phi(t,s)}{\partial t}$ is continuously differentiable from $\mathbb{R}^+$ to $\mathbb{R}$, therefore $\tilde{\Gamma}(t)x$ is strongly continuously differentiable on $\mathbb{R}^+$ for every $x \in \mathcal{H}$. So, $(\mathbf{W}_1)$ is fulfilled.
- Assumption $(\mathbf{W}_2)$: By virtue of the continuity of $\Phi(\cdot, \cdot)$, it follows the continuity of $\tilde{\Gamma}(t)$ on $\mathbb{R}^+$. In addition,

$$\begin{aligned} \left\|\tilde{\Gamma}(t)x\right\|_{\mathcal{E}} &= \sup_{s \in \mathbb{R}^+} \left\|\left(\tilde{\Gamma}(t)x\right)(s)\right\|_{\mathbb{X}} \\ &= \sup_{s \in \mathbb{R}^+} |\Phi(t + s, t)|\, \|\mathbb{A}x\|_{\mathbb{X}} \\ &\leq \sup_{s \in \mathbb{R}^+} |\Phi(t + s, t)| \left(\|\mathbb{A}x\|_{\mathbb{X}} + \|x\|_{\mathbb{X}}\right). \end{aligned}$$

  Then, $(\mathbf{W}_2)$ is satisfied.
- Assumption $(\mathbf{W}_3)$: Let $x \in \mathcal{H}$, then $\frac{d}{ds}(\tilde{\Gamma}(t)x)(s) = \frac{\partial}{\partial s}\Phi(t + s, t)\mathbb{A}x$, from which yields that $\tilde{\Gamma}(t)x \in \mathcal{D}(\mathbb{D})$ and

$$\left(\mathbb{D}\tilde{\Gamma}(t)x\right)(s) = \frac{\partial}{\partial s}\Phi(t + s, t)\,\mathbb{A}x.$$

  Moreover, $t \to \mathbb{D}\tilde{\Gamma}(t)$ is continuous on $\mathbb{R}^+$ and

$$\left\|\mathbb{D}\tilde{\Gamma}(t)x\right\|_{\mathcal{E}} \leq \sup_{s \in \mathbb{R}^+} \left|\frac{\partial}{\partial s}\Phi(t + s, t)\right| \left(\|\mathbb{A}x\|_{\mathbb{X}} + \|x\|_{\mathbb{X}}\right).$$

  Then, $\mathbb{D}\tilde{\Gamma}(t) \in \mathcal{L}(\mathcal{H}, \mathcal{E})$. Hence, $(\mathbf{W}_3)$ holds.

• **Assumptions:** $(\mathbf{R}_1) - (\mathbf{R}_3)$.

The assumption $(\mathbf{R}_1)$ follows from $(\mathbf{W}_1)$. Clearly, with $\alpha_1 = 1, \alpha_2 = \frac{L_\gamma}{3}$ and $\kappa = \delta$, one can see that $(\mathbf{R}_2) - (\mathbf{R}_3)$ hold. For more details, see [3, Section 5]. Hence, there exists a unique evolution system $\{\mathcal{U}(t,s) : 0 \leq s \leq t\}$ generated by the family $\{\mathbb{A}(t) : t \geq 0\}$. Moreover,

$$\mathcal{U}\,(t,s)\,x = \sum_{n=1}^{+\infty} \mathrm{e}^{-n^2(t-s)+\int_s^t \gamma\,(r)\mathrm{d}r} \langle x, \xi_n \rangle \xi_n, \quad \text{for } x \in \mathbb{X}.$$

Note that the maps $R(\omega, \mathbb{A}(t)), t \geq 0$ are compact for each $\omega > 0$ (see [37]), which implies that $\mathcal{U}(t,s)$ is compact for $t - s > 0$. Consequently, it follows that $\mathcal{U}(t,s)$ is norm-continuous for $t - s > 0$.

• **Assumptions:** $(\mathbf{C}_1) - (\mathbf{C}_5)$.

Since, $(\mathbf{W}_1) - (\mathbf{W}_3)$ hold, the linear part of (41) has a nique resolvent operator $\{\mathscr{R}(t,s) : 0 \leq s \leq t\}$. Moreover, since $\mathcal{U}(t,s)$ is norm-continuous for $t - s > 0$, it follows that $\mathscr{R}(t,s)$ is norm-continuous for $t - s > 0$ (see Theorem 2.14). Furthermore, due to the assumptions $(\mathbf{E}_1) - (\mathbf{E}_2)$, one can easily confirm that $(\mathbf{C}_2) - (\mathbf{C}_5)$ are fulfilled by the functions $g\sigma$ and h.

Thanks to [14], we have for some choice of λ and Φ the following estimate : $\|\mathscr{R}(t,s)\|_{\mathcal{L}(\mathbb{X})} \leq \mathrm{e}^{-\lambda(t-s)}$ for all $t \geq 0$. Considering all these assumptions, by Theorems 3.2 and 4.1, system (41) has a unique mild solution $u(t)$ which is exponentially stable provided that Δ_1 and Δ_2 are small enough.

6. Conclusion

This paper investigated a class of non-autonomous stochastic partial integro-differential equations in Hilbert space. In the first instance, we utilized the theory of resolvent operator, fixed point approach, and stochastic analysis to prove the existence of mild solutions for the suggested system and the uniqueness of these solutions. Moreover, we investigated the exponential stability of mild solutions for the system under consideration. Finally, we provided an example to illustrate the obtained results. Over the past few years, fractional calculus has gained significance and relevance for various knowledge domains, particularly physics, engineering, mechanical engineering, and biology. We intend to examine the fractional stochastic differential equations [5,29,40] to render our findings more applicable to a broader range of situations.

Acknowledgments

The authors would like to thank the editor and referees for their careful reading of this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Ravikumar Kasinathan http://orcid.org/0000-0002-5635-8517

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